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Gevrey Cohomology Groups for Confluent Hypergeometric Systems

KATSUNORI IWASAKI (岩崎 克則)

Graduate School of Mathematics
Kyushu University
Fukuoka 812-8581 Japan

1. CONTIGUITY RELATIONS

A "hypergeometric system" is a system of partial differential equations containing a parameter and admits a contiguity relation with respect to the parameter. In the langauge of $\mathcal{D}$-modules, this empirical fact is formulated as follows: Let $\mathcal{M}(c)$ be a left $\mathcal{D}_X$-module containing a parameter c, that is, a left $\mathcal{D}_X[c]$-module, where $\mathcal{D}_X[c]=\mathcal{D}_X\otimes\mathbb{C}[c]$. A *contiguity relation* for $\mathcal{M}(c)$ with respect to c is a commutative diagram:

$$\begin{array}{ccccccc}
\cdots \xrightarrow{Q^2(c)} & \mathcal{D}_X[c]^{m_2} & \xrightarrow{Q^1(c)} & \mathcal{D}_X[c]^{m_1} & \xrightarrow{Q^0(c)} & \mathcal{D}_X[c]^{m_0} \\
 & \downarrow P^2(c) & & \downarrow P^1(c) & & \downarrow P^0(c) \\
\cdots \xrightarrow{Q^2(c+1)} & \mathcal{D}_X[c]^{m_2} & \xrightarrow{Q^1(c+1)} & \mathcal{D}_X[c]^{m_1} & \xrightarrow{Q^0(c+1)} & \mathcal{D}_X[c]^{m_0}
\end{array} \tag{1.1}$$

of left $\mathcal{D}_X[c]$-modules such that the following sequence is a free resolution of $\mathcal{M}(c)$:

$$\cdots \longrightarrow \mathcal{D}_X[c]^{m_2} \xrightarrow{Q^1(c)} \mathcal{D}_X[c]^{m_1} \xrightarrow{Q^0(c)} \mathcal{D}_X[c]^{m_0} \longrightarrow \mathcal{M}(c) \longrightarrow 0$$

Here $Q^i(c)$ is an $m_{i+1}\times m_i$ matrix of holomorphic partial differential operators depending polynomially on c and acting on $\mathcal{D}_X[c]^{m_{i+1}}$ by right multiplication, where each element of $\mathcal{D}_X[c]^{m_{i+1}}$ is regarded as a row vector. As for the operators $P^i(c)$, we require that each $P^i(c)$ should be an $m_i\times m_i$ matrix of holomorphic *functions* (not of partial differential operators) depending polynomially on c and acting on $\mathcal{D}_X[c]^{m_i}$ by right multiplication.

Example 1.1. Consider Humbert's confluent hypergeometric system $\Phi_2(b_1,b_2;c)$:

$$\begin{cases} L_1(c)f := \{x\partial_x^2+y\partial_x\partial_y+(c-x)\partial_x-b_1\}f=0, \\ L_2(c)f := \{y\partial_y^2+x\partial_x\partial_y+(c-y)\partial_y-b_2\}f=0, \end{cases}$$

on $X=\mathbb{P}^1\times\mathbb{P}^1$ with parameters b_1,b_2 and c, (see [**2**]). Let $\mathcal{M}(c)$ be the $\mathcal{D}_X[c]$-module associated to the system $\Phi_2(b_1,b_2;c)$, where b_1,b_2 are regarded as fixed.

Then $\mathcal{M}(c)$ has a contiguity relation:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{D}_X^3 & \xrightarrow{Q^1(c)} & \mathcal{D}_X^9 & \xrightarrow{Q^0(c)} & \mathcal{D}_X^6 \\
 & & \downarrow{\scriptstyle P^2(c)} & & \downarrow{\scriptstyle P^1(c)} & & \downarrow{\scriptstyle P^0(c)} \\
0 & \longrightarrow & \mathcal{D}_X^3 & \xrightarrow{Q^1(c+1)} & \mathcal{D}_X^9 & \xrightarrow{Q^0(c+1)} & \mathcal{D}_X^6
\end{array}$$

