数理解析研究所講究録
1089 巻 1999 年 27-41

An Introduction to Markov Snakes in Dynkin-Kuznetsov's Work*

Isamu DÔKU (道工 勇) and Hiroyuki TAMURA (田村浩之)
Department of Mathematics, Saitama University
Urawa 338-8570 Japan
and
Graduate School of Education, Saitama University
Urawa 338-8570 Japan

1 Introduction

The purpose of this expository article is to introduce Dynkin-Kuznetsov's work on the Markov snake, which was recently studied in [DyK95]. The notion of Markov snake has been originally introduced by Le Gall [LG93], who calls it differently, say, the Brownian snake. Le Gall formulates the family of these newly introduced processes as a certain class of path-valued Markov processes, and it is well-known that he has been accomplishing so many remarkable interesting results by taking much advantage of the new object. For instance, 1) the study on sample path properties of superdiffusion, 2) the probabilistic research on partial differential equations with nonlinear operator $\Delta u - u^2$, etc. While, Dynkin and Kuznetsov [DyK95] are advocating the name, *Markov snake*, in connection with their series of works on branching measure-valued processes (cf. [Dy89], [Dy93], [Dy94] and [DyKSk94]). Successfully, they have recently established "an isomorphism theorem" that allows us to translate the results on continuous superprocesses into the language of Markov snakes, and vice versa. In addition, they have obtained new types of limit theorems for discrete Markov snakes by making use of this theorem. In what follows, we shall introduce mainly this isomorphism theorem for Markov snakes.

First of all, we begin with introducing notations. We denote by $\mathbf{W}$ the space $C([0,|w|], E)$ of all continuous paths in a Polish space E with its own domain $[0,u]$. Here we write

*Research supported in part by JMESC Grant-in-Aid SR(C) 07640280 and also by JMESC Grant-in-Aid CR(A)(1) 09304022, CR(A)(1) 10304006.

$|w|$ for the terminal time u and ∂w for the corresponding value $w(u)$. Put

$$\mathbf{W}_u \equiv \mathbf{W}_{[0,u]} := \{w \in \mathbf{W};\ |w| = u\}, \quad \mathbf{W}_{\leq b} := \{w \in \mathbf{W};\ |w| \leq b\}.$$

For $w \in \mathbf{W}_{[r,s]}$ and $\tilde{w} \in \mathbf{W}_{[s,t]}$, $r < s < t$, if $w(s) = \tilde{w}(s)$ holds, then we define

$$w\tilde{w} := w \quad \text{on} \quad [r,s]; \quad \text{and} \quad := \tilde{w} \quad \text{on} \quad [s,t].$$

When $|w| \geq a$, then $w_{\leq a}$ means that its domain is restricted on $[0,a]$. We denote by $\mathcal{F}_{\mathbf{W}}(\mathbf{R}_+)$ the σ-algebra in $\mathbf{W}_{[0,\infty]}$ generated by the cylinder sets. We sometimes write simply $\mathcal{F}_{\mathbf{W}}$ for $\mathcal{F}_{\mathbf{W}}(\mathbf{R}_+)$. The symbol $bp\mathcal{B}(\mathbf{R}_+)$ indicates the space of all positive bounded Borel-measurable functions on $\mathbf{R}_+$.

2 Snakes and Historical Superprocesses

Let $\xi = (\xi_t, \Pi_{r,x})$ be an E-valued continuous Markov process, and $\xi_{\leq t}$ denotes the path of ξ during time interval $[0,t]$. As a matter of fact, this $\xi_{\leq t}$ can be interpreted as the state of a path-valued process, and we call it the historical process for ξ. In [DyK95] ξ is assumed to be a right process, so we do. Here is the definition of historical process.

Definition 1 *We say that Z is the historical process for a Markov process $(\xi, \Pi_{r,x})$ if $\zeta_t = |Z_t| = t$ and if*

$$Z_t = Z_r\tilde{W}, \qquad \forall r < t,$$

where $\tilde{W}$ is the path of ξ on the interval $[r,t]$ started from the terminal point ∂Z_r.

The state space at time t is equal to $\mathbf{W}_t$. We often write $\xi_{\leq t}$ for Z_t. In other words, we consider Z as a Markov process with the transition probabilities determined by the formula

$$\Pi^Z_{r,w} f(\xi_{\leq t}) = \int_{\mathbf{W}_{[r,t]}} F(w\tilde{w})\Pi_{r,\partial w}(d\tilde{w}), \quad F \in bp\mathcal{B}(\mathbf{W}_t). \tag{1}$$

Definition 2 *$\mathcal{Z} = \{\mathcal{Z}_t\}$ is said to be a discrete random snake if $\mathcal{Z}_t$ is a $\mathbf{W}$-valued stochastic process with a discrete time parameter $t \in \mathbf{Z}_+$ such that*
(a) $\mathcal{Z}_{t+1}$ is the restriction of the path $\mathcal{Z}_t$ on the interval $[0, \zeta_{t+1}]$ if $\zeta_{t+1} \leq \zeta_t$, and
(b) $\mathcal{Z}_{t+1} = \mathcal{Z}_t\tilde{W}$ holds if $\zeta_{t+1} > \zeta_t$,
where $\tilde{W}$ is a random element of $\mathbf{W}[\zeta_t, \zeta_{t+1}]$.

Definition 3 *A simple snake $\mathcal{Z}^\beta$ is a discrete Markov snake for which ζ is the simple random walk on $\beta\mathbf{Z}_+$ with reflection at 0.*

Next we introduce the Markov snake.

Definition 4 *We say that $\mathcal{Z}$ is a Markov snake with parameters (ζ,ξ) (or simply a (ζ,ξ)-snake) if*
(a) ζ is a Markov process in $\mathbf{R}_+$,
(b) $\tilde{W}$ is the path of an E-valued continuous Markov process ξ on the interval $[\zeta_t,\zeta_{t+1}]$.

