数理解析研究所講究録
1090 巻 1999 年 116-125

On asymptotic estimates for coefficients of divergent solutions to second order non-homogeneous linear ordinary differential equations.

Nakamura Yayoi
Ochanomizu Univercity
中村弥生（お茶の水女子大）

1 Introduction

Conseder a second order linear ordinary differential equation

$$\frac{d^2y}{dx^2} - p(x)y = 0 \tag{1}$$

$$p(x) = x^m + a_1x^{m-1} + \ldots + a_m$$

This equation has a irregular singular point at infinity with irregularity $m+2$.

It is known that this equation has integral solutions in following four cases:

(i) $p(x) = x + a_1$ (i.e. $m = 1$)

(ii) $p(x) = x^2 + a_1x + a_2$ (i.e. $m = 2$)

(iii) $p(x) = x^m$

(iv) $p(x) = x^{2p} + cx^{p-1}$

In these cases, each equation will be transformed into confluent hypergeometric differential equation. Especially, the solution of the equation (1) with coefficient (i) is Airy function and (ii) parabolic cylinder function.

Let $\mathcal{O}$ and $\widehat{\mathcal{O}}$ be the ring of convergent power series and the ring of formal power series in x, respectively. Then, for a linear ordinary differential operator P with coefficients in holomorphic function, we see the following isomorphism of linear spaces due to Deligne;

$$H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0)) \cong \mathrm{Ker}(P; \widehat{\mathcal{O}}/\mathcal{O} \to \widehat{\mathcal{O}}/\mathcal{O})$$

where $\mathcal{A}_0$ is sheaf of germs of functions asymptotically developable to the formal power series 0 on the circle S^1.

To pay attension to this isomorphism, by vanishing theorem due to [3] in asymptotic analysis, we obtain the asymptotic estimates for coefficients of divergent solutions to non-homogeneous linear ordinary differential equations about the differential operators $P = \dfrac{d^2}{dx^2} - p(x)$.

In this note, we see the numerical calculation about the estimation about the case of (iii), (iv). By these calculation, we can see the estimate and the asymptotic estimate for coefficients of a solution of the non-homogeneous differential equation are almost the same.

2 About the operator $P = \dfrac{d^2}{dx^2} - x^m$

The differential equation

$$Py = (\frac{d^2}{dx^2} - x^m)y = 0 \tag{2}$$

has irregular singular point at $x = \infty$. The irregularity of P at $x = \infty$ equals $i_\infty(P) = m + 2$. Put

$$U_k = \left\{x \in \mathbf{C}; |x| > 0, \frac{2k-3}{m+2}\pi < \arg x < \frac{2k+1}{m+2}\pi\right\}, k = 0, 1, \ldots, m+1.$$

Then $\{U_k, k = 0, 1, \ldots, m+1\}$ forms an open sectorial covering at $x = \infty$, and put

$$S_k = U_k \cap U_{k+1} = \left\{x \in \mathbf{C}; |x| > 0, \left|\arg x - \frac{2k}{m+2}\pi\right| < \frac{\pi}{m+2}\right\}, k = 0, 1, \ldots, m+1,$$

where $U_{m+2} = U_0$.

We have now a unique entire solution $y = y_k$ of the differential equation (2) with the asymptotic representation:

$$y_k \sim \omega^{\frac{m}{4}k} x^{-\frac{m}{4}} (1 + \sum_{N=1}^{\infty} B_N \omega^{\frac{k}{2}N} x^{-\frac{N}{2}}) exp\{(-1)^{k+1} \frac{2}{m+2} x^{\frac{m+2}{2}}\}$$

as x tends to infinity in any closed subsector of the open sector U_k. Where $\omega = e^{\frac{2}{m+2}\pi i}$ and, for $N = 0, 1, \ldots$, the quantities B_N are decided by

$$B_{(m+2)(N+1)} = \prod_{\ell=0}^{N} \frac{-1}{(m+2)(\ell+1)} \{\frac{m}{4}(\frac{m}{4}+1) + \frac{\ell m(m+2)}{4} + \frac{\ell(m+2)}{2}(\frac{\ell(m+2)}{2}+1)\}.$$

