数理解析研究所講究録
1090 巻 1999 年 50-54

A dependence domain for a class of microdifferential operators with involutive double characteristics

Yasunori Okada (Chiba University) (岡田靖則)
Nobuyuki Tose (Keio University) (戸瀬信之)

In the study of the mathematical models for conical refraction, the theory of 2-microlocal analysis plays an impotant role. We give, in this note, a sharp microlocal dependence domain for a class of microdifferential operators, which is also an application of 2-microlocal analysis.

1 Statement of the Main Theorem

Let M be a real analytic manifold with complexification X, and P a microdifferential operator defined in a neighborhood U in T^*X of a point $\dot{q} \in T_M^*X \setminus M$. We assume that the characteristic variety of P satisfies

$$\mathrm{Char}(P) \subset \{q \in U; p_1(q) = p_2(q) = 0\}$$

with homogeneous holomorphic functions p_1 and p_2 on U with the following properties:

- p_1 and p_2 are real valued on T_M^*X,
- $dp_1 \wedge dp_2 \wedge \omega_X(q) \neq 0$ if $p_1(q) = p_2(q) = 0$,
- $\{p_1, p_2\}(q) = 0$ if $p_1(q) = p_2(q) = 0$.

Here ω_X is the canonical 1-form of T^*X, and $\{\cdot, \cdot\}$ the Poisson bracket on T^*X.

In this situation, we can define regular involutive submanifolds $V^{\mathbb{C}} \subset T^*X$ and $V \subset T^*_M X$ by

$$V^{\mathbb{C}} = \{q \in U; p_1(q) = p_2(q) = 0\},$$
$$V = V^{\mathbb{C}} \cap T^*_M X,$$

and we assume, for the simplicity, that $\dot{q} \in V$. Moreover Γ denotes the canonical leaf of V passing through $\dot{q}$.

A set $K \subset \Gamma$ is called a Γ-rectangle if there exists an injective real analytic map

$$\Phi : [0,1] \times [0,1] \longrightarrow \Gamma$$

with the following three properties:

- $\Phi([0,1] \times [0,1]) = K$
- $\Phi(\cdot, t)$ is an integral curve of the Hamiltonian vector field H_{p_1} for any fixed $t \in [0,1]$.
- $\Phi(s, \cdot)$ is an integral curve of the Hamiltonian vector field H_{p_2} for any fixed $s \in [0,1]$.

We give, in this situation,

Theorem 1.1. *There exists an open neighborhood U_0 of $\dot{q}$ in Γ with the property that for any Γ-rectangle K contained in U_0 with the four vertices q_0, q_1, q_2, and q_3 and for any microfuntion solution u to $Pu = 0$ on K,*

$$q_1,\ q_2,\ q_3 \notin \mathrm{supp}(u) \Longrightarrow q_0 \notin \mathrm{supp}(u).$$

This theorem can be deduced from the model case given in the next section.

2 Theorem in the model case

Let M be an open subset of $\mathbb{R}^n$ with a complex neighborhood X in $\mathbb{C}^n$ ($n \geq 3$). We take a coordinate system of M (resp. X) as $x = (x_1, \cdots, x_n)$ (resp. $z = (z_1, \cdots, z_n)$). Then $(x; \sqrt{-1}\xi \cdot dx)$ (resp. $(z; \zeta \cdot dz)$) denotes a point in $T^*_M X$ (resp. T^*X) with $\xi = (\xi_1, \cdots, \xi_n)$ (resp. $\zeta = (\zeta_1, \cdots, \zeta_n)$).

We take a point $q_0 = (0; \sqrt{-1}dx_n) \in T^*_M X$. Let P be a microdifferential operator defined in a neighborhood of q_0 whose principal symbol is of the form

$$\zeta_1^{m_1}\zeta_2^{m_2}$$

with $m_1, m_2 \geq 1$. We define an involutive manifold V of $T^*_M X$ by

$$V = \{(x; \sqrt{-1}\xi \cdot dx); \xi_1 = \xi_2 = 0\}$$

and denote by Γ the leaf of V passing through the point q_0. We take a rectangle K on Γ defined by

$$K = \{(x_1, x_2, x'' = 0; \sqrt{-1}dx_n); 0 \leq x_1 \leq t_1, \quad 0 \leq x_2 \leq t_2\}.$$

Here $x'' = (x_3, \cdots, x_n)$. The vertices of K are denoted by

$$q_0, q_1 = (t_1, 0, 0; \sqrt{-1}dx_n), q_2 = (0, t_2, 0; \sqrt{-1}dx_n), q_3 = (t_1, t_2, 0; \sqrt{-1}dx_n).$$

Then we have

Theorem 2.1. *Let u be a microfunction defined in a neighborhood of K. We assume that u satisfies*

$$Pu = 0$$

and that the three points q_1, q_2, q_3 are not in supp(u)*:*

$$q_1, q_2, q_3 \notin \text{supp}(u).$$

Then

$$q_0 \notin \text{supp}(u).$$

Remark 2.2. The phenomenon in Theorem 2.1 was first observed by Y. Okada [O] for C^∞ wavefront set of microdistribution solutions. His result concerns with the case $m_1 = m_2 = 1$ under a Levi condition on the lower order term of P. He employed a microlocal version of Goursat problem in the complex domain.