As a consequence, the Markov property of (ζ,ξ)-snake (= Markov snake) can be thought to be transplanted from those of (ζ,ξ). In fact we have the following observation. Before stating the result, we need some notations. Let $\Pi^\zeta_{r,x}$ be the transition probabilities of ζ, and $\Pi_{r,x}$ the transition probabilities of ξ. For simplicity, we write

$$a^* := \min_{u\in[r,t]} \zeta_u, \qquad b^* := \zeta_t.$$

The symbol $\zeta[r,\infty)$ stands for a path of ζ on $[r,\infty)$, i.e., for a map from $[r,\infty)\cap\mathbf{Z}_+$ to $\mathbf{R}_+$, and $\mathbf{W}^\zeta[r,\infty)$ is the space of all such paths.

Theorem 1 ([DyK95],Theorem 1.1, §1.2, p.444) *The (ζ,ξ)- snake $\mathcal{Z}$ is a Markov process with transition probabilities $\mathbf{P}_{r,w}$ connected with $\Pi^\zeta_{r,x}$ and $\Pi_{r,x}$ by the formula*

$$\mathbf{P}_{r,w}\varphi(\zeta[r,\infty))F(\mathcal{Z}_t) = \int_{\mathbf{W}^\zeta[r,\infty)} \varphi(\zeta)\Pi^\zeta_{r,|w|}(d\zeta)\int_{\mathbf{W}[a^*,b^*]} F(w_{\le a^*}\tilde{w})\Pi_{a^*,w(a^*)}(d\tilde{w}), \tag{2}$$

where φ, F are positive measurable functions.

The Brownian snake is a special case of the Markov snake in Dynkin-Kuznetsov's sense.

Definition 5 *The Brownian snake $\mathcal{Z}$ with parameter c is a Markov snake for which ζ is the reflectiong Brownian motion in $\mathbf{R}_+$ with generator $(c/2)(d^2/ds^2)$. For the special case $c=1$, it is called the standard Brownian snake.*

Lastly we introduce the so-called historical superprocess. We denote by $M_F(D)$ the space of all finite measures on D. The historical (ξ,ψ)-superprocess is an $M_F(\mathbf{W}_t)$-valued Markov process $(X_t,P_{r,\mu})$ such that the Laplace functional is given by

$$P_{r,\mu}\mathrm{e}^{-\langle f,X_t\rangle} = \mathrm{e}^{-\langle v^r,\mu\rangle}, \quad \forall\, r,\mu, \quad \forall\, f\in p\mathcal{B}(\mathbf{W}_t), \tag{3}$$

where v^r solves the following log-Laplace equation

$$v^r(w) + \Pi_{r,w}\int_r^t \psi(v^s)(\xi_{\le s})ds = \Pi_{r,w}f(\xi_{\le t}), \quad r\le t, \quad w\in\mathbf{W}_r. \tag{4}$$

Remark. In [DyK95] a process ξ is fixed, and terminology "a superprocess with branching parameter c" is used for the historical (ξ,ψ)-superprocess with branching mechanism $\psi(z) = (2/c)z^2$.

To state a limit theorem for discrete Markov snakes later, we need to define a specific type of snake. That is, we deal with a simple snake $\mathcal{Z}^\beta$ started at time 0 from point $x\in E$ and killed at the first return to x. The paths are loops in $\mathbf{W}$. More precisely,

Definition 6 *A simple snake $\mathcal{Z}^\beta$ is called a simple L-snake if each path is a continuous mapping $z : [0,u] \to \mathbf{W}$ such that*

$$z_0 = z_u = x \quad \textit{and} \quad z_s \neq x \quad \textit{for} \quad 0 < s < u.$$

We call such mappings $\mathbf{W}$-excursions from x, and denote by $\mathbf{Z}_x$ the space of all such excursions. The law of $\mathcal{Z}^\beta$ is a probability $\mathbf{P}_x^\beta$ on $\mathbf{Z}_x$.

Remark. With respect to the measure $\mathbf{P}_x^\beta$, the process $\zeta_t = |\mathcal{Z}_t^\beta|$ is for $t \geq 1$ the simple random walk killed at 0. However, the transition from $\zeta_0 = 0$ to $\zeta_1 = \beta$ just corresponds not to killing but to reflection at 0.

Analogously, we can define the Brownian L-snake as a Markov process $(\mathcal{Z}_t, \mathrm{N}_x)$, in which N_x is an infinite measure on the $\mathbf{W}$-excursion space $\mathbf{Z}_x$, closely related to the so-called Itô measure dn on the space of positive Brownian excursions. dn was originally introduced by K. Itô in his theory of the Brownian excursions [IM74]. We set

$$n_t(y) := \frac{y}{\sqrt{2\pi t^3}} \exp\left(-\frac{y^2}{2t}\right), \qquad n_t(dy) = n_t(y)dy.$$

Then N_x is a measure on $\mathbf{Z}_x$ with finite-dimensional distributions

$$\mathrm{N}_x\{z_{t_1} \in dw_1, \cdots, z_{t_k} \in dw_k\} = \nu_{t_1}(dw_1)\mathcal{P}(t_1, w_1; t_2, dw_2) \cdots \mathcal{P}(t_{k-1}, w_{k-1}; t_k, dw_k) \quad (5)$$

for every $0 < t_1 < \cdots < t_k$ where $\mathcal{P}$ is the transition function of the Brownian snake $\mathcal{Z}$ killed at $\mathbf{W}_0$ and the $\mathcal{P}$-entrance law ν is given by the formula

$$\nu_t(\cdot) = \int_0^\infty \Pi_x\{\xi_{\leq y} \in (\cdot)\}\, n_t(dy). \quad (6)$$

Definition 7 *A Markov process $(\mathcal{Z}_t, \mathrm{N}_x)$ is called the Brownian L-snake if it has $\mathbf{Z}_x$-valued paths such that ζ_t is the Brownian motion in $\mathbf{R}_+$ killed at 0 and started with an entrance law*

$$n_t(B) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} p(0, \varepsilon; t, B),$$

where p is the transition function of the killed Brownian motion.

