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SOME APPROXIMATIONS ON $L^1(\mathbb{R}^n)$

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ABSTRACT. In this note, we approximate $L^1(\mathbb{R})$ by the linear subspace constructed by translations of the functions $x^k e^{-x^{2m}}$ and $x^{k+1}e^{-x^{2m}}$, where m is a natural number ≥ 2 and k is a nonnegative integer. We give an analogous result in the general dimensional case. This result is induced by the simpleness of all zeros of certain entire functions in the Laguerre-Pólya class.

1. INTRODUCTION

This study is a joint work with Professor Haseo Ki (Yonsei University) and Professor Young-One Kim (Sejong University).

Given a certain set $\mathfrak{M}$ in the space $L^1(\mathbb{R}^n)$. Let us consider all possible functions of the form

$$\sum_{j,k} c_{j,k} f_j(x + \lambda_{j,k}), \tag{1.1}$$

where $c_{j,k}$ are complex numbers, $\lambda_{j,k}$ are in $\mathbb{R}^n$, f_j are in $\mathfrak{M}$ and the sum is finite. Every function of the form (1.1) lies in $L^1(\mathbb{R}^n)$ and the totality of these functions constitutes a linear subspace in $L^1(\mathbb{R}^n)$. The closure of this set in $L^1(\mathbb{R}^n)$ is denoted by $I(\mathfrak{M})$. $I(\mathfrak{M})$ is closed and translation-invariant in $L^1(\mathbb{R}^n)$ (i.e. if $f \in I(\mathfrak{M})$ and $\lambda \in \mathbb{R}^n$, then $f(\cdot+\lambda) \in I(\mathfrak{M})$) and moreover it becomes an ideal in $L^1(\mathbb{R}^n)$ (see [9]). It is an important problem to find necessary and sufficient conditions for the set $\mathfrak{M}$ so that $I(\mathfrak{M}) = L^1(\mathbb{R}^n)$. N. Wiener [10] solved this problem in the following.

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Theorem 1.1 (Wiener). *$I(\mathfrak{M}) = L^1(\mathbb{R}^n)$ if and only if there does not exist any point $x^0 = (x^0_1, \ldots, x^0_n) \in \mathbb{R}^n$ at which the Fourier transforms of all functions in $\mathfrak{M}$ become zero.*

Next let us consider more actual problem for approximation on $L^1(\mathbb{R}^n)$. Given ϕ in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. Our question is: How many monomials (or polynomials) p_j are necessary to satisfy $I(\{p_j\phi\}_j) = L^1(\mathbb{R}^n)$? The minimum number of these monomials (or polynomials) is denoted by $M(\phi)$ (or $N(\phi)$). In this note we answer this question in the special case.

We consider the case that $\phi(x) = \phi_m(x) = \exp\{-\sum_{j=1}^n x_j^{2m_j}\}$ $(m_j \in \mathbb{N})$. Note that if $m_j = 1$ $(j = 1, \cdots, n)$, then $I(\{\phi_m\}) = L^1(\mathbb{R}^n)$ and so $M(\phi_m) = N(\phi_m) = 1$. In fact the Fourier transform of ϕ_m is $c\exp\{-\sum_{j=1}^n \xi_j^2/4\}$, which has no zero. We denote by R the set of indices in $\{1, \ldots, n\}$ such that m_j is not 1 and by r the cardinality of R. We are interested in the case $R \neq \emptyset$ $(r > 1)$. The set $\mathfrak{M}_{m,k}$ is defined by

$$\mathfrak{M}_{m,k} = \left\{ \prod_{j \in R} x_j^{k_j + \delta_j} \cdot \phi_m(x)\,;\, \delta_j = 0, 1 \ (j \in R) \right\},$$

where k_j are nonnegative integers.

Theorem 1.2. $I(\mathfrak{M}_{m,k}) = L^1(\mathbb{R}^n)$.

As a corollary, we obtain $M(\phi_m) \leq 2^r$ and $N(\phi_m) = 1$. In fact if we take the polynomial $p_k(x) = \prod_{j \in R}(x_j^{k_j} + x_j^{k_j+1})$, then we have $I(\{p_k\phi_m\}) = L^1(\mathbb{R}^n)$ by Wiener's theorem. The above theorem can be easily obtained by the following theorem.

Theorem 1.3. *All zeros of the Fourier transform of $x^k e^{-x^{2m}}$:*

$$\varphi_k(\xi) := \int_{-\infty}^{\infty} x^k e^{-x^{2m} + i\xi x} dx \tag{1.2}$$

are real and simple (i.e. if $\varphi_k(a) = 0$, then $\varphi_k'(a) \neq 0$), where m is a natural number and k is a nonnegative integer.

