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HIGHER-CODIMENSIONAL BOUNDARY VALUE PROBLEMS AND F-MILD MICROFUNCTIONS —LOCAL AND MICROLOCAL UNIQUENESS—

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INTRODUCTION.

In this article, we shall state results of our recent paper [O-Y] about the higher-codimensional boundary value problem for a general system of linear partial differential equations with analytic coefficients. In general, we must impose some regularity condition on the solutions in order to define their boundary values. We introduce the notion of F-mild hyperfunctions as this regularity condition. Let M be a real analytic manifold and N a closed real analytic submanifold of M of codimension $d \geqslant 2$. Then the sheaf $\mathcal{B}^F_{N|M}$ of F-mild hyperfunctions is defined on the normal bundle T_NM of N (strictly speaking, the sheaf $\mathcal{B}^F_{N|M}$ depends on a partial complexification L of M). Let us take a local coordinate system $(t,x) = (t_1,\dots,t_d,x_1,\dots,x_n)$ of M such that N is defined by $t=0$. The restriction of $\mathcal{B}^F_{N|M}$ to the zero-section of T_NM coincides with the sheaf of hyperfunctions defined on a neighborhood of N which have t as real analytic parameters. Moreover, a section of $\mathcal{B}^F_{N|M}$ which is defined on T_NM with the zero section removed has also t as real analytic parameters on a neighborhood of N. Hence we may regard $\mathcal{B}^F_{N|M}$ as a tangential decomposition (specialization) of the sheaf of hyperfunctions with real analytic parameters. We also define the notion of F-mild microfunctions as a microlocalization of that of F-mildness.

We take complexifications X and Y of M and N respectively such that Y is a closed submanifold of X. We denote by $\mathcal{D}_X$ the sheaf on X of rings of linear partial differential operators (of finite order) with holomorphic coefficients.

Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-Module; that is, a system of linear partial differential equations with holomorphic coefficients (in this article, we shall write Module with a capital letter, instead of sheaf of modules). Then, our main result is the local and microlocal uniqueness of F-mild hyperfunction solutions of a system of linear partial differential equations which is Fuchsian along Y in the sense of Y. Laurent and T. Monteiro Fernandes [L-MF] or in the sense of N. S. Madi [M] and S. Yamazaki [Y].

First, suppose that $\mathcal{M}$ is Fuchsian along Y in the sense of Laurent and Monteiro Fernandes [L-MF]. In this case, not all the hyperfunction solutions of $\mathcal{M}$ are necessarily F-mild, but we can obtain the local and microlocal uniqueness for F-mild solutions.

Next, assume that $\mathcal{M}$ is a Fuchs-Goursat system in the sense of Yamazaki [Y], which is a generalization of a Fuchs-Goursat operator due to Madi [M]. In this case, we consider a kind of Goursat problem and prove the local and microlocal uniqueness of the F-mild solution of $\mathcal{M}$ whose Goursat data are zero. Note that Yamazaki [Y] proved the (micro-)local solvability of this Goursat problem for microfunctions with real analytic parameters under a kind of (micro-)hyperbolicity condition.

We should remark the following: The higher-codimensional boundary value problem for hyperfunctions was initiated by M. Kashiwara and T. Kawai [K-K] for elliptic systems of differential equations from the microlocal point of view. After that, M. Kashiwara and T. Oshima ([K-Os], [Os]) defined the boundary values of an arbitrary hyperfunction solution of $\mathcal{M}$ which is defined in $\{(t,x) \in \mathbb{R}^d \times \mathbb{R}^n;\, t_i > 0\,(1 \leqslant i \leqslant d)\}$ under a condition stronger than that of Fuchsian system in the sense of Laurent-Monterio Fernandes [L-MF].

§1. F-Mild Hyperfunctions.

We denote the sets of integers, real numbers and complex numbers by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively as usual. Further, we set $\mathbb{N} := \{n \in \mathbb{Z};\, n \geqslant 1\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Let M be a $(d+n)$-dimensional real analytic manifold and N an n-dimensional closed real analytic submanifold of M. In this paper, we always assume that $d \geqslant 2$. There exist complexifications X and Y of M and N respectively such that Y is a closed submanifold of X. We assume that there exists a $(d+2n)$-dimensional real analytic submanifold L of X containing both M and Y such that the triplet (N, M, L) is locally isomorphic to the triplet $(\{0\} \times \mathbb{R}^n, \mathbb{R}^{d+n}, \mathbb{R}^d \times \mathbb{C}^n)$ by a local coordinate system (τ, z) of X around each point of N. We call such a local coordinate system *admissible*. We use the notation $\tau = t + \sqrt{-1}\,s$ $(t,\, s \in \mathbb{R}^d)$, $z = x + \sqrt{-1}\,y$ $(x,\, y \in \mathbb{R}^n)$, $|z| = \max\{|z_k|;\, 1 \leqslant k \leqslant n\}$ and so on for an admissible local coordinate system (τ, z). Hence by an admissible local coordinate system the following inclusion relations are obtained:

$$\begin{tikzcd} N = \{0\} \times \mathbb{R}^n_x \arrow[r, hook] \arrow[d, hook] & M = \mathbb{R}^d_t \times \mathbb{R}^n_x \arrow[d, hook, "i"] \arrow[dr, hook, "i_M"] & \\ Y = \{0\} \times \mathbb{C}^n_z \arrow[r, hook, "i_Y"] & L = \mathbb{R}^d_t \times \mathbb{C}^n_z \arrow[r, hook, "i_L"] & X = \mathbb{C}^d_\tau \times \mathbb{C}^n_z. \end{tikzcd}$$

We shall mainly follow the notation of Kashiwara-Schapira [K-S]; we denote the normal deformation of N in M by $\widetilde{M}_N$ and $\widetilde{L}_Y$. By an admissible coordinate system, we see that $\widetilde{M}_N = \{(r,t,x);\, r \in \mathbb{R},\, (r\,t, x) \in M\}$, $\Omega_M = \widetilde{M}_N \cap \{(r,t,x);\, r > 0\}$, $T_N M \simeq \widetilde{M}_N \cap \{(r,t,x);\, r = 0\}$ and $p_M \colon \widetilde{M}_N \ni (r,t,x) \longmapsto (r\,t, x) \in M$.