3 $\mathcal{Z}$-Additive Functional and X-Additive Functional

Let $\mathcal{Z}$ denote a Brownian snake, and X a historical superprocess. $w_{\leq s}$ stands for the restriction of w to $[0,s]$. We say that a function $A(w)$ on $\mathbf{W}$ is monotone increasing if

$$A(w_{\leq s}) \leq A(w) \quad \text{holds} \quad \text{for} \quad \text{all} \quad s < |w|,$$

and that $A(w)$ is left continuous if

$$A(w_{\leq s}) \to A(w) \quad \text{as} \quad s \uparrow |w| \quad \text{for} \quad \text{every} \quad w \in \mathbf{W}.$$

Definition 8 *The class $\mathcal{A}$ is the set of all monotone increasing left continuous measurable functions $A(w)$ on $\mathbf{W}$ such that*

$$A(w_{\leq s}) \to 0 \quad ass \to 0$$

and $A(w)$ is bounded on each set $\mathbf{W}_{\leq b} = \{w\colon |w| \leq b\}$.

For every $A \in \mathcal{A}$ we define a time-homogeneous additive functional I_A of $\mathcal{Z}$ and an additive functional J_A of X. Recall that $\mathcal{Z}$ is a time-homogeneous process and X is a inhomogeneous process.

Definition 9 *We say that A belongs to the class $\mathcal{A}^0$ if $A \in \mathcal{A}$ and if there exists b such that $A(w) = A(w_{\leq b})$ for all w with $|w| > b$.*

Put $A \in \mathcal{A}_b$ if $A \in \mathcal{A}$ and if $A(w) = A(w_{\geq b})$ holds for $|w| \geq b$ and $A(w) \leq b$ holds for all w. Note that

$$\mathcal{A}^0 = \bigcup_b \mathcal{A}_b.$$

Write $A_n \Rightarrow A$ if $A_n \to A$ pointwise and if there exists b such that $A_n \in \mathcal{A}_b$ for all n. $A_n \Uparrow A$ means that $A_n \to A$ pointwise and that $A_{n+1} - A_n \in \mathcal{A}$ for all n. For every subclass $\tilde{\mathcal{A}}$ of class $\mathcal{A}$, we denote by $c(\tilde{\mathcal{A}})$ the minimal subset of $\mathcal{A}$ which contains $\tilde{\mathcal{A}}$ and is closed under operations $\Rightarrow$ and $\Uparrow$.

The followings are important typical examples of $\mathcal{Z}$-additive functionals I_A and X-additive functionals J_A.

Example 1 *If $A(w) = \mathbf{I}_{a<|w|}$, $a > 0$, then $L^a = I_A$ is the local time at point a for the reflecting Brownian motion ζ. If*

$$A(w) = \mathbf{I}_{a<|w|} f(w_{\leq a}),$$

then $I_A(ds) = f(\mathcal{Z}_s) L^a(ds)$, resp. $J_A(ds) = \langle f, X_s \rangle \delta_a(ds)$ gives respectively an important example of $\mathcal{Z}$-additive functional (resp. X-additive functional).

Definition 10 *The class $\mathcal{A}^1$ is the set of functions*

$$A_\Lambda(w) = \sum_{t \in \Lambda} \mathbf{I}_{t<|w|} f^t(w_{\leq t}), \tag{7}$$

where $\Lambda = \{t_1 < \cdots < t_n\}$ is a finite subset of $(0, \infty)$ and $f^t \in bp\mathcal{F}_{\mathbf{W}_t}$ for each $t \in \Lambda$.

Example 2 *If*

$$A(w) = \int_0^{|w|} f(w_{\leq s}) ds, \tag{8}$$

then the corresponding $\mathcal{Z}$-additive functional and X-additive functional are given respectively

$$I_A(ds) = f(\mathcal{Z}_s) ds, \qquad J_A(ds) = \langle f, X_s \rangle ds. \tag{9}$$

Definition 11 *The class $\mathcal{A}^2$ is the set of $a \in \mathcal{A}$ of the form Eq.(8) with a positive bounded function f vanishing on $|w| > b$ for some $b > 0$.*

Then we have the following equivalence

$$\mathcal{A} = c(\mathcal{A}^1) = c(\mathcal{A}^2)$$

(cf. Lemma 2.1, p.446, §2.2, [DyK95]).

Example 3 *Let $Q \subset \mathbf{W}$ be finely open and define the first exit time*

$$\tau := \inf\{t > 0 : \ w_{\leq t} \neq Q\} \tag{10}$$

similarly as usual one. Suppose that $f(w) = 0$ if $|w| = 0$. Let

$$A(w) = \begin{cases} 0, & \text{if } w_{\leq s} \in Q \quad \text{for all } s < |w|, \\ f(w_{\leq \tau}) & \text{otherwise.} \end{cases} \tag{11}$$

Then we have an X-additive functional $J_A(\mathbf{R}_+) = \langle f, X_\tau \rangle$.

4 Construction of $\mathcal{Z}$-Additive Functional I_A

In this section let ζ stand for the standard Brownian motion on $\mathbf{R}_+$ with reflection at 0 and let m_{st} be the minimum of ζ_u on the interval $[s,t]$. The symbol e.g. $\bar{\zeta}$ with bar is used for the case of standard Brownian motion killed at 0. The function

$$h(s,x) := \frac{2}{\sqrt{2\pi s}} \exp\left\{-\frac{x^2}{2s}\right\} \qquad \text{for} \quad 0 < s, x < \infty \tag{12}$$

is the probability density of ζ_s given $\zeta_0 = 0$ and

$$\theta(t,x;c,y) := \frac{2(x+y-2c)}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(x+y-2c)^2}{2t}\right\} \quad \text{for} \quad 0 < c < x \wedge y \tag{13}$$

is the joint probability density of $\bar{m}_{0t}$ and $\bar{\zeta}_t$ given $\bar{\zeta}_0 = x$ (cf. [IM74]).