where

$$P^0(c) = \begin{pmatrix} c & x & y & 0 & 0 & 0 \\ b_1 & x & 0 & 0 & 0 & 0 \\ b_2 & 0 & y & 0 & 0 & 0 \\ 0 & 1+b_1 & 0 & x & 0 & 0 \\ 0 & \frac{b_2}{2} & \frac{b_1}{2} & 0 & \frac{1}{2}(x+y) & 0 \\ 0 & 0 & 1+b_2 & 0 & 0 & y \end{pmatrix}$$

$$P^1(c) = \begin{pmatrix} c & 0 & x & 0 & y & 0 & 1 & 0 & 0 \\ 0 & c & 0 & x & 0 & y & 0 & 1 & 0 \\ b_1 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & x & 0 & 0 & 0 & 0 & \frac{1}{2} \\ b_2 & 0 & 0 & 0 & y & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & b_2 & 0 & 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & -\frac{y}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & \frac{x}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(x+y) \end{pmatrix}$$

$$P^2(c) = \begin{pmatrix} c & 1 & 0 \\ b_1x+b_2y & x+y & -1 \\ (b_1+b_2)xy & xy & 0 \end{pmatrix}$$

$$Q^0(c) = \begin{pmatrix} \partial_x & -1 & 0 & 0 & 0 & 0 \\ \partial_y & 0 & -1 & 0 & 0 & 0 \\ 0 & \partial_x & 0 & -1 & 0 & 0 \\ 0 & \partial_y & 0 & 0 & -1 & 0 \\ 0 & 0 & \partial_x & 0 & -1 & 0 \\ 0 & 0 & \partial_y & 0 & 0 & -1 \\ -b_1 & c-x & 0 & x & y & 0 \\ -b_2 & 0 & c-y & 0 & x & y \\ 0 & -b_2 & b_1 & 0 & x-y & 0 \end{pmatrix}$$

$$Q^1(c) = \begin{pmatrix} \partial_y & -b_2 & -b_2x \\ -\partial_x & b_1 & b_1y \\ 0 & x\partial_y & x(\delta_y+b_2) \\ 1 & -(\delta_x-x+c) & -y(\delta_x-x+c-b_2) \\ -1 & \delta_y-y+c & x(\delta_y-y+c-b_1) \\ 0 & -y\partial_x & -y(\delta_x+b_1) \\ 0 & \partial_y & \delta_y+b_2 \\ 0 & -\partial_x & -(\delta_x+b_1) \\ 0 & 1 & \delta_x+\delta_y+c \end{pmatrix}^T$$

Here $\partial_y = \partial/\partial y$, $\delta_y = y\partial_y$, and T stands for the transpose of a matrix.

2. Mapping Cones

From the contiguity relation (1.1), one obtains a $\mathcal{D}_{X\times\mathbb{P}^1}$-module $\mathcal{N}(c)$ containing a parameter as follows: Let y be an inhomogeneous coordinate of $\mathbb{P}^1$ and set $\partial_y = \partial/\partial y$, $\delta_x = y\partial_y$. Given a nonzero polynomial $\phi(c) \in \mathbb{C}[c]$ independent of i, set

$$f^i(c) = \phi(\delta_y) - P^i(\delta_y + c)\partial_y.$$

Then the contiguity relation (1.1) induces a commutative diagram

$$\begin{array}{ccccccc}
\cdots \xrightarrow{Q^2(\delta_y+c)} & \mathcal{D}_{X\times\mathbb{P}^1}[c]^{m_2} & \xrightarrow{Q^1(\delta_y+c)} & \mathcal{D}_{X\times\mathbb{P}^1}[c]^{m_1} & \xrightarrow{Q^0(\delta_y+c)} & \mathcal{D}_{X\times\mathbb{P}^1}[c]^{m_0} \\
 & \downarrow f^2(c) & & \downarrow f^1(c) & & \downarrow f^0(c) \\
\cdots \xrightarrow{Q^2(\delta_y+c)} & \mathcal{D}_{X\times\mathbb{P}^1}[c]^{m_2} & \xrightarrow{Q^1(\delta_y+c)} & \mathcal{D}_{X\times\mathbb{P}^1}[c]^{m_1} & \xrightarrow{Q^0(\delta_y+c)} & \mathcal{D}_{X\times\mathbb{P}^1}[c]^{m_0},
\end{array} \tag{2.1}$$