We denote by $\mathcal{C}_M$ and $\mathcal{B}_M$ the sheaf of *microfunctions* on $T^*_M X$ and that *of hyperfunctions* on M respectively as usual. Further we set

$$\mathcal{B}_{N|M} := \nu_N(\mathcal{B}_M),$$

where $\nu_N(\cdot)$ denotes the specialization functor along N. We shall define the sheaf of F-mild hyperfunctions for the higher-codimensional boundary case. Note that the results of this section were essentially contained in Oaku [O 4]. The main difference is that we use the notion of normal deformation (cf. Kashiwara-Schapira [K-S]) here instead of the real monoidal transform adopted in Oaku [O 4]. Let us set

$$D(V, \varepsilon, \Gamma) := \{(t, z) \in \mathbb{R}^d \times \mathbb{C}^n;\ (t, \operatorname{Re} z) \in V,\ |\operatorname{Im} z| < \varepsilon,\ \operatorname{Im} z \in \Gamma\}$$

for a subset V of M, a constant $\varepsilon > 0$ and a cone Γ of $\mathbb{R}^n$.

In general, we mean by Cl and Int *the closure* and *the interior* of a set respectively.

1.1 Definition. Let x^* be a point of $T_N M$ and (τ, z) an admissible local coordinate system of X around $\mathring{x} := \tau_N(x^*)$ such that $z(\mathring{x}) = 0$. Then a germ $u(t, x)$ of $\mathcal{B}_{N|M}$ at x^* is said to be *F-mild* (with respect to a partial complexification L) at x^* if there exist a natural number J and holomorphic functions $F_j(\tau, z)$ $(1 \leqslant j \leqslant J)$ defined on a neighborhood of $D(p_M(U \cap \operatorname{Cl} \Omega_M), \varepsilon, \Gamma_j)$ in X such that

$$u(t, x) = \sum_{j=1}^{J} F_j(t, x + \sqrt{-1}\,\Gamma_j\, 0)$$

as a hyperfunction on $\widetilde{p}_M(U \cap \Omega_M)$. Here U is an open neighborhood of x^* in $\widetilde{M}_N$ such that the all the fibers of the mapping $\widetilde{p}_M \colon U \cap \Omega_M \longrightarrow M$ are connected, ε is a positive constant and $\Gamma_1, \ldots, \Gamma_J$ are open convex cones in $\mathbb{R}^n$. We denote by $\mathcal{B}^F_{N|M}$ the sheaf of sections of $\mathcal{B}_{N|M}$ which are F-mild at each point of their defining domains. Sections of $\mathcal{B}^F_{N|M}$ are called *F-mild hyperfunctions.*

1.2 Remark. Let x^* be a point of $T_N M$. Then F-mildness (with respect to L) at x^* of a section of $\mathcal{B}_{N|M}$ does not depend on an admissible local coordinate system (τ, z) of X taken in Definition 1.1.

We denote the natural inclusion $\mathcal{B}^F_{N|M} \rightarrowtail \mathcal{B}_{N|M}$ by $\beta^F_{N|M}$.

Let us denote by $\mathcal{B}^A_M$ *the sheaf of hyperfunctions which have t as real analytic parameters* on M. Moreover, set

$$\mathcal{B}^A_{N|M} := \mathcal{B}^A_M\big|_N.$$

By the lemma below, we can regard an F-mild hyperfunction as a refinement of a hyperfunction with real analytic parameters:

1.3 Lemma. (1) *The following equality holds:*

$$\mathcal{B}^F_{N|M}\big|_N = \mathcal{B}^A_{N|M}.$$

(2) *There exists a natural monomorphism*

$$\alpha_{N|M}\colon \tau_N^{-1}\,\mathcal{B}^A_{N|M} \rightarrowtail \mathcal{B}^F_{N|M}.$$

In particular, any germ of $\mathcal{B}^A_{N|M}$ is F-mild on a whole fiber of τ_N.

(3) *Let $u(t,x)$ be a hyperfunction on $M \setminus N$ which is F-mild at any point of $\dot{T}_N M := T_N M \setminus T_N N$. Then there exists a unique hyperfunction $v(t,x)$ on M such that $v(t,x) = u(t,x)$ on $M \setminus N$ and $\mathrm{SS}_M(v) \cap \dot{T}^*_N M = \emptyset$.*