In fact, the simpleness of zeros of φ_k implies that φ_k and φ_{k+1} $(= -i\varphi_k')$ have no common zero in $\mathbb{R}$. Since the Fourier transforms of $\prod_{j\in R} x_j^{k_j+\delta_j} \cdot \phi_m(x)$ ($\delta_j = 0$ or 1) are $c\exp\{-\sum_{j\notin R}\xi_j^2/4\}\cdot\prod_{j\in R}\varphi_{k_j+\delta_j}(\xi_j)$, which have no common zeros in $\mathbb{R}^n$. By Wiener's theorem, we have Theorem 1.3.

We briefly explain the difficulty of the proof of Theorem 1.3. The function φ_0 satisfies an ordinary differential equation (see Section 2), but the order of this equation is greater than two. For example, the simpleness of zeros of the Bessel and Airy functions can be seen from *second* order differential equations. Unlike these case, another properties of φ_k are necessary for our purpose. The fact that φ_k belong to the *Laguerre-Pólya class* plays a key role.

This note is organized as follows. In Section 2 we briefly review the definitions of the Laguerre-Pólya class and those properties of functions in this class which will be used in the proof of Theorem 1.3. Next the properties of the function φ_0 are studied in detail in [2]. We recall important properties of φ_0, and moreover show that φ_k are in the Laguerre-Pólya class. In Section 3, we will prove Theorem 1.3.

Last we remark that the above theorems positively solves the conjectures which are given in [2], Section 4, in more general case.

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2. Known results

2.1. The Laguerre-Pólya class. An entire function ψ is said to be in the *Laguerre-Pólya class* if ψ can be expressed in the form

$$\psi(x) = cx^n e^{-\alpha x^2+\beta x} \prod_{j=1}^{\infty}(1+x/a_j)e^{-x/a_j},$$

where c, β, a_j are real, $\alpha \geq 0$, n is a nonnegative integer and $\sum a_j^{-2} < \infty$. By the classical results of Laguerre [5] and Pólya [7], ψ is in the Laguerre-Pólya class if and only if ψ can be uniformly approximated on disks about the origin by a sequence of polynomials with only real zeros. (For a modern proof of this theorem see Levin [6], Chapter 8.) Thus, it follows from this result that the class is closed under differentiation; that is, if ψ is in the Laguerre-Pólya class, then $\psi^{(n)}$ are in this class for $n \geq 0$. Moreover, any easy calculation shows that the logarithmic derivative of a function ψ in , $\psi(x) \neq ce^{ax}$, is strictly decreasing:

$$\frac{d}{dx}\left(\frac{\psi'}{\psi}(x)\right) < 0, \qquad x \in \mathbb{R}.$$

The details about the Laguerre-Pólya class are seen in [1],[4].

2.2. The function φ_k. The properties of $\varphi := \varphi_0$, which is sometimes called as an *integral of Hardy and Littlewood*, are studied in detail. For studies of this integral and the proof of the results below in (iii),(iv), see the paper [2].

(i) It is easy to check that φ is an entire and even function (i.e. $\varphi(-\xi) = \varphi(\xi)$). The restriction of φ on the real axis belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$.

(ii) The function φ satisfies the following ordinary differential equation:

$$\varphi^{(2m-1)}(\xi) - \frac{(-1)^m}{2m}\xi\varphi(\xi) = 0, \tag{2.1}$$

where $\varphi^{(k)}$ means k-times derivatives of φ.

(iii) By saddle point method, we obtain the asymptotic expansion of φ at infinity.

$$\varphi(\xi) \sim \Phi(\alpha\xi^{\frac{2m}{2m-1}}) \quad \text{as } \xi \to \infty,\ 0 < \arg\xi < \pi, \tag{2.2}$$

where $\Phi(X) = \sqrt{\frac{2\pi}{2m(2m-1)}} X^{\frac{1-m}{2m}} e^{(2m-1)X} \cdot \sum_{j=1}^{\infty} c_j X^{-j}$ ($c_j \in \mathbb{R}$ and $c_0 = 1$) and $\alpha = (2m)^{\frac{-1}{2m-1}} \cdot \exp\{(2m-3/2)\cdot\frac{\pi i}{2m-1}\}$. Note that the Stokes phenomenon occurs on the lines $\arg\xi = n\pi$ ($n \in \mathbb{Z}$).