where the horizontal lines are exact. Let $\mathcal{N}(c)$ be the $\mathcal{D}_{X\times\mathbb{P}^1}[c]$-module having $M(f(c))[-1]$ as its free resolution, where $M(f(c))$ is the *mapping cone* of the morphism (1.2). Namely, $\mathcal{N}(c)$ is the $\mathcal{D}_{X\times\mathbb{P}^1}[c]$-module such that

$$\cdots \longrightarrow \mathcal{D}_{X\times\mathbb{P}^1}[c]^{m_4+m_3} \xrightarrow{D^3(c)} \mathcal{D}_{X\times\mathbb{P}^1}[c]^{m_3+m_2} \xrightarrow{D^2(c)} \mathcal{D}_{X\times\mathbb{P}^1}[c]^{m_2+m_1}$$

$$\xrightarrow{D^1(c)} \mathcal{D}_{X\times\mathbb{P}^1}[c]^{m_1+m_0} \xrightarrow{D^0(c)} \mathcal{D}_{X\times\mathbb{P}^1}[c]^{m_0} \longrightarrow \mathcal{N}(c) \longrightarrow 0$$

is a free resolution of $\mathcal{N}(c)$, where the operator $D^i(c)$ is given by

$$D^i(c) = \begin{pmatrix} Q^i(\delta_y + c) & 0 \\ \phi(\delta_y) - P^i(\delta_y + c)\partial_y & -Q^{i-1}(\delta_y + c) \end{pmatrix}.$$

In this situation, we say that $\mathcal{N}(c)$ is obtained as the *mapping cone of a contiguity relation* for $\mathcal{M}(c)$. We observe that $\mathcal{N}(c)$ is a system of partial differential equations on $X \times \mathbb{P}^1$ having singularities along the hypersurface $X \times \{\infty\}$. It is an empirical fact that a confluent hypergeometric system $\mathcal{N}(c)$ often appears as the mapping cone of a contiguity relation for another hypergeometric system $\mathcal{M}(c)$, at least locally around an irregular singular point.

Example 2.1. Let $\Phi_2^{(n)}(b_1, \ldots, b_n; c)$ denote Humbert's confluent hypergeometric system on $X = (\mathbb{P}^1)^n$ with parameters $b_1, \ldots, b_n$ and c, (see [**1**]). Note that $\Phi_2^{(1)}(b_1; c)$ is Kummer's equation and $\Phi_2(b_1, b_2; c) = \Phi_2^{(2)}(b_1, b_2; c)$ is considered in Example 1.1. If $\mathcal{M}(c)$ is the system $\Phi_2^{(n)}(b_1, \ldots, b_n; c)$ and $\phi(c) = c - b_{n+1}$, then $\mathcal{N}(c)$ is the system $\Phi_2^{(n+1)}(b_1, \ldots, b_{n+1}; c)$.

3. Gevrey Cohomology Groups

Let $\mathcal{N}(c)$ be a $\mathcal{D}_{X\times\mathbb{P}^1}[c]$-module obtained as the mapping cone of a contiguity relation for a $\mathcal{D}_X[c]$-module $\mathcal{M}(c)$. We are interested in computing the extension groups $\mathrm{Ext}^i_{\mathcal{D}_{X\times\mathbb{P}^1}}(\mathcal{N}(c), \mathcal{O}_X[[1/y]]_{s,a})$ for generic values of $c \in \mathbb{C}$. Here $\mathcal{O}_X[[1/y]]_{s,a}$

is the sheaf of (formal) *Gevrey* functions, that is, $\mathcal{O}_X[[1/y]]_{s,a}$ consists of the functions $f = \sum_{n=0}^{\infty} u_n(x) y^{-n}$ with $u_n(x) \in \mathcal{O}_X$ such that for any n,

$$||u_n|| \leq C(f,b)\, b^n (n!)^{s-1} \qquad (\forall\, b > a),$$

where $C(f,b)$ is a constant depending only on f and b. It can easily been seen that all the *formal* extension groups $\mathrm{Ext}^i(\mathcal{N}(c), \mathcal{O}_X[[1/y]])$ are trivial, but the Gevrey extention groups $\mathrm{Ext}^i(\mathcal{N}(c), \mathcal{O}_X[[1/y]]_{s,a})$ are, in general, nontrivial.