1.4 Theorem (the Edge of the Wedge Theorem for F-Mild Hyperfunctions). *Let x^* be a point of $T_N M$ and (τ, z) an admissible local coordinate system around $\tau_N(x^*)$ such that $\tau_N(x^*) = 0$ in this system. Let ε be a positive constant, U an open neighborhood x^* in $\widetilde{M}_N$, and Γ_j $(1 \leqslant j \leqslant J)$ open convex cones of $\mathbb{R}^n$. Let $F_j(\tau,z)$ be holomorphic functions defined on a neighborhood of $D(p_M(U \cap \mathrm{Cl}\,\Omega_M), \varepsilon, \Gamma_j)$ such that*

$$\sum_{j=1}^{J} F_j(t, x + \sqrt{-1}\,\Gamma_j\, 0) = 0$$

holds as a hyperfunction on $\widetilde{p}_M(U \cap \Omega_M)$. Then for any open convex cones Γ'_j such that $\Gamma'_j \Subset \Gamma_j$, there exist a positive constant δ, an open neighborhood V of x^* in $\widetilde{M}_N$, and holomorphic functions $F_{jk}(\tau,z)$ defined on a neighborhood of $D(p_M(V \cap \mathrm{Cl}\,\Omega_M), \delta, \Gamma'_j + \Gamma'_k)$ such that

$$F_j(\tau,z) = \sum_{k=1}^{J} F_{jk}(\tau,z), \qquad F_{jk}(\tau,z) + F_{kj}(\tau,z) = 0 \quad (1 \leqslant j,\, k \leqslant J).$$

We can prove this theorem by using the theory of Radon transformations for hyperfunctions (cf. A. Kaneko [Kn], K. Kataoka [kt]).

By this theorem, we obtain:

1.5 Proposition. *There exists a morphism*

$$\gamma^F_{N|M}\colon \mathcal{B}^F_{N|M} \longrightarrow \tau_N^{-1}\,\mathcal{B}_N$$

defined by

$$\gamma^F_{N|M}(u)(x) := \sum_{j=1}^{J} F_j(0, x + \sqrt{-1}\,\Gamma_j\, 0)$$

if a section $u(t,x)$ of $\mathcal{B}^F_{N|M}$ is expressed as in Definition 1.1. *In particular, $\gamma^F_{N|M}$ induces an isomorphism*:

$$\mathcal{B}^F_{N|M} \Big/ \sum_{j=1}^{d} t_j\, \mathcal{B}^F_{N|M} \simeq \tau_N^{-1}\, \mathcal{B}_N$$

This proposition means that the boundary value $\gamma^F_{N|M}(u)(x)$ does not depend on the direction along which the boundary value is taken. More precisely, let $u(t,x)$ be a section of $\mathcal{B}^F_{N|M}$ on an open set U of $T_N M$ with connected fibers. Then there exists a section $v(x)$ of $\mathcal{B}_N$ on $\tau_N(U)$ such that $\gamma^F_{N|M}(u|_V) = \tau_N^{-1}(v)|_V$ for any open subset V of U.

§2. F-Mild Microfunctions.

In this section, we microlocalize the F-mildness property. To this end, we introduce new sheaves.

Let us set $\iota_Y := i_L\, i_Y \colon Y \longrightarrow X$. Then we have the following commutative diagram:

$$\begin{tikzcd}
T^*_M X \arrow[r, two heads, "\pi_M"] & M & N \arrow[l] \arrow[dr, phantom, "\square"] & T_N M \arrow[l, "\tau_N"'] & T^*_{T_N M} T_Y L \arrow[l, "\pi_{N|M}"'] \\
M \underset{M}{\times} T^*_M X \arrow[u, "i_{L\pi}", "\wr"'] & N \underset{M}{\times} T^*_M X \arrow[l, hook', "i_{Y\pi}"'] \arrow[r, "{}^t\iota'_L"'] & T^*_N Y \arrow[u, "\pi_N"'] & T_N M \underset{N}{\times} T^*_N Y \arrow[l, "\tau_{Y\pi}"'] \arrow[u] \arrow[ur, "\sim", "{}^t\iota'_Y"'] &
\end{tikzcd}$$

2.1 Definition. We set:

$$\begin{aligned}
\widetilde{\mathcal{C}}^A_{N|M} :=& \mathcal{H}^n\big(\mu_N(\iota_Y^{-1}\, \mathcal{O}_X \otimes or_{N/Y})\big), \\
\widetilde{\mathcal{B}}^A_{N|M} :=& \widetilde{\mathcal{C}}^A_{N|M}\big|_N = \mathcal{H}^n_N(\iota_Y^{-1}\, \mathcal{O}_X) \otimes or_{N/Y} . \widetilde{\mathcal{A}}^A_{N|M}
\end{aligned}$$

Thus $\widetilde{\mathcal{C}}^A_{N|M}$ and $\widetilde{\mathcal{B}}^A_{N|M}$ are sheaves on $T^*_N Y$ and N respectively.

By the same arguments as in the theory of microfunctions (cf. Oaku [O 3]), we can obtain a natural epimorphism

$$\widetilde{\mathrm{sp}}^A_{N|M} \colon \pi_N^{-1}\, \widetilde{\mathcal{B}}^A_{N|M} \longrightarrow\!\!\!\!\!\rightarrow \widetilde{\mathcal{C}}^A_{N|M}.$$

2.2 Lemma. *There exists a natural monomorphism*

$$\alpha^F_{N|M} \colon \mathcal{B}^F_{N|M} \rightarrowtail \tau_N^{-1}\, \widetilde{\mathcal{B}}^A_{N|M}.$$

By this lemma, we can regard $\mathcal{B}^F_{N|M}$ as a subsheaf of $\tau_N^{-1}\, \widetilde{\mathcal{B}}^A_{N|M}$.