If $A \in \mathcal{A}$, then to every $w \in \mathbf{W}$ there corresponds a measure $A(w,\cdot)$ on $\mathbf{R}_+$ concentrated on the interval $[0,|w|)$ such that

$$A(w,[0,s)) = A(w_{\leq s}) \quad \text{for} \quad \text{all} \quad 0 < s \leq |w|.$$

Note that, if A is given by formula Eq.(8) in Example 2, then

$$\int_\Delta \varphi(u) A(du) = \int_\Delta \varphi(u) f(w_{\leq u}) du$$

holds. For brevity's sake, put

$$H_0(s,t;u,v) := \begin{cases} h(s,u)h(t-s,u+v), & \text{if } \ s<t, \\ h(t,v)h(s-t,u+v), & \text{otherwise,} \end{cases} \tag{14}$$

and

$$H(s,t;u,c,v) := \begin{cases} h(s,u)\theta(t-s,u;c,v), & \text{if } \ s<t, \\ h(t,v)\theta(s-t,v;c,u), & \text{otherwise.} \end{cases} \tag{15}$$

Take $A, B \in \mathcal{A}^2$ associted with f, g respectively by formula Eq.(8). We may use the standard Brownian snake $\mathcal{Z}$ to define

$$I_A(ds) = f(\mathcal{Z}_s)ds, \qquad I_B(ds) = g(\mathcal{Z}_s)ds. \tag{16}$$

Then it is easy to see that

$$\mathbf{P}_x \int_0^\infty \varphi(t) I_A(dt) = \Pi_{0,x} \int_0^\infty A(du) \int_0^\infty h(t,u)\varphi(t)dt \tag{17}$$

holds for every $\varphi, \psi \in p\mathcal{B}_{R+}$. Moreover, a simple calculation together with the Markov property of $(\zeta_s, \Pi^\zeta_{r,x})$ gives the equality

$$\begin{aligned} \mathbf{P}_x \quad & \int_0^\infty \varphi(s) I_A(ds) \int_0^\infty \psi(t) I_B(dt) \\ = \quad & \int_0^\infty \varphi(s)ds \int_0^\infty \psi(t)dt \int_0^\infty dc \Pi_{0,x} \Pi_{c,\xi_{\le c}} \int_0^\infty A(du) \Pi_{c,\xi_{\le c}} \int_0^\infty H(s,t;u,c,v)B(dv) \\ + \quad & \int_0^\infty \varphi(s)ds \int_0^\infty \psi(t)dt \Pi_{0,x} \int_0^\infty A(du) \Pi_{0,x} \int_0^\infty H_0(s,t;u,v)B(dv), \end{aligned} \tag{18}$$

(cf. Lemma 2.2, p.447, §2.3, [DyK95]) when we make use of the formula (2) in Theorem 1.

Theorem 2 ([DyK95], Theorem 2.1, p.449) *There exists a mapping : $A \mapsto I_A$ from $\mathcal{A}$ to the set of homogeneous additive functionals of the standard Brownian snake $\mathcal{Z}$ which satisfies conditions (17), (18). This $\mathcal{Z}$-additive functional I_A is defined uniquely up to $\mathbf{P}_x$-equivalence for all x.*

The existence of an additive functional I_A is greatly due to Lemma 2.4.2 in [Dy94, p.33]. Dynkin's general theory of additive functionals shows that there exists a homogeneous version of I_A. Furthermore, this I_A has the following properties.

Proposition 1 *We have the following equalities in the sense of $\mathbf{P}_x$-equivalence.*
(a) $I_{cA} = cI_A$ holds for all $c \ge 0$, $A \in \mathcal{A}$.
(b) $I_{A+B} = I_A + I_B$ holds for all $A, B \in \mathcal{A}$.

If $A_n \Rightarrow A$, then for every t and every $\varphi \in bp\mathcal{B}_{R+}$,

$$\int_0^t \varphi(s) I_{A_n}(ds) \to \int_0^t \varphi(s) I_A(ds), \qquad \text{in} \quad \mathbf{P}_x - \text{probability}.$$

If $A_n \Uparrow A$, then for every $\varphi \in p\mathcal{B}_{R+}$,

$$\int_0^\infty \varphi(s) I_{A_n}(ds) \nearrow \int_0^\infty \varphi(s) I_A(ds), \qquad \mathbf{P}_x - \text{a.s.}$$

If

$$B(w) = \int_{[0,|w|)} f(w_{\leq s}) dA(w_{\leq s}) \qquad \text{in the sense of } \mathbf{P}_x - \text{equivalence}$$

where $f \in p\mathcal{F}_W$ and $A, B \in \mathcal{A}$, then $I_B(ds) = f(\mathcal{Z}_s) I_A(ds)$.

Proposition 2 *$I_A(ds)$ does not charge the set $\{s : \zeta_s = 0\}$.*

Note that the trivial case $I_A(ds) = ds$ corresponds to $A(w) = |w|$.

5 Branching Particle Systems

Let us consider a system of particles moving in E according to the law of a continuous Markov process $(\xi_t, \Pi_{r,x})$. Each particle lives for a constant time β and it produces at death time offspring of size n with probability p_n, $n = 0, 1, 2, \cdots$ Here we assume only that

$$\sum_{n=0}^{\infty} n p_n = 1, \qquad p_1 < 1.$$

The birth place of offspring coincides with the death place of the parent. There is no other intersection between particles. We write $b \ll b'$ if b is the parent of b'. The historical path of b is the combination

$$w^b = w(b_0) w(b_1) \cdots w(b_n)$$

of paths of $b = b_n$ and all its ancestors $b_0 \ll b_1 \ll \cdots \ll b_n$. We also assume that the process starts at time $a \geq 0$ by a finite number of progenitors with prehistories $w_i \in \mathbf{W}_a$.