On the other hand, we have the solution with above properties:

$$y_{2k} = \frac{(\frac{4}{m+2})^{\frac{1}{2}+\frac{1}{m+2}}}{\Gamma(\frac{1}{2}-\frac{1}{m+2})} \int_0^{+\infty} e^{-x^{\frac{m+2}{2}}(\zeta+\frac{2}{m+2})} \zeta^{-\frac{1}{2}-\frac{1}{m+2}} (\zeta + \frac{4}{m+2})^{-\frac{1}{2}-\frac{1}{m+2}} d\zeta,$$

$$y_{2k+1} = \frac{(-1)^{\frac{2}{m+2}}(\frac{4}{m+2})^{\frac{1}{2}+\frac{1}{m+2}}}{\Gamma(\frac{1}{2}-\frac{1}{m+2})} \int_0^{+\infty} e^{x^{\frac{m+2}{2}}(\zeta+\frac{2}{m+2})} \zeta^{-\frac{1}{2}-\frac{1}{m+2}} (\zeta + \frac{4}{m+2})^{-\frac{1}{2}-\frac{1}{m+2}} d\zeta.$$

Then each y_k gives subdominant solution in S_k.

2.1 Coefficient and its asymptotic estimation of the solution

By using above representations of solutions, we can choose a basis of $H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0))$ in the following way:
Put 1-cocycle about $\{U_0, U_1, \ldots, U_{m+1}\}$ by

$$\{u_{ij}^{(k)}, (i,j) = (0,1), (1,2), \ldots, (m, m+1), (m+1, 0)\},$$

$$u_{ij}^{(k)} = \begin{cases} y_i & x \in S_k \quad i = k, \\ 0 & x \in S_k \quad i \neq k, \end{cases}$$

for $k = 0, 1, \ldots, m+1$.

In this situation, the pair of cohomology classes of $\{u_{ij}^{(0)}\}, \{u_{ij}^{(1)}\}, \ldots, \{u_{ij}^{(m+1)}\}$ forms a basis of $H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0))$.

By vanishing theorem in asymptotic analysis, we have 0-cochain $\{v_0^{(k)}, v_1^{(k)}, \ldots, v_{m+1}^{(k)}\}, k = 0, 1, \ldots, m+1$, such that

$$u_{ij}^{(k)} = -v_i^{(k)} + v_j^{(k)}, (i,j) = (0,1), (1,2), \cdot, (m, m+1), (m+1, 0),$$

where each $v_j^{(k)}$ are defined in U_j for $j = 0, 1, \ldots, m+1$ and asymptotically developable to formal power-series $\hat{v}^{(k)}(x) = \sum_{r=0}^{\infty} v(m,k,r)x^{-r}, k = 0, 1, \ldots, m+1$, respectively. Then

$$Pu_{ij}^{(k)} = -Pv_i^{(k)} + Pv_j^{(k)}.$$

In S_i, we have

$$Pv_i^{(k)} = Pv_j^{(k)}.$$

Put

$$g^{(k)}(x) = \begin{cases} Pv_i^{(k)} & x \in U_i \\ Pv_j^{(k)} & x \in U_j. \end{cases}$$

Then

$$f^{(k)}(x) = \begin{cases} g^{(k)}(x) & x \in \mathbf{C} \\ \lim_{x \to \infty} g^{(k)}(x) & x = \infty \end{cases}$$

define holomorphic function, and

$$P\hat{v}_i^{(k)}(x) = f^{(k)}(x).$$

Hence, by the vanishing theorem due to [3] in asymptotic analysis, $\langle [\hat{v}^{(0)}], [\hat{v}^{(1)}], \ldots, [\hat{v}^{(m+1)}] \rangle$ forms a basis of $\mathrm{Ker}(\frac{d^2}{dx^2} - x^m; \widehat{\mathcal{O}}/\mathcal{O})$.