The main idea for tackling the problem is to introduce an auxiliary complex C of $\mathcal{D}_X$-modules (called the *harmonic complex*), quasi-isomorphic to the solution complex $\mathbb{R}\mathrm{Hom}(\mathcal{N}(c), \mathcal{O}_X[[1/y]]_{s,a})$, in such a manner that computing the cohomology groups $H^i(C)$ is more accessible than computing $\mathrm{Ext}^i(\mathcal{N}(c), \mathcal{O}_X[[1/y]]_{s,a})$ directly. In the next section we construct such a complex C by expressing it *combinatorially* in terms of the contiguity operators $P^i(c)$ as well as the differential operators $Q^i(c)$. The construction of C is formal, that is, it does not require analysis. However, determing admissible indices (s,a) for which C is quasi-isomorphic to $\mathbb{R}\mathrm{Hom}(\mathcal{N}(c), \mathcal{O}_X[[1/y]]_{s,a})$ depends strongly upon hard analysis, that is, upon Gevrey estimates of solutions to certain finite difference equations arising from the contiguity relation. The necessary analysis is developed in [**3**]. In this report we restrict our attention to the algebraic aspect of the theory, leaving the analytic aspect to the above-mentioned paper.

4. Harmonic Complex

To construct the harmonic complex C, we first set

$$P_n^i = \frac{P^i(n-c)}{(n-c)^{\deg P^i(c)}}, \qquad Q_n^i = Q^i(n-c) \qquad (n = 0, 1, 2, \dots),$$

where $\deg P^i(c)$ is the degree of $P^i(c)$ as a polynomial of c. Then P_n^i and Q_n^i define the operators $P_n^i : \mathcal{O}_X^{m_i} \to \mathcal{O}_X^{m_i}$ and $Q_n^i : \mathcal{O}_X^{m_i} \to \mathcal{O}_X^{m_{i+1}}$. The following assumption is very natural for the operators P_n^i and Q_n^i arising from hypergeometric systems.

Assumption 4.1. Assume that P_n^i and Q_n^i admit factorial asymptotic expansions:

$$\begin{cases} P_n^i \sim \sum\limits_{j=0}^{\infty} P^{i,j} (n-c)_j & (n \to +\infty), \\ Q_n^i = \sum\limits_{j=0}^{N^i} Q^{i,j} \langle n-c \rangle_j + o(1) & (n \to +\infty), \end{cases}$$

where $(x)_j$ and $\langle x \rangle_j$ are defined by

$$(x)_j = \frac{(-1)^j j!}{x(x+1)\cdots(x+j-1)}, \qquad \langle x \rangle_j = \frac{x(x-1)\cdots(x-j+1)}{(-1)^j j!},$$

and that there exists a direct sum decomposition $\mathcal{O}_X^{m_i} = U_0^i \oplus U_1^i$ with the associated projections $X^i : \mathcal{O}_X^{m_i} \to U_0^i$ and $Y^i : \mathcal{O}_X^{m_i} \to U_1^i$ such that

$$\begin{cases} X^i P^{i,0} X^i = X^i, & X^i P^{i,0} Y^i = O, \\ Y^i P^{i,0} X^i = O, & X^i P^{i,1} X^i = O, \\ I_1 - Z^i : U_1^i \to U_1^i & \text{is invertible}, \end{cases}$$

where I_1 is the identity operator on U_1^i and $Z^i := Y^i P^{i,0} Y^i$.