Remark. Under these assumptions, all particles disappear almost surely after a finite number of generations.

Let $\mathcal{M}_Z^a \equiv \mathcal{M}(\mathbf{W}_a, \mathbf{Z}_+)$ denote the space of all finite integer-valued measures on $\mathbf{W}_a$. $P_{a,\nu}$ is the measure on the space of diagrams $\mathbf{D}$ which corresponds to the particle system started at time a by particles with prehistories $w_1, \cdots, w_n$. Put

$$P_{a,w} = P_{a,\delta_w}, \quad P_{a,M} = \int P_{a,\nu} M(d\nu) \quad \text{for} \quad M \in \mathcal{M}_Z^a.$$

Let t_b stand for the death time of b. The formula

$$X_t(B) = \sum_{\{b: t_b = t\}} \mathbf{I}_B(w^b), \quad t \in \beta \mathbf{Z}_+$$

determines a Markov process in the state space $\mathcal{M}_Z^t$ with the transition Probabilities $P_{a,\nu}$. Here the state space changes in time.

Remark. The probability distribution of w^b under $P_{a,w}$ is identical to the probability distribution of $\xi[0,t_b]$ under $\Pi_{a,w}$.

A random measure $\mathcal{X}$ on $\mathbf{W}$ is defined by

$$\mathcal{X}(C) = \sum_b \mathbf{I}_C(w^b).$$

Notice that the measure X_t is the restriction of $\mathcal{X}$ to $\mathbf{W}_t$.

$$\varphi(z) = \sum_{n=0}^{\infty} p_n z^n$$

is the offspring generating function, and put

$$\psi_\beta(z) = \frac{1}{\beta^2}\{\varphi(1-\beta z) - 1 + \beta z\}.$$

σ_β is a measure on $\beta\mathbf{Z}_+$ given by the formula $\sigma_\beta(s) = \beta$ for all $s \in \beta\mathbf{Z}_+$. Then the quantity

$$\beta\Pi_{a,w}\sum_s \psi_\beta(u)(\xi_{\leq s})$$

is well-defined for $0 \leq u \leq 1/\beta$ and may be expressed alternatively by

$$\Pi_{a,w}\int_{(a,\infty)} \psi_\beta\left(u(\xi_{\leq s})\right)\sigma_\beta(ds).$$

Here $a \in \beta\mathbf{Z}_+$ and the sum is taken over $s \in \beta\mathbf{Z}_+ \cup (a,\infty)$.

Now we think of passing to the limit $\beta \to 0$. We suppose that a positive function F on $\mathbf{W}$ depends on β and also that $\psi_\beta(z) \to \psi(z)$. We set

$$\psi(z) := z^2, \qquad \Psi(z)(w) := \Pi_{a,w}\int_a^{\infty} \psi(z)(\xi_{\leq s})ds.$$

In addition, for $A \in \mathcal{A}^0$, we define

$$j_A^\beta = 2\beta\sum_{b' \ll b}\left\{A(w^b) - A(w^{b'})\right\}.$$

$N(c)$ is the Poisson random variable with mean c. First of all, we obtain

Proposition 3 *For every positive function F on $\mathbf{W}$, the Laplace functional of $\mathcal{X}$ is given by*

$$P_{a,N(\rho/\beta)w}e^{-\beta\langle F,\mathcal{X}\rangle} = \exp\{-\rho v(w)\}, \tag{19}$$

where v solves the following log-Laplace equation

$$v + \Pi_{a,w}\int_a^{\infty}\psi_\beta\left(v(\xi_{\leq s})\right)\sigma_\beta(ds) = \Pi_{a,w}\sum_s F(\xi_{\leq s}) + \Pi_{a,w}\int_a^{\infty} G_\beta(v)(\xi_{\leq s})\sigma_\beta(ds) \tag{20}$$

as long as $0 \leq v \leq 1/\beta$, where

$$G_\beta(v) := \{\beta\psi_\beta(v) - v\}\frac{1-e^{-\beta F}}{\beta} - \frac{1}{\beta^2}\left(e^{-\beta F} - 1 + \beta F\right).$$

See [Da93]. As a matter of fact, the passage to the limit $\beta \to 0$ is justified for a special family F_β, namely,

$$F_\beta(w) = A(w) - A(w_{b-\beta}).$$

Theorem 3 ([DyK95], Theorem 4, p.456, §4.2) *The following limit exists*

$$\lim_{\beta\downarrow 0} P_{a,N(\rho/\beta)w} \exp\left\{-\frac{1}{2}j_A^\beta\right\} = e^{-\rho v(w)}, \tag{21}$$

where v is a solution of the equation

$$v + \Psi(v) = \Pi_{a,w}\left\{A(\xi_{<\infty}) - A(w)\right\}. \tag{22}$$

An application of Theorem 3.4.2(p.53) in [Dy94] together with Proposition 3 guarantees that Eq.(22) has a positive solution v. As a corollary of Theorem 3, we can readily obtain the following result, whereby the existence of historical (ψ, ξ)-superprocess $\hat{X}$ is guaranteed in accordance with the definition (3), (4). Let $X_{\hat{t}_\beta}$ be a $\mathcal{M}_Z(\hat{t}_\beta)$-valued Markov process defined in the former part of Section 5, with time

$$\hat{s}_\beta := k\beta, \qquad \text{for} \quad (k-1)\beta \le s < k\beta.$$

Let $\hat{X}_t$ be the image of $X_{\hat{t}_\beta}$ under the mapping : $w \mapsto w_{\le t}$ from $\mathbf{W}_{\hat{t}_\beta}$ to $\mathbf{W}_t$.