2.3 Definition. The sheaf of *F-mild microfunctions* on $T^*_{T_N M} T_Y L$ is defined by

$$\mathcal{C}^F_{N|M} := \mathrm{Image}(\widetilde{\mathrm{sp}}^A_{N|M}\, \alpha^F_{N|M} \colon \pi^{-1}_{N|M}\, \mathcal{B}^F_{N|M} \longrightarrow ({}^t\tau'_Y)_!\, \tau^{-1}_{Y\pi}\, \widetilde{\mathcal{C}}^A_{N|M}).$$

Sections of $\mathcal{C}^F_{N|M}$ are called *F-mild microfunctions*. The morphism $\widetilde{\mathrm{sp}}^A_{N|M}\, \alpha^F_{N|M}$ induces an epimorphism

$$\mathrm{sp}^F_{N|M} \colon \pi^{-1}_{N|M}\, \mathcal{B}^F_{N|M} \longrightarrow\!\!\!\!\rightarrow \mathcal{C}^F_{N|M}.$$

For a section u of $\mathcal{B}^F_{N|M}$, $\mathrm{SS}^F_{N|M}(u)$ denotes $\mathrm{supp}(\mathrm{sp}^F_{N|M}(u))$.

We remark that by definition $\mathcal{C}^F_{N|M}\big|_{T_N M} = \mathcal{B}^F_{N|M}$.

We also denote the natural inclusion $\mathcal{C}^F_{N|M} \rightarrowtail ({}^t\tau'_Y)_!\, \tau^{-1}_{Y\pi}\, \widetilde{\mathcal{C}}^A_{N|M}$ by $\alpha^F_{N|M}$.

2.4 Lemma. *Let $u(t,x)$ be a germ of $\mathcal{B}^F_{N|M}$ at a point x^* of $T_N M$. Then a point $p = (x^*; \sqrt{-1}\,\xi^*)$ of $T^*_{T_N M} T_Y L$ is not contained in $\mathrm{SS}^F_{N|M}(u)$ if and only if $u(t,x)$ has an expression as in Definition* 1.1 *such that ξ^* does not contained in ${\Gamma_j}^\circ$ for any j.*

Let us set

$$\mathcal{C}^A_{N|M} := ({}^t\iota'_Y)_!\, \iota^{-1}_{Y\pi}\, \mathcal{C}_M.$$

Then we see that $\mathcal{C}^A_{N|M}\big|_N = \mathcal{B}^A_{N|M}$. Let us denote by $\mathrm{sp}^A_{N|M} \colon \pi_N^{-1}\, \mathcal{B}^A_{N|M} \longrightarrow\!\!\!\!\rightarrow \mathcal{C}^A_{N|M}$ the spectral morphism. Therefore we obtain a natural morphism

$$\alpha^A_{N|M} \colon \mathcal{C}^A_{N|M} \longrightarrow \widetilde{\mathcal{C}}^A_{N|M}.$$

2.5 Lemma. *The morphism $\alpha^A_{N|M}$ induces a natural monomorphism*

$$\alpha_{N|M} \colon ({}^t\tau'_Y)_!\, \tau^{-1}_{Y\pi}\, \mathcal{C}^A_{N|M} \rightarrowtail \mathcal{C}^F_{N|M},$$

such that the restriction of this morphism to the zero-section coincides with $\alpha_{N|M}$: $\tau_N^{-1}\, \mathcal{B}^A_{N|M} \rightarrowtail \mathcal{B}^F_{N|M}$ of Lemma 1.3 (2).

Let $\gamma^A_{N|M} \colon \mathcal{C}^A_{N|M} \longrightarrow \mathcal{C}_N$ and $\gamma^A_{N|M} \colon \mathcal{B}^A_{N|M} \longrightarrow \mathcal{B}_N$ be the restriction morphisms. Then, these morphisms induce isomorphisms

$$\mathcal{C}^A_{N|M}\Big/\sum_{j=1}^d t_j\, \mathcal{C}^A_{N|M} \simeq \mathcal{C}_N, \qquad \mathcal{B}^A_{N|M}\Big/\sum_{j=1}^d t_j\, \mathcal{B}^A_{N|M} \simeq \mathcal{B}_N.$$

We shall define restriction and boundary value morphisms.

First, induced by a natural morphism $\iota_Y^{-1}\, \mathcal{O}_X \longrightarrow \mathcal{O}_Y$ there exists a natural morphism

$$\widetilde{\gamma}^A_{N|M} \colon \widetilde{\mathcal{C}}^A_{N|M} \longrightarrow \mathcal{C}_N.$$

We also denote the restriction of this morphism to the zero-section by the same notation:

$$\widetilde{\gamma}^{A}_{N|M}\colon \widetilde{\mathcal{B}}^{A}_{N|M} \longrightarrow \mathcal{B}_N.$$

Then, these morphisms induce isomorphisms

$$\widetilde{\mathcal{C}}^{A}_{N|M}\Big/\sum_{j=1}^{d} t_j\, \widetilde{\mathcal{C}}^{A}_{N|M} \simeq \mathcal{C}_N, \qquad \widetilde{\mathcal{B}}^{A}_{N|M}\Big/\sum_{j=1}^{d} t_j\, \widetilde{\mathcal{B}}^{A}_{N|M} \simeq \mathcal{B}_N,$$

Next, by Lemma 2.4, we see that $\gamma^{F}_{N|M}\colon \mathcal{B}^{F}_{N|M} \longrightarrow \tau_N^{-1}\mathcal{B}_N$ induces a morphism

$$\gamma^{F}_{N|M}\colon \mathcal{C}^{F}_{N|M} \longrightarrow ({}^{t}\tau'_Y)_!\, \tau_{Y\pi}^{-1}\, \mathcal{C}_N,$$

which in turn induces an isomorphism

$$\mathcal{C}^{F}_{N|M}\Big/\sum_{j=1}^{d} t_j\, \mathcal{C}^{F}_{N|M} \simeq ({}^{t}\tau'_Y)_!\, \tau_{Y\pi}^{-1}\, \mathcal{C}_N.$$