(iv) Pólya [8] shows that all zeros of φ exist on real axis. The set of zeros of φ is denoted by $\{\pm a_j; 0 < a_j \leq a_{j+1}\,(j \in \mathbb{N})\}$. The asymptotic expansion (2.2) implies that all but finitely many zeros are simple and the asymptotic distribution of zeros of φ is the following:

$$j = c a_j^{\frac{2m}{2m-1}} + \frac{m}{2(2m-1)} + O(j^{-1}) \quad \text{as } j \to \infty, \tag{2.3}$$

where $c = \pi^{-1}\{(2m)^{\frac{-1}{2m-1}} - (2m)^{\frac{-2m}{2m-1}}\} \cos\frac{\pi}{2(2m-1)}$. Moreover the differential equation (2.1) directly implies that the order of zeros of φ is not greater than $2m-2$ (i.e. there does not exit any point $a \in \mathbb{R}$ such that $\varphi(a) = \cdots = \varphi^{(2m-2)}(a) = 0$). Therefore we have $I(\{e^{-x^{2m}}, \ldots, x^{2m-2}e^{-x^{2m}}\}) = L^1(\mathbb{R})$ by Wiener's theorem.

(v) By (2.2),(2.3), we obtain the infinite product representation:

$$\varphi(\xi) = \frac{1}{m}\Gamma\left(\frac{1}{2m}\right)\prod_{j=1}^{\infty}\left(1 - \frac{\xi^2}{a_j^2}\right) \tag{2.4}$$

The formulas (2.3),(2.4) imply that φ is in the Laguerre-Pólya class. In fact, the formula (2.3) yields $\sum a_j^{-2} < \infty$. Note that $\varphi_k(\xi) = (-i)^k\varphi^{(k)}(\xi)$ ($k \in \mathbb{N}$), then φ_k are also in this class.

神本 丈 (JOE KAMIMOTO) 熊本大学自然科学研究科

3. Proof of Theorem 1.3

First we prepare the following lemma.

Lemma 3.1. *Suppose F is in the Laguerre-Pólya class and does not take the form ce^{ax}. For any $a \in \mathbb{R}$ and $k \in \mathbb{N}$, if $F^{(k-1)}(a) \neq 0$ and $F^{(k)}(a) = 0$, then $F^{(k+1)}(a) \neq 0$.*

Proof. Now $F^{(k)}$ are also in the Laguerre-Pólya class. As mentioned in Section 2, since $F^{(k)}(\xi) \neq ce^{a\xi}$, the derivative of $F^{(k)}(\xi)/F^{(k-1)}(\xi)$ is negative for $x \in \mathbb{R}$. Thus $F^{(k+1)} \cdot F^{(k-1)} - (F^{(k)})^2$ at a is also negative, and so $F^{(k+1)}(a) \neq 0$. □

Now we will prove Theorem 1.3.

Proof of Theorem 1.3. First let us show the simpleness of zeros of φ $(= \varphi_0)$. Suppose that there is a point $a \in \mathbb{R}$ such that $\varphi(a) = \varphi'(a) = 0$. By the differential equation (2.1),

$$\varphi^{(2m-1+k)}(\xi) - \frac{(-1)^m}{2m}\{\xi\varphi^{(k)}(\xi) + k\varphi^{(k-1)}(\xi)\} = 0 \quad \text{for } k \geq 0.$$

From this equation, we have $\varphi^{(2m-1)}(a) = \varphi^{(2m)}(a) = 0$. Then Lemma 3.1 implies that $\varphi^{(2m-2)}(a)$ must be zero. In a similar fashion, we obtain $\varphi^{(2m-3)}(a) = \cdots = \varphi^{(2)}(a) = 0$. By the differential equation (2.1), this implies that φ identically equals zero. This is a contradiction.

Next suppose that there is a point $a \in \mathbb{R}$ such that $\varphi^{(k)}(a) = \varphi^{(k+1)}(a) = 0$ for $k \in \mathbb{N}$. Then Lemma 3.1 implies $\varphi(a) = \varphi'(a) = 0$. Therefore the above argument induces a contradiction. Thus all zeros of φ_k $(= (-i)^k\varphi^{(k)})$ are also simple.

The proof of Theorem 1.3 is complete. □

4. Question

Let us consider the value of $M(\phi_m)$ in more detail. In the case $r = 1$, $M(\phi_m)$ is 2, which is the best possible. But there is a room for improving the inequality $M(\phi_m) \leq 2^r$

for $r \geq 2$. Actually we can easily obtain $M(\phi_m) \leq 3 \cdot 2^{r-2}$ by differential equation (2.1). If the following question is solved positively, we obtain $M(\phi_m) = r - 1$.

Question 1. *Let $l \geq 4$ and $m \geq 1$ be integers. Do there exist nonnegative integers $k_1, \ldots, k_l$ such that the zero sets of $\psi^{(k_1)}, \ldots, \psi^{(k_l)}$ are mutually disjoint?*

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