Moreover, we can have estimates for the coefficients $v(m,k,r)$.

$$\begin{aligned} v(m, 2k+1, r) &= (-1)^{\frac{2r}{m+2}+1} \frac{1}{2\pi i} \Gamma(\frac{2r}{m+2}) (\frac{4}{m+2})^{-\frac{2}{m+2}} (\frac{2}{m+2})^{1-\frac{2r}{m+2}} \\ &\times \frac{\Gamma(\frac{1}{2} - \frac{1}{m+2}) \Gamma(\frac{2r+2}{m+2})}{\Gamma(\frac{2r+1}{m+2} + \frac{1}{2})} F(\frac{2r}{m+2}, \frac{1}{2} - \frac{1}{m+2}, \frac{2r+1}{m+2} + \frac{1}{2}; -1). \end{aligned}$$

$$\begin{aligned} v(m,2k,r) &= (-1)^{\frac{1}{2}-\frac{1}{m+2}}\frac{1}{2\pi i}\Gamma(\frac{2r}{m+2})(\frac{4}{m+2})^{-\frac{2}{m+2}}(\frac{2}{m+2})^{1-\frac{2r}{m+2}} \\ &\times \frac{\Gamma(\frac{1}{2}-\frac{1}{m+2})\Gamma(\frac{2r+2}{m+2})}{\Gamma(\frac{2r+1}{m+2}+\frac{1}{2})}F(\frac{2r}{m+2},\frac{1}{2}-\frac{1}{m+2},\frac{2r+1}{m+2}+\frac{1}{2};-1). \end{aligned}$$

And we can have asymptotic estimates for any sufficiently large number r,

$$\begin{aligned} v(m,k,r) &\sim v_M(m,k,r) \\ &= \frac{1}{2\pi i}(-\omega^{r+1+k})\sum_{N=0}^{M-1} B_{m,N}(\frac{m+2}{2})^{\frac{2}{m+2}(r-\frac{m}{4}-\frac{N}{2})-1}\Gamma(\frac{2}{m+2}(r-\frac{1}{4}m-\frac{N}{2})) \\ &\quad + O(\Gamma(\frac{2}{m+2}(r-\frac{m}{4}-\frac{M}{2}))), \end{aligned}$$

provided $1 \leq M < r$.

2.2 About the operator $P = \dfrac{d^2}{dx^2} - (x^{2p} + cx^{p-1})$

The differential equation

$$Py = (\frac{d^2}{dx^2} - (x^{2p} + cx^{p-1}))y = 0 \tag{3}$$

has irregular singular point at $x = \infty$. The irregularity of P at $x = \infty$ equals $i_\infty(P) = 2p + 2$. Put

$$U_k = \left\{ x \in \mathbf{C}; |x| > 0, \frac{2k-3}{2p+2}\pi < \arg x < \frac{2k+1}{2p+2}\pi \right\}, k = 0, 1, \ldots, 2p+1.$$

Then $\{U_k, k = 0, 1, \ldots, 2p+1\}$ forms an open sectorial covering at $x = \infty$, and put

$$S_k = U_k \cap U_{k+1} = \left\{ x \in \mathbf{C}; |x| > 0, \left| \arg x - \frac{k}{p+1}\pi \right| < \frac{\pi}{2p+2} \right\}, k = 0, 1, \ldots, 2p+1,$$

where $U_{2p+2} = U_0$.

We have now a unique entire solution $y = y_k$ of the differential equation (3) with the asymptotic representation:

$$y_k \sim \omega^{\frac{p+1}{2}k} x^{-\frac{p+1}{2}}(1 + \sum_{N=1}^{\infty} B_{2N(p+1)}(\omega^{-(p+1)k}c)\omega^{kN}x^{-N})e^{-\frac{1}{p+1}(\omega^{-k}x)^{p+1}}$$

as x tends to infinity in any closed subsector of the open sector U_k. Where $\omega = e^{\frac{1}{p+1}\pi i}$ and, for $N = 0, 1, \ldots$, the quantities B_N are decided by