2.6 Lemma. *The following diagram is commutative*:

$$\begin{tikzcd}
\pi_{N|M}^{-1}\tau_N^{-1}\mathcal{B}^{A}_{N|M} \arrow[rr, two heads, "\mathrm{sp}^{A}_{N|M}"] \arrow[dr, tail, "\alpha^{A}_{N|M}"] \arrow[dd, "\gamma^{A}_{N|M}"'] & & ({}^{t}\tau'_Y)_!\, \tau_{Y\pi}^{-1}\, \mathcal{C}^{A}_{N|M} \arrow[dr, tail, "\alpha^{A}_{N|M}"] \arrow[dd, "\gamma^{A}_{N|M}"' near start] & \\
& \pi_{N|M}^{-1}\mathcal{B}^{F}_{N|M} \arrow[rr, two heads, "\mathrm{sp}^{F}_{N|M}" near start] \arrow[dl, "\gamma^{F}_{N|M}"] \arrow[dd, tail, "\alpha^{F}_{N|M}" near start] & & \mathcal{C}^{F}_{N|M} \arrow[dl, "\gamma^{F}_{N|M}"] \arrow[dd, tail, "\alpha^{F}_{N|M}"] \\
\pi_{N|M}^{-1}\tau_N^{-1}\mathcal{B}_N \arrow[rr, two heads, "\mathrm{sp}_N" near end] & & ({}^{t}\tau'_Y)_!\, \tau_{Y\pi}^{-1}\, \mathcal{C}_N & \\
& \pi_{N|M}^{-1}\tau_N^{-1}\widetilde{\mathcal{B}}^{A}_{N|M} \arrow[ul, "\widetilde{\gamma}^{A}_{N|M}"] \arrow[rr, two heads, "\widetilde{\mathrm{sp}}^{A}_{N|M}"] & & ({}^{t}\tau'_Y)_!\, \tau_{Y\pi}^{-1}\, \widetilde{\mathcal{C}}^{A}_{N|M}. \arrow[ul, "\widetilde{\gamma}^{A}_{N|M}"]
\end{tikzcd}$$

§3. Fuchsian Systems of Partial Differential Equations.

Let $\mathcal{D}_X$ be *the sheaf on X of rings of linear partial differential operators* (*of finite order*) *with holomorphic coefficients*. Let $\mathcal{M}$ be a coherent (left) $\mathcal{D}_X$-Module on X; that is, a system of linear partial differential equation with with holomorphic coefficients. Recall that Y is *non-characteristic* for $\mathcal{M}$ if $T^*_Y X \cap \operatorname{char}(\mathcal{M}) \subset T^*_X X$, where $\operatorname{char}(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$. We denote *the inverse image of* $\mathcal{M}$ *by* ι_Y in $\mathcal{D}$-Modules by $\underline{\iota_Y^{-1}}\,\mathcal{M}$; that is,

$$\underline{\iota_Y^{-1}}\,\mathcal{M} := \mathcal{D}_{Y\to X} \overset{L}{\underset{\iota_Y^{-1}\mathcal{D}_X}{\otimes}} \iota_Y^{-1}\mathcal{M} \simeq \mathcal{O}_Y \overset{L}{\underset{\iota_Y^{-1}\mathcal{O}_X}{\otimes}} \iota_Y^{-1}\mathcal{M}.$$

Let us set $\mathcal{M}_Y := \mathcal{H}^0(\underline{\iota_Y^{-1}}\,\mathcal{M})$.

First, we recall the non-characteristic case: If Y is non-characteristic for $\mathcal{M}$, then $\underline{\iota_Y^{-1}}\,\mathcal{M}$ is concentrated in degree zero; that is, we can identify $\underline{\iota_Y^{-1}}\,\mathcal{M}$ with $\mathcal{M}_Y$, and that $\mathcal{M}_Y$ is a coherent $\mathcal{D}_Y$-Module. Moreover, we can prove that any hyperfunction solution of $\mathcal{M}$ which is defined on a wedge domain with edge N is always F-mild, thus having boundary values with no further assumption. This case was studied by P. Schapira ([Sc 1], [Sc 2]) by using the theory of microlocalization of sheaves. The local uniqueness in this boundary value problem was proved in T. Oaku [O 4]. K. Takeuchi [T] formulated microlocal boundary value problem by using the theory of second microlocalization and proved the microlocal uniqueness in this case. We can obtain another proof to the microlocal uniqueness by a natural extension of the method used in Oaku [O 4] (see [O-Y] for details).

Now in this section, we shall state the uniqueness theorem in the boundary value problems for $\mathcal{D}$-Modules of a Fuchsian type and of a Fuchs-Goursat type in the frame work of F-mild microfunctions.

First, assume that $\mathcal{M}$ is a Fuchsian system along Y in the sense of Laurent-Monterio Fernandes [L-MF]. Recall that a coherent $\mathcal{D}_X$-Module $\mathcal{M}$ is a Fuchsian along Y if and only if for any (local) section $u \in \mathcal{M}$, there exists a differential operator P such that $Pu = 0$ and that P can be written in a coordinate system (τ, z) with $Y = \{(\tau, z);\ \tau = 0\}$ as follows:

$$P(\tau, z; \partial_\tau, \partial_z) = \sum_{0 \leqslant |\alpha| = |\beta| \leqslant \operatorname{ord} P} P_{\alpha\beta}(z)\, \tau^\alpha\, {\partial_\tau}^\beta + Q(\tau, z; \partial_\tau, \partial_z),$$

where ord denotes the (usual) order of a differential operator, and the conditions below hold:

(a) For any $\eta \in \mathbb{C}^d \setminus \{0\}$, it follows that $\sum_{|\alpha|=|\beta|=\operatorname{ord} P} P_{\alpha\beta}(z)\, \eta^\alpha\, \overline{\eta}^\beta \neq 0$;

(b) For any $j \in \mathbb{Z}$, it follows that $Q\, \mathcal{I}_Y{}^j \subset \mathcal{I}_Y{}^{j+1}$. Here $\mathcal{I}_Y$ denotes the defining ideal of Y in X with a convention $\mathcal{I}_Y{}^j = \mathcal{O}_X$ for $j \leqslant 0$.
Note that all the cohomologies of $\underline{\iota_Y^{-1}}\, \mathcal{M}$ are coherent $\mathcal{D}_Y$-Modules by Théorème 3.3 of Laurent-Schapira [L-S] and that we may choose the coordinate system above as admissible.