Corollary 1 *For every $a < t$ and every $f \in bp\mathcal{F}_{W_t}$, we have*

$$\lim_{\beta\downarrow 0} P_{a,\pi_\mu/\beta} \exp\left\{-\beta\langle f, \hat{X}_{\hat{t}}\rangle\right\} = e^{-\langle v,\mu\rangle}, \tag{23}$$

where v is a solution of the equation

$$v(w) - \Psi(v)(w) = \Pi_{a,w} f(\xi_{\le t}) \qquad \textit{for all} \quad a < t. \tag{24}$$

6 Construction of X-Additive Functional J_A

It is well-known that Eq.(4) has the unique solution (cf. Theorem I.2.1 in [Dy93]) for a large class of functions ψ which includes $\psi(v) = cv^2$, $(c > 0)$ and for all $f \in bp\mathcal{F}_{W_t}$. Note that the aforementioned Eq.(24) is equivalent to Eq.(4). We denote the unique solution by $V_t^r f$. Operators V_t^r preserve bounded convergence and they form a semigroup (cf. [Da93]; see also [Dy94, §1.1]), which we call the log-Laplace semigroup generated by ψ. By Eq.(23)-(24) in Corollary 1, $\exp\{-\langle V_t^a f, \mu\rangle\}$ is the limit, as $\beta \to 0$, of the Laplace functionals

$$\int e^{-\langle f,\nu\rangle} M_\beta(d\nu),$$

where M_β is the probability distribution of $\beta\hat{X}_t$ relative to $P_{a,\pi_\mu/\beta}$. Therefore, by virtue of arguments in [Dy94, Chap. 3, 3.4C, p.51](see also [T99]), it follows that functional $\exp\{-$

$\langle V_t^a f, \mu\rangle\}$ is the Laplace transform of a measure $\mathcal{P}$ on $\mathbf{W}_t$ which depends on a, w and t. Hence we denote it by

$$\mathcal{P}(a, w; t, \cdot).$$

The semigroup property of V_t^a implies that $\mathcal{P}$ satisfies the Chapman-Kolmogorov equation, and therefore $\mathcal{P}$ proves to be the transition function of a Markov process $X = (X_t, P_{r,\mu})$. (For the detail of the proof, see [Dy93]). Clearly, X satisfies the definition of a historical superprocess in Section 2 (cf. (3),(4)).

It is known [F88] that there exists a version of X_t such that $\langle f, X_t\rangle$ is right continuous a.s. relative to all measure $P_{a,w}$, for every R-continuous function f. Moreover, $P_{t,X_t}Y$ is a.s. right continuous on interval $[0, u]$ for every positive Y which is measurable with respect to the σ-algebra generated by X_s, $s \geq u$. The next assertion easily yields from Theorem 3 and discussions in [Dy94]. The proof goes similarly as in the proof of Theorem 2.

Theorem 4 ([DyK95], Theorem 4.2, p.459, §4.4) *Let X be the historical (ξ, ψ)-superprocess with twice continuously differentiable ψ. There exists a mapping : $A \mapsto J_A$ from $\mathcal{A}$ to the set of additive functionals of X such that, for all $r \geq 0$, $\mu \in M_F(\mathbf{W}_t)$,*

$$P_{r,\mu} e^{-J_A[r,\infty)} = e^{-\langle v,\mu\rangle} \tag{25}$$

where v is a solution of Eq.(22) in Theorem 3.

Moreover, this X-additive functional J_A has quite similar properties as $\mathcal{Z}$-additive functional has in Section 4.

Proposition 4 *(a) $J_{cA} = cJ_A$ a.s. for all $c \geq 0$.*
(b) $J_{A+B} = J_A + J_B$ a.s. for $A, B \in \mathcal{A}$.

We have $P_{r,\mu}J_A[r,t) = \Pi_{r,\mu}A[r,t)$, where $A[r,t) = A(\xi_{\leq t}) - A(\xi_{\leq r})$. If $q = \Psi''(0)$, then

$$P_{r,\mu}J_A[r,t)J_{\tilde{A}}[r,\tilde{t}) = \Pi_{r,\mu}A[r,t)\Pi_{r,\mu}\tilde{A}[r,\tilde{t}) + q\Pi_{r,\mu}\int_r^{t\wedge\tilde{t}} \Pi_{s,\xi_s}A[s,t)\Pi_{s,\xi_s}\tilde{A}[s,\tilde{t})ds.$$

Proposition 5 *(a) If $A_n \Rightarrow A$, then for all $r < t \in \mathbf{R}_+$, $\mu \in M_F(\mathbf{W}_r)$,*

$$J_{A_n}[r,t) \to J_A[r,t) \qquad \text{in} \quad L^2(P_{r,\mu}).$$

(b) If $A_n \Uparrow A$, then $J_{A_n}[r,t) \nearrow J_A[r,t)$ holds $P_{r,\mu}$-a.s. for all $r < t \in \mathbf{R}_+$, $\mu \in M_F(\mathbf{W}_r)$.

7 The Isomorphism Theorem

Suppose that X_i is a positive measurable function on a probability space $(\Omega_i, \mathcal{F}_i, P_i)$, for $i = 1, 2$. Then we write

$$(X_1, P_1) \Longleftrightarrow (X_2, P_2)$$

if the probability distribution of X_1 relative to P_1 is identical to the probability distribution of X_2 relative to P_2.

Through the whole section, let $\mathcal{Z}$ be a Brownian snake with parameter c and X a superprocess with branching parameter c. We denote by $L^0[0,s)$ the local time at 0 for the reflecting Brownian motion $\zeta_t = |\mathcal{Z}_t|$. Define

$$\sigma_\rho := \inf\{t > 0 : \ L^0[0,t) \geq \rho\}.$$

Then $\mathbf{P}_{r,w}$ are the transition probabilities of $\mathcal{Z}$. We are now in a position to state one of the main results in [DyK95].