$$B_{2N(p+1)}(c) = \frac{1}{8(p+1))^N}\frac{1}{k!}\prod_{\ell=0}^{N-1}(p + c + 2\ell(p+1))(p + c + 2\ell(p+1) - 1),$$

$$B_\ell(c) = 0, \ell \neq 2N(p+1).$$

On the other hand, we have the solution with above properties:

$$y_{2k} = \frac{(\frac{2}{p+1})^{\frac{1}{2}(\frac{c+1}{p+1}+1)}}{\Gamma(\frac{1}{2}(\frac{c-1}{p+1}+1))} \int_0^{+\infty} e^{-x^{p+1}(\zeta+\frac{1}{p+1})} \zeta^{\frac{1}{2}(\frac{c-1}{p+1}-1)} (\zeta + \frac{2}{p+1})^{-\frac{1}{2}(\frac{c+1}{p+1}+1)} d\zeta,$$

$$y_{2k+1} = \frac{(-1)^{\frac{1}{2}(\frac{c+1}{p+1}+1)+1} (\frac{2}{p+1})^{\frac{1}{2}(1-\frac{c-1}{p+1})}}{\Gamma(-\frac{1}{2}(\frac{c+1}{p+1}+1)+1)} \int_0^{+\infty} e^{x^{p+1}(\zeta+\frac{1}{p+1})} \zeta^{-\frac{1}{2}(\frac{c+1}{p+1}+1)} (\zeta + \frac{2}{p+1})^{\frac{1}{2}(\frac{c-1}{p+1}-1)} d\zeta.$$

Then each y_k gives subdominant solution in the sector S_k.

2.3 Coefficient and its asymptotic estimation of the solution

By using above representations of solution, we can choose a basis of $H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0))$ with following way:

Put 1-cocycle about $\{U_0, U_1, \ldots, U_{2p+1}\}$ by

$$\{u_{i,j}^{(k)}, (i,j) = (0,1), (1,2), \ldots, (2p, 2p+1), (2p+1, 0)\},$$

$$u_{i,j}^{(k)} = \begin{cases} y_j & x \in S_j \quad j = k, \\ 0 & x \in S_j \quad j \neq k, \end{cases}$$

for $k = 0, 1, ldots, 2p+1$.

In this situation, the pair of the cohomology class of $\{u_{j,j+1}^{(0)}\}, \{u_{j,j+1}^{(1)}\}, \ldots, \{u_{j,j+1}^{(2p+1)}\}$ forms a basis of $H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0))$.

By vanishing theorem in asymptotic analysis, we have 0-cochain $\{v_0^{(k)}, v_1^{(k)}, \ldots, v_{2p+1}^{(k)}\}, k = 0, 1, \ldots, 2p+1$ such that

$$u_{i,j}^{(k)} = -v_i^{(k)} + v_j^{(k)}, (i,j) = (0,1), (1,2), \ldots, (2p, 2p+1), (2p+1, 0),$$

where each $v_j^{(k)}$ are defined in U_j, for $j = 0, 1, \ldots, 2p+1$ and asymptotically developable to formal power-series $\hat{v}^{(k)}(x) = \sum_{r=0}^{\infty} v(2p+1, c, k, r) x^{-r}, k = 0, 1, \ldots, 2p+1$, respectively. Then

$$Pu_{ij}^{(k)} = -Pv_i^{(k)} + Pv_j^{(k)}.$$

In S_i, we have

$$Pv_i^{(k)} = Pv_j^{(k)}.$$

Put

$$g^{(k)}(x) = \begin{cases} Pv_i^{(k)} & x \in U_i \\ Pv_j^{(k)} & x \in U_j. \end{cases}$$

Then

$$f^{(k)}(x) = \begin{cases} g^{(k)}(x) & x \in \mathbf{C} \\ \lim_{x\to\infty} g^{(k)}(x) & x = \infty \end{cases}$$

define holomorphic function, and

$$P\hat{v}_i^{(k)}(x) = f^{(k)}(x).$$

Hence, by the vanishing theorem due to [3] in asymptotic analysis, $\langle[\hat{v}^{(0)}],[\hat{v}^{(1)}],\ldots,[\hat{v}^{(2p+1)}]\rangle$ forms a basis of $\mathrm{Ker}(\frac{d^2}{dx^2}-(x^{2p}+cx^{p-1});\widehat{\mathcal{O}}/\mathcal{O})$.