3.1 Theorem. *Let $\mathcal{M}$ be a Fuchsian system along Y. Then, the boundary value morphism $\gamma^F_{N|M}\colon \mathcal{C}^F_{N|M} \longrightarrow ({}^t\tau'_Y)_!\, \tau_{Y\pi}^{-1}\, \mathcal{C}_N$ induces a monomorphism*

$$\gamma^F_{N|M}\colon \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}^F_{N|M}) \rightarrowtail ({}^t\tau'_Y)_!\, \tau_{Y\pi}^{-1}\, \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N).$$

Note that not all the hyperfunction solutions of $\mathcal{M}$ are necessarily F-mild, contrary to the non-characteristic case.

We can prove this theorem by the Cauchy-Kovalevskaja type theorem (Théorème 3.2.2 of Laurent-Monterio Fernandes [L-MF]).

By virtue of Lemma 2.5 we have the following corollary:

3.2 Corollary. *Let $\mathcal{M}$ be a Fuchsian system along Y. Then, the restriction morphism $\gamma^A_{N|M}\colon \mathcal{C}^A_{N|M} \longrightarrow \mathcal{C}_N$ induces a monomorphism*

$$\gamma^A_{N|M}\colon \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}^A_{N|M}) \rightarrowtail \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N).$$

Next, we shall give similar theorems for a matrix of Fuchs-Goursat type introduced by Madi [M] and Yamazaki [Y]. To state the results, we define boundary value morphisms, which we shall regard as Goursat data, as follows: By an admissible coordinate system, we may assume that $X = \mathbb{C}^d_\tau \times \mathbb{C}^n_z$, $Y = \mathbb{C}^n_z$, $L = \mathbb{R}^d_t \times \mathbb{C}^n_z$, $M = \mathbb{R}^d_t \times \mathbb{R}^n_x$ and $N = \mathbb{R}^n_x$. For $1 \leqslant i \leqslant d$, let us set $L_i := \{(t,z) \in L;\, t_i = 0\}$ and $M_i := M \cap L_i$. Then the inclusion $L_i \hookrightarrow L$ induces the following commutative diagrams:

Moreover, we have the following commutative diagram:

$$\begin{tikzcd}
T_Y L_i \arrow[r, hook, "\varphi_i"] & T_Y L \\
T_N M_i \arrow[u, hook] \arrow[r, hook, "\varphi_{iN}"] \arrow[ur, phantom, "\circlearrowleft"] & T_N M \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T_N M \arrow[dl, "\tau_N"'] & T^*_{T_N M} T_Y L \arrow[l, "\pi_{N|M}"'] \\
N & T_N M_i \arrow[l, "\tau_{iN}"'] \arrow[u, "\varphi_{iN}"] \arrow[ur, phantom, "\square"] & T_N M_i \underset{T_N M}{\times} T^*_{T_N M} T_Y L \arrow[l] \arrow[u, "\varphi_{i\pi}"'] \arrow[d, "{}^t\varphi'_i", "\wr"'] \\
T^*_N Y \arrow[u, "\pi_N"] & T_N M_i \underset{N}{\times} T^*_N Y \arrow[l, "\tau_{iY\pi}"'] \arrow[u] \arrow[ul, phantom, "\square"] \arrow[r, "\sim", "{}^t\varphi'_{iY}"'] & T^*_{T_N M_i} T_Y L_i . \arrow[ul, "\pi_{N|M_i}"]
\end{tikzcd}$$

Then we have the following:

3.3 Lemma. *There exist natural morphisms*

$$\begin{aligned}\tilde{\gamma}^{A,i}_{N|M}\colon \tilde{\mathcal{C}}^A_{N|M} &\longrightarrow \tilde{\mathcal{C}}^A_{N|M_i},\\ \tilde{\gamma}^{A,i}_{N|M}\colon \tilde{\mathcal{B}}^A_{N|M} &\longrightarrow \tilde{\mathcal{B}}^A_{N|M_i},\end{aligned}$$

and

$$\begin{aligned}\gamma^{F,i}_{N|M}\colon ({}^t\varphi'_i)_!\,\varphi^{-1}_{i\pi}\,\mathcal{C}^F_{N|M} &\longrightarrow \mathcal{C}^F_{N|M_i},\\ \gamma^{F,i}_{N|M}\colon \varphi^{-1}_{iN}\,\mathcal{B}^F_{N|M} &\longrightarrow \mathcal{B}^F_{N|M_i},\end{aligned}$$

such that the following diagram is commutative:

$$\begin{tikzcd}
\pi^{-1}_{N|M_i}\,\varphi^{-1}_{iN}\,\mathcal{B}^F_{N|M} \arrow[rr, two heads, "\mathrm{sp}^F_{N|M}"] \arrow[dr, tail, "\alpha^F_{N|M}"] \arrow[dd, "\gamma^{F,i}_{N|M}"'] & & ({}^t\varphi'_i)_!\,\varphi^{-1}_{i\pi}\,\mathcal{C}^F_{N|M} \arrow[dr, tail, "\alpha^F_{N|M}"] \arrow[dd, "\gamma^{F,i}_{N|M}"' near start] & \\
& \pi^{-1}_{N|M_i}\,\pi^{-1}_{iN}\,\tilde{\mathcal{B}}^A_{N|M} \arrow[rr, two heads, "\mathrm{sp}^A_{N|M}" near start] \arrow[dd, "\tilde{\gamma}^{A,i}_{N|M}" near start] & & ({}^t\varphi'_{iY})_!\,\tau^{-1}_{iY\pi}\,\tilde{\mathcal{C}}^A_{N|M} \arrow[dd, "\tilde{\gamma}^{A,i}_{N|M}"] \\
\pi^{-1}_{N|M_i}\,\mathcal{B}^F_{N|M_i} \arrow[rr, two heads, "\mathrm{sp}^F_{N|M_i}" near end] \arrow[dr, tail, "\alpha^F_{N|M_i}"'] & & \mathcal{C}^F_{N|M_i} \arrow[dr, tail, "\alpha^F_{N|M_i}"'] & \\
& \pi^{-1}_{N|M_i}\,\tau^{-1}_{iN}\,\tilde{\mathcal{B}}^A_{N|M_i} \arrow[rr, two heads, "\widetilde{\mathrm{sp}}^A_{N|M_i}"] & & ({}^t\varphi'_{iY})_!\,\tau^{-1}_{iY\pi}\,\tilde{\mathcal{C}}^A_{N|M_i}.
\end{tikzcd}$$

3.4 Lemma. *Let $l = (l_1, \dots, l_d)$ be a d-tuple of non-negative integers and $f(t,x)$ a germ of $\tilde{\mathcal{C}}^A_{N|M}$ at $p = (x_0; \sqrt{-1}\,\langle \xi^*, dx\rangle) \in T^*_N Y$. Then the following conditions are equivalent:*

(1) *There exists a germ $g(t,x)$ of $\tilde{\mathcal{C}}^A_{N|M}$ at p such that $f(t,x) = t^l\, g(t,x)$.*

(2) *For any $0 \leqslant k_i \leqslant l_i - 1$ $(1 \leqslant i \leqslant d)$*

$$\tilde{\gamma}^{A,i}_{N|M}({\partial_{t_i}}^{k_i}\, f(t,x)) = 0.$$

Moreover in this case, $g(t,x)$ is unique.

For a vector $l = (l_1, \dots, l_d) \in \mathbb{R}^d$, we set $[l]_+ := ([l_1]_+, \dots, [l_d]_+)$, where $[l_j]_+ = \max\{l_j, 0\}$. We fix $J \in \mathbb{N}$, $m^{(\nu)} = (m^{(\nu)}_1, \dots, m^{(\nu)}_d)$ and $k^{(\nu)} = (k^{(\nu)}_1, \dots, k^{(\nu)}_d) \in {\mathbb{N}_0}^d$ with $m^{(\nu)} \geqslant k^{(\nu)}$ $(1 \leqslant \nu \leqslant J)$ and set $m = (m^{(1)}, \dots, m^{(J)})$ and $k = (k^{(1)}, \dots, k^{(J)}) \in ({\mathbb{N}_0}^d)^J$. Set $\mathbf{1}_d := (1, \dots, 1) \in \mathbb{N}^d$.

3.5 Definition. Let $P(\tau, z; \partial_\tau, \partial_z) = (P^{(\mu,\nu)}(\tau, z; \partial_\tau, \partial_z))_{\mu,\nu=1}^{J}$ be a matrix of size $J \times J$ whose components is in $\mathcal{D}_X$ defined in a neighborhood of the origin. Then, P is said to be *of Fuchs-Goursat type with weight* (k, m) (*with respect to* τ*-variables*) if it can be written as a form

$$P^{(\mu,\nu)}(\tau, z; \partial_\tau, \partial_z) = \sum_{0 \leqslant \alpha \leqslant m^{(\nu)}} P_\alpha^{(\mu,\nu)}(\tau, z; \partial_z)\, \partial_\tau{}^\alpha,$$

where each $P_\alpha^{(\mu,\nu)}$ is a differential operator satisfying the following:

(1) The order $\operatorname{ord} P_\alpha^{(\mu,\nu)}$ of $P_\alpha^{(\mu,\nu)}$ is at most $|m^{(\nu)}| - |\alpha|$;

(2) There exist $P_\alpha^{1,(\mu,\nu)}(\tau, z)$ and $P_\alpha^{2,(\mu,\nu)}(\tau, z; \partial_z)$ $(0 \leqslant \alpha \leqslant m^{(\nu)})$ such that

$$P_\alpha^{(\mu,\nu)}(\tau, z; \partial_z) = \tau^{[\alpha - m^{(\nu)} + k^{(\nu)}]_+} P_\alpha^{1,(\mu,\nu)}(\tau, z) + \tau^{[\alpha - m^{(\nu)} + k^{(\nu)} + \mathbf{1}_d]_+} P_\alpha^{2,(\mu,\nu)}(\tau, z; \partial_z).$$