Theorem 5 (The Isomorphism Theorem) *For all $x \in E$ and $A \in \mathcal{A}$, we have the equivalence*

$$(I_A[0,\sigma_\rho), \mathbf{P}_{0,w}) \Longleftrightarrow (J_A, P_{0,\rho\delta_x}), \qquad \text{for} \quad w = x \in \mathbf{W}_0. \tag{26}$$

Simple applications of this isomorphism theorem allow us to derive several important assertions.

Application 1. For every $A_1, \cdots, A_n \in \mathcal{A}$, the joint probability distribution of

$$I_{A_1}[0,\sigma_\rho), \cdots, I_{A_n}[0,\sigma_\rho)$$

relative to $\mathbf{P}_{0,w}$ (for $w = x \in \mathbf{W}_0$) is the same as the joint probability distribution of $J_{A_1}, \cdots, J_{A_n}$ relative to $P_{0,\rho\delta_x}$.

Application 2. For all $x \in E$, $t \in \mathbf{R}_+$ and all positive measurable functions f, the following equivalence holds, i.e.,

$$\left(\int_0^{\sigma_\rho} f(\mathcal{Z}_s) L^t(ds), \mathbf{P}_{0,x}\right) \Longleftrightarrow (\langle f, X_t \rangle, P_{0,\rho\delta_x}). \tag{27}$$

This relation has been established by Le Gall(1993) in [LG93; Theorem 2.1].

Application 3. Let τ be the first exit time of $\xi_{\leq t}$ from a finely open set $Q \subset \mathbf{W}$. Let L^Q be the additive functional I_A corresponding to

$$A(w) = \begin{cases} 0, & \text{if } w_{\leq s} \in Q \quad \text{for all } s < |w|, \\ 0, & \text{if } w_{\leq 0} \notin Q, \\ 1, & \text{otherwise.} \end{cases} \tag{28}$$

Then the additive functional $f(\mathcal{Z}_s)L^Q(ds)$ corresponds to Eq.(11) in Example 3 of §3. For all $x \in E$ and all positive measurable functions f such that $f(w) = 0$ if $|w| = 0$, we readily obtain

$$\left(\int_0^{\sigma_1} f(\mathcal{Z}_s) L^Q(ds), \mathbf{P}_{0,x}\right) \Longleftrightarrow (\langle f, X_\tau \rangle, P_{0,\delta_x}). \tag{29}$$

Here functional L^Q is called the exit local time of $\mathcal{Z}$ from Q.

Remark. If $Q = \{w : \ |w| < a\}$, then formula Eq.(28) is identical to the simple case given in Example 1 of §3. In this case, $L^Q = L^a$ is nothing but the Brownian local time at a.

8 Limit Theorems for Simple Snakes

8.1 The Case of $\mathcal{Z}$-Additive Functional

Suppose that a measurable function X^β is given on a probability space $(\Omega^\beta, \mathcal{F}^\beta, P^\beta)$ for every $\beta > 0$, and let X be a measurable function on a probability space $(\Omega, \mathcal{F}, P)$. We write

$$(X^\beta, P^\beta) \Longrightarrow (X, P)$$

if the probability distribution of X^β relative to P^β converges weakly to the probability distribution of X relative to P as $\beta \to 0$.

We consider a simple snake $\mathcal{Z}^\beta$. To every $A \in \mathcal{A}$, there correponds a random measure i_A^β on $\mathbf{Z}_+$ given by the formula

$$i_A^\beta(t) = \beta \left| A(\mathcal{Z}_t^\beta) - A(\mathcal{Z}_{t-1}^\beta \right|. \tag{30}$$

We denote by $\mathcal{L}^\beta$ the measure on $\mathbf{Z}_+$ which charges every point of the set $\{t : \zeta(t) = 0\}$ by 2β. For simplicity we put

$$l^\beta(t) := \mathcal{L}^\beta(0, t), \qquad l(t) := L^0[0, t).$$

Then we have the following limit theorem for discrete Markov snakes, in terms of $\mathcal{Z}$-additive functional I_A, namely,

Theorem 6 ([DyK95], Theorem B, p.437) *Let $(\mathcal{Z}_t^\beta, \mathbf{P}_{0,x}^\beta)$ be a simple snake and $(\mathcal{Z}_t, \mathbf{P}_{0,x})$ the standard Brownian snake. If $A \in \mathcal{A}^0$ and if φ is integrable and piecewise continuous, then*

$$\left(\int_0^\infty \varphi(l^\beta(s)) i_A^\beta(ds), \mathbf{P}_{0,x}^\beta \right) \Longrightarrow \left(\int_0^\infty \varphi(l(s)) I_A(ds), \mathbf{P}_{0,x} \right) \tag{31}$$

holds as $\beta \to 0$, where I_A is a $\mathcal{Z}$-additive functional constructed in Theorem 2 of §4.

8.2 The Case of X-Additive Functional

It is obvious that the path-valued process $W_t = w^{b_t}$ for $t = 0, 1, \cdots$ is a simple L-snake. To prove this, we have only to pay attention to the key fact that $|W_t|$ is the simple random walk killed at the first return to 0. Therefore, for every $A \in \mathcal{A}$, we get the equivalence

$$(i_A^\beta(0, \gamma_1], \mathbf{P}_{0,x}) \Longleftrightarrow (j_A^\beta, P_{0,\delta_x}),$$

where γ_1 is the time of the first return of ζ to 0, and j_A^β is defined in Section 5.

P_{Nx} stands for the probability law of the branching system started by N independent progenitors at point x. Then it follows immediately from the above-mentioned equivalence that

$$\mathbf{E}\left[\mathbf{P}_{0,x}[\exp\left\{-i_A^\beta(0, \gamma_1]\right\}]\right]^N = P_{Nx} e^{-j_A^\beta}.$$

Theorem 7 ([DyK95], Theorem B', p.440) *Let X be the standard historical superprocess. Then for all $x \in E$, $A \in \mathcal{A}^0$ and $\rho > 0$,*

$$(j_A^\beta, P_{N(\rho/2\beta)x}) \Longrightarrow (J_A, P_{0,\rho x}), \tag{32}$$

where J_A is an X-additive functional constructed in Theorem 4 of §6.