Moreover, we can have estimates for the coefficients $v(2p+1,c,k,r)$.

$$\begin{aligned}
&v(2p+2,c,2k+1,r)\\
=\;&(-1)^{\frac{r}{p+1}+1}\frac{1}{2\pi i}\Gamma(\frac{r}{p+1})(\frac{2}{p+1})^{-\frac{1}{p+1}}(\frac{1}{p+1})^{\frac{r}{p+1}+1}\\
&\times\frac{\Gamma(-\frac{1}{2}(\frac{c+1}{p+1}-1))\Gamma(\frac{-r+1}{p+1})}{\Gamma(\frac{r}{p+1}-\frac{1}{2}(\frac{c-1}{p+1}-1))}F(\frac{r}{p+1},-\frac{1}{2}(\frac{c+1}{p+1}-1),\frac{r}{p+1}-\frac{1}{2}(\frac{c-1}{p+1}-1);-1).
\end{aligned}$$

$$\begin{aligned}
&v(2p+2,c,2k,r)\\
=\;&\frac{1}{2\pi i}(-1)^{-\frac{1}{p+1}+1}\Gamma(\frac{r}{p+1})(p+1)^{-\frac{r}{p+1}+1}(\frac{2}{p+1})^{-\frac{1}{p+1}}\\
&\times\frac{\Gamma(\frac{1}{2}(\frac{c-1}{p+1}+1))\Gamma(\frac{1+r}{p+1})}{\Gamma(\frac{r}{p+1}+\frac{1}{2}(\frac{c+1}{p+1}+1))}F(\frac{r}{p+1},\frac{1}{2}(\frac{c-1}{p+1}+1),\frac{r}{p+1}+\frac{1}{2}(\frac{c+1}{p+1}+1);-1).
\end{aligned}$$

And we can have asymptotic estimates for any sufficiently large number r,

$$\begin{aligned}
&v(2p+2,c,k,r)\\
\sim\;&v_M(2p+2,c,k,r)\\
=\;&-\frac{1}{2\pi i}\omega^{kr}\sum_{N=0}^{M-1}B_{2N(p+1)}(\omega^{-(p+1)k}c)(p+1)^{\frac{1}{p+1}(r-(p+1)(N+\frac{1}{2}))-1}\\
&\times\Gamma(\frac{1}{p+1}(r-(p+1)(N+\frac{1}{2})))+O(\Gamma(\frac{1}{p+1}(r-(p+1)(M+\frac{1}{2})))),
\end{aligned}$$

provided $1\le M<r$.

3 numerical calculation

We shall confirm that the estimate and the asymptotic estimate for a coefficient are almost the same by numerical calculation by using symbolic computation system *Mathematica.*

3.1 About the operator $P=\dfrac{d^2}{dx^2}-x^m$

3.1.1 $\dfrac{d^2}{dx^2}-x^5$ and $r=200$

Put $m=5$ and $r=200$.

When k is even number, we can have

$$\begin{aligned} v(5,2k,200) &= 5.7809195499603844406905645 3 \times 10^{103} \\ &\quad -1.3194571632529001115674583 4 \times 10^{103}10 \times i. \end{aligned}$$

If we take M equals 50, we have

$$\begin{aligned} v_{50}(5,2k,200) &= 5.780919549960384444069056422 9 \times 10^{103} \\ &\quad -1.319457163252900111567458271 \times 10^{103}10 \times i. \end{aligned}$$

Hence we have the module $v(5,2k,200)$ of $v_{50}(5,2k,200)$

$$v(5,2k,200)/v_{50}(5,2k,200) = 1.0000000000000000000000000053.$$

This shows that $v(5,2k,200)$ and $v_{50}(5,2k,200)$ are almost the same.