Let $T^{(\nu)} = (T_1^{(\nu)}, \dots, T_d^{(\nu)})$ $(1 \leqslant \nu \leqslant J)$ be indeterminates and set

$$\widetilde{T} := (T^{(1)}, \dots, T^{(J)}).$$

If P is of Fuchs-Goursat type with weight (k, m), we define *the indicial polynomial of* P by

$$\mathcal{I}_P(z; \widetilde{T}) := \det \Bigl(\sum_{m^{(\nu)} - k^{(\nu)} \leqslant \alpha \leqslant m^{(\nu)}} P_\alpha^{1,(\mu,\nu)}(0, z)\, \mathcal{I}_\alpha(T^{(\nu)}) \Bigr)$$

where $\mathcal{I}_\alpha(T^{(\nu)}) := \prod_{j=1}^{d} \mathcal{I}_{\alpha_j}(T_j^{(\nu)})$ with

$$\mathcal{I}_{\alpha_j}(T_j^{(\nu)}) := \begin{cases} T_j^{(\nu)}(T_j^{(\nu)} - 1) \cdots (T_j^{(\nu)} - \alpha_j + 1) & (\alpha_j \geqslant 1), \\ 1 & (\alpha_j = 0). \end{cases}$$

Consider the following condition:

(A). *There exist a positive constant $C > 0$ and a neighborhood W of the origin in $\mathbb{C}^n$ such that for any $z \in W$ and $\beta \in \mathbb{N}_0{}^d$*

$$|\mathcal{I}_P(z; \beta + m^{(1)} - k^{(1)}, \dots, \beta + m^{(J)} - k^{(J)})| \geqslant C \prod_{\nu=1}^{J} (\beta + \mathbf{1}_d)^{m^{(\nu)}}.$$

Under the notation above, we can prove the following theorems:

3.6 Theorem. *Let P be a matrix of Fuchs-Goursat type of size $J \times J$ with weight (k, m). Let $p = (x^*; \sqrt{-1}\langle \xi^*, dx\rangle)$ be a point of $T^*_{T_N M} T_Y L$ with $\tau_N(x^*) = 0$. Assume that P satisfies* (A). *Let $u(t,x) = {}^t(u_1(t,x), \dots, u_J(t,x))$ be a germ of $(\mathcal{C}^F_{N|M})^{\oplus J}$ at p. Suppose that $u(t,x)$ satisfies*

$$\begin{cases} \quad P(t,x;\partial_t,\partial_x)\, u(t,x) = 0, \\ \tilde{\gamma}^{A,i}_{N|M}\, \alpha^F_{N|M}(\partial_{t_i}{}^{j_i}\, u_\nu(t,x)) = 0 \qquad (1 \leqslant \nu \leqslant J,\ 1 \leqslant i \leqslant d,\ 0 \leqslant j_i \leqslant m_i^{(\nu)} - k_i^{(\nu)} - 1). \end{cases}$$

Then it follows that $u(t,x) = 0$ at p.

3.7 Theorem. *Let U be an open set of $T^*_{T_N M} T_Y L$ such that each fiber of*

$$\tau_{Y\pi} \colon ({}^t\tau'_Y)^{-1}(U) \longrightarrow T^*_N Y$$

is connected and intersects with $T^*_{T_N M_i} T_Y L_i$ for any $1 \leqslant i \leqslant d$. Let P be a matrix of Fuchs-Goursat type of size $J \times J$ with weight (k,m). Assume that P satisfies (A) *and that a section $u(t,x) \in \Gamma(U; \mathcal{C}^F_{N|M})^{\oplus J}$ satisfies*

$$\begin{cases} P(t,x;\partial_t,\partial_x)\, u(t,x) = 0, \\ \gamma^{F,i}_{N|M}(\partial_{t_i}{}^{j_i}\, u_\nu(t,x)) = 0 \in \Gamma(U \cap T^*_{T_N M_i} T_Y L_i; \mathcal{C}^F_{N|M_i}) \begin{pmatrix} 1 \leqslant \nu \leqslant J,\ 1 \leqslant i \leqslant d, \\ 0 \leqslant j_i \leqslant m_i^{(\nu)} - k_i^{(\nu)} - 1 \end{pmatrix}. \end{cases}$$

Then it follows that $u(t,x) = 0$.

The proofs of Theorems 3.6 and 3.7 are based on Lemma 3.4 and The Cauchy-Kovalevskaja type theorem (Theorem 1.3 of [Y] which is an extension of Théorème (1.1) of Madi [M]).

By Lemma 2.5 we have also the following corollary:

3.8 Corollary. *Let P be a matrix of Fuchs-Goursat type of size $J \times J$ with weight (k,m). Assume that P satisfies* (A) *and a germ $u(t,x)$ of $(\mathcal{C}^A_{N|M})^{\oplus J}$ at $(0; \sqrt{-1}\langle \xi^*, dx\rangle) \in T^*_M X$ satisfies*

$$\begin{cases} P(t,x;\partial_t,\partial_x)\, u(t,x) = 0, \\ \qquad \partial_{t_i}{}^{j_i}\, u_\nu\big|_{t_i=0} = 0 \qquad (1 \leqslant \nu \leqslant J,\ 1 \leqslant i \leqslant d,\ 0 \leqslant j_i \leqslant m_i^{(\nu)} - k_i^{(\nu)} - 1). \end{cases}$$

Then it follows that $u(t,x) = 0$ at $(0; \sqrt{-1}\langle \xi^*, dx\rangle)$.

3.9 Remark. (1) Since the induced system $\mathcal{M}_Y$ is not necessarily a coherent $\mathcal{D}_Y$-Module in cases of Theorems 3.6 and 3.7, we must impose boundary (or rather initial) conditions on each hypersurface M_i, rather than the boundary conditions on N. This might be regarded as a hyperfunction (or microfunction) version of the Goursat problem, rather than the higher-codimensional boundary value problem.

(2) In [Y], we discussed the solvability of the Goursat problem for a Fuchs-Goursat type in the framework of microfunctions. By Corollary 6.8, in the differential case we can conclude a uniqueness of each solution of Theorem 4.2 and Corollary 4.5 of [Y].

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