To give an implication of Theorem 2.1, we recall a result obtained by N. Tose [T2].

Theorem 2.3. *Let u be a microfunction solution to $Pu = 0$ on an open subset U of Γ. Then there exist a family $\{b_\lambda^{(1)}\}_{\lambda\in\Lambda_1}$ of integral curves on Γ of $\partial/\partial x_1$ and another family $\{b_\lambda^{(2)}\}_{\lambda\in\Lambda_2}$ of integral curves on Γ of $\partial/\partial x_2$ which satisfy the properties that the set*

$$\bigcup_{\lambda\in\Lambda_1} b_\lambda^{(1)} \cup \bigcup_{\lambda\in\Lambda_2} b_\lambda^{(2)}$$

is included in $\mathrm{supp}(u)$ *and that* $\mathrm{supp}(u)$ *has unique continuation property on the set*

$$\Omega = U \setminus \left(\bigcup_{\lambda\in\Lambda_1} b_\lambda^{(1)} \cup \bigcup_{\lambda\in\Lambda_2} b_\lambda^{(2)} \right).$$

More precisely, if a point $q \in \Omega$ is not in $\mathrm{supp}(u)$*, then the connected component of Ω containing q is disjoint with* $\mathrm{supp}(u)$.

In the situation of Theorem 2.3, we take a point

$$\dot{q} = (s_1, s_2, x'' = 0; \sqrt{-1}dx_n) \in \Gamma.$$

We assume that, for a neighborhood U_1 of $\dot{q}$, the only one integral curve $b_{\lambda_1}^{(1)}$ of $\partial/\partial x_1$ and the only one $b_{\lambda_2}^{(2)}$ of $\partial/\partial x_2$ pass U_1. We assume, for simplicity, that the both two curves pass $\dot{q}$:

$$\dot{q} \in b_{\lambda_j}^{(j)} \quad (j = 1, 2).$$

We assume that

$$\mathrm{supp}(u) \cap U_1 \cap \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n);\ x_1 < s_1,\ x_2 > s_2\} = \emptyset$$

and that

$$\mathrm{supp}(u) \cap U_1 \cap \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n);\ x_1 > s_1,\ x_2 < s_2\} = \emptyset.$$

In this situation, if a point

$$\dot{q}' \in \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n);\ x_1 < s_1,\ x_2 < s_2\}$$

does not belong to $\mathrm{supp}(u)$, then it follows from Theorem 2.1 that

$$\mathrm{supp}(u) \cap \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n);\ x_1 > s_1,\ x_2 > s_2\} = \emptyset.$$

We also give the following decomposition proposition.

Proposition 2.4. *Let $u \in \mathcal{C}_M(K)$ be a solution to $Pu = 0$. Then there exist two microfunctions u_1 and u_2 in $\mathcal{C}_M(K)$ with the properties*

- $u = u_1 + u_2$,
- $Pu_j = 0$ $(j = 1, 2)$,
- $\mathrm{SS}^2_V(u_j) \setminus V \subset \{(x; \sqrt{-1}\xi'' \cdot dx''; \sqrt{-1}(x_1^* \cdot dx_1 + x_2^* \cdot dx_2)); x_j^* = 0\}$.

Here $\mathrm{SS}^2_V(\cdot)$ is the second singular spectrum along V. Using this decomposition, we can prove the Theorem 2.1, but we omit the details.

References

[K-L] Kashiwara, M. and Y. Laurent, Théorèm d'annulation et deuxième microlocalisation, Prépublication d'Orsay, 1983.

[K-O-T] Kataoka, K., Y. Okada and N. Tose, Decomposition of second microlocal singularities, "Microlocal Geometry" (ed. by T. Monteiro-Fernandes and P. Schapira), 1990.

[K-S] Kashiwara, M. and P. Schapira, Sheaves on manifolds, Grundlehren der Math., Springer, 1990.

[O] Okada, Y., On distribution solutions of microdifferential equations with double involutive characteristics, preprint.

[O-T] Okada, Y. and N. Tose, FBI transformation and second microlcoalization, J. de math. pures et appl. 70(4) (1991), 427-455.

[T1] Tose, N., On a class of microdifferential equations with involutive characteristics – as an application of second microlocalization, J. Fac. Sci., Univ. of Tokyo, Sect. IA, Math. 33(1986), 619-634.

[T2] —–, On a class of 2-microhyperbolic systems, J. de Math. pures et appl. 67 (1988), 1–15.