When k is odd number, we can have

$$\begin{aligned} v(5,2k+1,200) &= -2.572751234767002988887166675 \times 10^{103} \\ &\quad -5.342372987051580391092454 17 \times 10^{103}10 \times i. \end{aligned}$$

If we take M equals 50, we have

$$\begin{aligned} v_{50}(5,2k+1,200) &= -2.572751234767002988887166 54 \times 10^{103} \\ &\quad -5.342372987051580391092453892 \times 10^{103}10 \times i. \end{aligned}$$

Hence we have the module $v(5,2k+1,200)$ of $v_{50}(5,2k+1,200)$

$$v(5,2k+1,200)/v_{50}(5,2k+1,200) = 1.0000000000000000000000000053.$$

This shows that $v(5,2k+1,200)$ and $v_{50}(5,2k+1,200)$ are almost the same.

3.1.2 $\frac{d^2}{dx^2} - x^{19}$ and $r = 400$

Put $m = 19$ and $r = 400$.

When k is even number, we can have

$$\begin{aligned} v(19,2k,400) &= 3.4530770657838575579184387 05 \times 10^{78} \\ &\quad -7.170388411113402004415825685 \times 10^{78} \times i. \end{aligned}$$

If we take M equals 37, we have

$$\begin{aligned} v_{37}(19,2k,400) &= 3.4530770657838931372039674 99 \times 10^{78} \\ &\quad -7.170388411113475885563500981 \times 10^{78} \times i. \end{aligned}$$

Hence we have the module $v(19,2k,400)$ of $v_{37}(19,2k,400)$

$$v(19,2k,400)/v_{37}(19,2k,400) = 0.9999999999999989696353469 39.$$

This shows that $v(19, 2k, 400)$ and $v_{37}(19, 2k, 400)$ are almost the same.
When k is odd number, we can have

$$\begin{aligned} v(19, 2k+1, 400) &= 6.8922897785390505750339919 7 \times 10^{78} \\ &\quad -3.97926535897242699497453915 3 \times 10^{78} \times i. \end{aligned}$$

If we take M equals 37, we have

$$\begin{aligned} v_{37}(19, 2k+1, 400) &= 6.8922897785391215907516565 74 \times 10^{78} \\ &\quad -3.9792653589724679959182495 07 \times 10^{78} \times i. \end{aligned}$$

Hence we have the module $v(19, 2k+1, 400)$ of $v_{37}(19, 2k+1, 400)$

$$v(19, 2k+1, 400)/v_{37}(19, 2k+1, 400) = 0.99999999999998969635346939.$$

This shows that $v(19, 2k+1, 400)$ and $v_{37}(19, 2k+1, 400)$ are almost the same.

3.2 $P = \dfrac{d^2}{dx^2} - (x^{2p} + cx^{p-1})$

3.2.1 $\dfrac{d^2}{dx^2} - (x^8 + 2x^3)$ and $r = 100$

Put $p = 4$ and $r = 100$.
When k is even number, we can have

$$\begin{aligned} v(10, 2, 2k, 100) &= -3.7670034170747439392516658 48 \times 10^28 \\ &\quad -5.1848353976148091750773271 19 \times 10^28 \times i. \end{aligned}$$

If we take M equals 19, we have

$$\begin{aligned} v_{19}(10, 2, 2k, 100) &= -3.7670034200446482007086506 66 \times 10^28 \\ &\quad -5.1848354017025317060770141 79 \times 10^28 \times i. \end{aligned}$$

Hence we have the module $v(10, 2, 2k, 100)$ of $v_{19}(10, 2, 2k, 100)$

$$v(10, 2, 2k, 100)/v_{19}(10, 2, 2k, 100) = 0.99999999921160032782189929 9 + 0.10^{-38} \times i.$$