Definition 4.2. Under Assumption 4.1 we define C^i and $d^i : C^i \to C^{i+1}$ by

$$\begin{cases} C^i = U_0^{i-1}, \\ d^i = Q^{i-1,0} + \sum_{j=1}^{N^{i-1}} Q^{i-1,j} \sum_{J \in S_j} A_J^{i-1}, \end{cases}$$

where S_j is the set of all nonempty subsets of $\{1, 2, \dots, j\}$. The operators A_J^i : $U^i \to U^i$ $(J \in S_j)$ are defined as follows. We first set

$$P_{jk}^i = \sum_{m=1}^{j-k} \frac{(k-1)_+!(m-1)!}{(k+m-1)!} \begin{bmatrix} m-1 \\ j-k-m \end{bmatrix} P^{i,m},$$

for $0 \le k < j$, where $a_+ = \max\{a, 0\}$ and

$$\begin{bmatrix} a \\ j \end{bmatrix} = \begin{cases} 1 & (j = 0), \\ \frac{1}{j!} a(a+1) \cdots (a+j-1) & (j = 1, 2, 3, \dots). \end{cases}$$

Using the operators P_{jk}^i defined above, we next set

$$A_{jk}^i = X^i P_{j+1,k}^i + (I + \frac{1}{j} X^i P^{i,1})(I_1 - Z^i)^{-1}(Y^i P_{jk}^i - \delta_{j,k+1} Z^i),$$

for $0 \le k < j$, where I is the identity operator on U^i and δ_{ij} is Kronecker's symbol. Then for each $J = \{j_1, j_2, \dots, j_k\} \in S_j$ with $j_1 < j_2 < \cdots < j_k$, the operator A_J^i is defined by $A_J^i = A_{j_k j_{k-1}}^i A_{j_{k-1} j_{k-2}}^i \cdots A_{j_2 j_1}^i A_{j_1 0}^i$.

Lemma 4.3. *C so defined is a complex, i.e., d^i maps C^i into C^{i+1} and $d^{i+1} d^i = 0$.*

5. Quasi-Isomorphism

Theorem 5.1. *For suitable Gevrey indices (s, a), we have for any $c \in \mathbb{C} \setminus \mathbb{Z}$,*

$$\mathbb{R}\mathrm{Hom}_{\mathcal{D}_{X \times \mathbb{P}^1}}(\mathcal{N}(c), \mathcal{O}_X[[1/y]]_{s,a}) \underset{\text{qis}}{\simeq} C. \tag{5.1}$$

A Gevrey index (s, a) for which (5.1) holds is said to be *admissible.* To describe admissible Gevrey indices, we set

$$\begin{cases} \underline{s} = \max_i \{\deg P^i(c) - s^i\} - \deg \phi(c) + 2, \\ \bar{s} = \min_i \deg P^i(c) - \deg \phi(c) + 2, \end{cases}$$

where

$$\begin{cases} p^i = \min\{ j \,;\, X^i P^{i,j} Y^i \neq O \}, \\ q^i = \min\{ j \,;\, Y^i P^{i,j} X^i \neq O \}, \\ r^i = \min\{ j \,;\, Y^i P^{i,j} Y^i \neq O \}, \\ s^i = \min\{ p^i + q^i - 1, r^i \}. \end{cases}$$

- Case $\underline{s} < s < \bar{s}$: (s, a) is admissible for any $a \ge 0$.
- Case $s = \underline{s}$ or $\bar{s}$: admissible values of a can be determined explicitly in terms of the coefficients $P^{i,j}$ of the asymptotic expansion of P_n^i, though the description of them are rather complicated (and hence omitted). See [4] for details.

Example 5.2. Recall that if $\mathcal{M}(c) = \Phi_2^{(n)}(b_1, \ldots, b_n; c)$ and $\phi(c) = c - b_{n+1}$, then $\mathcal{N}(c) = \Phi_2^{(n)}(b_1, \ldots, b_{n+1}; c)$, (Example 2.1). In this case the harmonic complex C is isomorphic to the de Rham complex $\Omega^{\cdot}_{(\mathbb{P}^1)^n}[-1]$ shifted by one, and $\underline{s} = 1$, $\bar{s} = 2$. Theorem 5.1 implies that

$$\dim \operatorname{Ext}^i(\mathcal{N}(c), \mathcal{O}_X[[1/y]]_{s,a}) = \dim H^i(C) = \begin{cases} 1 & (i = 1) \\ 0 & (i \neq 0). \end{cases}$$

where the second equality follows from Poincaré's lemma.

H. Majima [**5**] also computed the Gevrey extension groups for the Humbert system $\Phi_2^{(n)}(b_1, \ldots, b_n; c)$.

References

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