9 Limit Theorems for L-Snakes

By virtue of some discussions on Poisson random measures (cf. Appendix, pp.469-471, [DyK95]), we can easily verify with Proposition 2 of §4 that

$$-\log \mathbf{P}_{0,x} \exp\left(-\int_0^\infty \varphi(l(s)) I_A(ds)\right) = \int_0^\infty \int_{\mathbf{Z}_x} \left\{1 - \mathrm{e}^{-\varphi(t) I_A(z)}\right\} dt \mathrm{N}_x(dz) \tag{33}$$

holds for every $A \in \mathcal{A}$ and for every positive Borel function φ. Here $I_A(z)$ indicates the value of the additive functional of the Brownian L-snake on its life interval. By applying Eq.(33) to $\varphi(u) = \mathbf{I}_{u<1}$, we get

$$-\log \mathbf{P}_{0,x} \mathrm{e}^{-I_A[0,\sigma_1)} = \int_{\mathbf{Z}_x} \left\{1 - \mathrm{e}^{-I_A(z)}\right\} \mathrm{N}_x(dz). \tag{34}$$

Hence we may employ Eq.(26) in Theorem 5 of Section 7 together with (34) to obtain

$$\mathrm{N}_x[1 - \mathrm{e}^{-I_A}] = -\log P_{0,\delta_x} \mathrm{e}^{-J_A}. \tag{35}$$

Thus we attain

Theorem 8 ([DyK95], Theorem C, p.442) *Let $(\mathcal{Z}^\beta, \mathbf{P}_{0,x}^\beta)$ be a simple L-snake and let $(\mathcal{Z}, \mathrm{N}_x)$ be the Brownian L-snake. Put*

$$i_A^\beta := \beta \sum_s \left| A(\mathcal{Z}_s^\beta) - A(\mathcal{Z}_{s-1}^\beta) \right|.$$

If $A \in \mathcal{A}^0$ and if g is a positive continuous function on $\mathbf{R}_+$ such that $g(u)/u$ is bounded, then

$$\lim_{\beta \to 0} \frac{1}{2\beta} \mathbf{P}_{0,x}^\beta [g(i_A^\beta)] = \mathrm{N}_x g(I_A), \tag{36}$$

where $I_A = I_A[0, i^*)$ with the terminal point i^* of its life interval.

References

[Da93] Daeson, D.A. : Measure-valued Markov processes, *LNM* **1541**(1993, Springer), 1-260.

[Dy89] Dynkin, E.B. : Superprocesses and their linear additive functionals, *Trans. Amer. Math. Soc.* **314**(1989), 255-282.

[Dy93] Dynkin, E.B. : Superprocesses and partial differential equations, *Ann. Probab.* **21**(1993), 1185-1262.

[Dy94] Dynkin, E.B. : *An Introduction to Branching Measure-Valued Processes*, AMS, Providence, 1994.

[DyK95] Dynkin, E.B. and Kuznetsov, S.E. : Markov snakes and superprocesses, *Prob. Th. Rel. Fields* **103**(1995), 433-473

[DyKSk94] Dynkin, E.B., Kuznetsov, S.E. and Skorokhod, A.V. : Branching measure-valued processes, *Prob. Th. Rel. Fields* **99**(1994), 55-96.

[DST98] Dôku, I., Sameshima, H. and Tamura, H. : The ultimate value of Brownian local time and the elastic boundary value problem (in Japanese), *Collection of Abstracts of Math.Soc.Japan Annual Meeting*, Meijo Univ.(Nagoya), March 26-29,1998, pp.3-4.

[DT98] Dôku, I. and Tamura, H. : The Brownian local time and the elastic boundary value problem, *J. SU Math. Nat. Sci.* **47**(1998), 1-5.

[DTn98a] Dôku, I. and Tanuma, F. : Explicit representations of the quadratic variational process of semimartingale term associated with a historical martingale problem (in Japanese), *Collection of Abstracts of Math. Soc. Japan Annual Meeting 1998*, Meijo Univ. (Nagoya), March 26-29, 1998, pp.5-6.

[DTn98b] Dôku, I. and Tanuma, F. : Example of representation of quadratic variation processes related to the historical martingale problem, *J.SU Math.Nat.Sci.***47**(1998),7-16.

[F88] Fitzsimmons, P.J. : Construction and regularity of measure-valued Markov branching processes, *Israel J. Math.* **64**(1988), 337-361.

[IM74] Itô, K. and McKean, H.P.Jr. : *Diffusion Processes and their Sample Paths*, Springer-Verlag, Berlin, 1974.

[LG93] Le Gall, J.-F. : A class of path-valued Markov processes and its applications to superprocesses, *Prob. Th. Rel. Fields* **95**(1993), 25-46.

[S98] Sameshima, H. : On the weak solutions for McKean-Vlasov equations and nonlinear martingale problems (in Japanese), Master Thesis, Saitama Univ., Urawa, January, 1998.

[T98] Tamura, H. : Introduction to Dynkin-Kuznetsov's work on Markov snakes, *Collection of Abstracts of RIMS Workshop on Stoch. Anal. MVP*, Nov.4-6, 1998, p.5.

[T99] Tamura, H. : Research on branching measure-valued processes: On construction of the branching mesure-valued process and its mathematical structures (in Japanese), Master Thesis, Saitama Univ., Urawa, February, 1999.

[Tn98] Tanuma, F. : Research on probability measure-valued diffusion processes (in Japanese), Master Thesis, Saitama Univ., Urawa, January, 1998.