This shows that $v(10, 2, 2k, 100)$ and $v_{19}(10, 2, 2k, 100)$ are almost the same.
When k is odd number, we can have

$$\begin{aligned} v(10, 2, 2k+1, 100) &= -2.3582519678612478831939528 33 \times 10^29 \\ &\quad +3.2458553724797885787477273 95 \times 10^29 \times i. \end{aligned}$$

If we take M equals 19, we have

$$\begin{aligned} v_{19}(10, 2, 2k+1, 100) &= -2.3582519540410331435767629 207 \times 10^29 \\ &\quad +3.2458553534578948741094000 01 \times 10^29 \times i. \end{aligned}$$

Hence we have the module $v(10,2,2k+1,100)$ of $v_{19}(10,2,2k+1,100)$

$$v(10,2,2k+1,100)/v_{19}(10,2,2k+1,100) = 1.0000000058603639513306731 08.$$

This shows that $v(10,2,2k+1,100)$ and $v_{19}(10,2,2k+1,100)$ are almost the same.

3.2.2 $\dfrac{d^2}{dx^2} - (x^{40} + 10x^{19})$ and $r = 500$

Put $p = 20$ and $r = 500$.

When k is even number, we can have

$$\begin{aligned} v(42,10,2k,500) &= -1.0334155667293746746106228 75 \times 10^{5}0 \\ &\quad -2.3587035943917831294865334 42 \times 10^{4}9 \times i. \end{aligned}$$

If we take M equals 19, we have

$$\begin{aligned} v_{19}(42,10,2k,500) &= -1.0334155666234282085136772 88 \times 10^{5}0 \\ &\quad -2.3587035941499672342732825 53 \times 10^{4}9 \times i. \end{aligned}$$

Hence we have the module $v(42,10,2k,500)$ of $v_{19}(42,10,2k,500)$

$$v(42,10,2k,500)/v_{19}(42,10,2k,500) = 1.0000000001025206795008072 33 + 0.10^{-37} \times i.$$

This shows that $v(42,10,2k,500)$ and $v_{19}(42,10,2k,500)$ are almost the same.

When k is odd number, we can have

$$\begin{aligned} v(42,10,2k+1,500) &= -3.0181882931369859209278254 71 \times 10^{5}0 \\ &\quad +2.0024370939702928182207631 6 \times 10^{5}1 \times i. \end{aligned}$$

If we take M equals 19, we have

$$\begin{aligned} v_{19}(42,10,2k+1,500) &= -3.0181882931360153102778485 65 \times 10^{5}0 \\ &\quad +2.0024370939696488601304956 088 \times 10^{5}1 \times i. \end{aligned}$$

Hence we have the module $v(42,10,2k+1,500)$ of $v_{19}(42,10,2k+1,500)$

$$v(42,10,2k+1,500)/v_{19}(42,10,2k+1,500) = 1.0000000000003215871760500 42 + 0.10^{-35} \times i.$$

This shows that $v(42,10,2k+1,500)$ and $v_{19}(42,10,2k+1,500)$ are almost the same.

References

[1] Inui. T. : Special functions (Tokusyukansuu written in japanese), Iwanami-shoten (1962)

[2] Majima. H.: Resurgent Equations and Stokes Multipliyers for Generalized Confluent Hypergeometric Differential Equations of the Second Order, in the Proceedings of Hayashibara Forum'90 International Symposium on Special Functions, ICM Satellite Conference Proceedings, Springer-Verlag (1991), pp.222 – 233.

[3] Majima. H., Howls, C. J. and Olde Daalhuis, A. B.: Vanishing Theorem in Asymptotic Analysis III (to appear in "Structure of Solutions of Differential Equations" edited by T.Kawai and M. Morimoto, World Science, May 1996)

[4] Mizuno. T.: On estimates of coefficients of formal solutions to inhomogeneous linear ordinary differential equations, Master's Dissertation, Ochanomizu University, (1996), (in japanese).

[5] Sibuya. Y.: Global Theory of a Second Order Linear Differential Equation with a Polynomial Coefficient,North-Holland Mathematics studies,No 18.