

## SOLVABILITY OF A CLASS OF DIFFERENTIAL OPERATORS IN $\mathcal{CO}$

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### Introduction

Let  $V$  and  $\Sigma$  be an involutive submanifold and a lagrangian submanifold of  $\sqrt{-1}T^*\mathbb{R}^n$  respectively given as follows:

$$V = \{(x; i\xi) \in \sqrt{-1}T^*\mathbb{R}^n; \xi_1 = \cdots = \xi_{n-1} = 0\}, \quad (1)$$

$$\Sigma = \{(x; i\xi) \in V; x_n = 0\}. \quad (2)$$

In [3], Grigis-Schapira-Sjöstrand obtained a result on the propagation of micro-analyticity of solutions along  $\Sigma$  for transversally elliptic operators  $P$ ; that is, the principal symbol  $\sigma(P)$  of  $P$  satisfies

$$|\sigma(P)(x, i\eta)| \sim (|\eta'| + |x_n| \cdot |\eta_n|)^\ell \quad \text{near } \Sigma, \quad (3)$$

where  $\ell$  is some positive integer and  $\eta' = (\eta_1, \dots, \eta_{n-1})$ . On the other hand, by using an elementary functorial construction of the sheaf  $\tilde{\mathcal{C}}_V^2$  of small second microfunctions, the first author Funakoshi proved in [2] the solvability of those operators in the space of small second microfunctions as follows:

**Theorem 1.** *Let  $P(x, \partial_x)$  be a differential operator with real analytic coefficients defined at  $x = 0$ . We suppose that*

$$|\sigma(P)(x, i\eta)| \sim (|\eta'| + |x_n|^k \cdot |\eta_n|)^\ell \quad \text{near } \Sigma \quad (4)$$

for some positive integers  $k, \ell$ . Then we have a sheaf isomorphism:

$$\tilde{\mathcal{C}}_V^2 \xrightarrow{P} \tilde{\mathcal{C}}_V^2 \quad \text{on } \overset{\circ}{\pi}^{-1}(\Sigma).$$

Here,  $\tilde{\mathcal{C}}_V^2$  is called the sheaf on  $T_V^* \tilde{V}$  of small second microfunctions along  $V$ , which satisfies the following exact sequence:

$$0 \longrightarrow \mathcal{A}_V^2 \longrightarrow \mathcal{C}_{\mathbb{R}^n}|_V \longrightarrow \overset{\circ}{\pi}_* \tilde{\mathcal{C}}_V^2 \longrightarrow 0, \quad (5)$$

where  $\overset{\circ}{\pi} : T_V^* \tilde{V} = T_V^* \tilde{V} \setminus V \rightarrow V$  is the canonical projection,  $\tilde{V}$  is the partial complexification of  $V$  along each leaf of  $V$ , and

$$\mathcal{A}_V^2 := \mathcal{C}_{\tilde{V}}|_V = \mathcal{C}_{x_n} \mathcal{O}_{z'}|_V \quad (6)$$

is the sheaf on  $V$  of second analytic functions along  $V$ . Since any section of  $\mathcal{A}_V^2$  has a unique continuation property along each leaf of  $V$ , Theorem 1 implies the above-mentioned result of Grigis-Schapira-Sjöstrand. However, to get a solvability result in microfunctions, Theorem 1 is not sufficient. We need a solvability result in  $\mathcal{A}_V^2$ . Though we have a general result due to Bony and Schapira [1] on solvability in  $\mathcal{C}_{\tilde{V}}$  for non-micro-characteristic operators, our operators as in (4) do not fall in such a class of operators.

In this paper, we introduce some special class of differential operators satisfying the property (4), which admit the solvability in  $\mathcal{A}_V^2|_\Sigma = \mathcal{C}_{\tilde{V}}|_\Sigma = \mathcal{C}_{x_n} \mathcal{O}_{z'}|_\Sigma$ .

## Our Main Results

**Theorem 2.** *Let  $P(z, \partial_z)$  be a holomorphic differential operator written in the form:*

$$P(z, \partial_z) = \sum_{|\alpha|+\beta=m} C_{\alpha,\beta} \partial_{z'}^\alpha (z_n \partial_{z_n})^\beta. \quad (7)$$

Here the  $C_{\alpha,\beta}$ 's are complex constants satisfying

$$C_{0,m} \neq 0. \quad (8)$$

Then, the morphism

$$P : \mathcal{C}_{x_n} \mathcal{O}_{z'} \rightarrow \mathcal{C}_{x_n} \mathcal{O}_{z'}$$

is surjective on  $\{x_n = 0\}$ .

As a direct corollary of Theorem 1 and Theorem 2, we have

### Theorem 3.

Let  $P(z, \partial_z)$  be a holomorphic differential operator written as in (7). Suppose that

$$\left| \sum_{|\alpha|+\beta=m} C_{\alpha,\beta} (\eta')^\alpha (x_n)^\beta \right| \sim (|\eta'| + |x_n|)^m \quad (9)$$

for any real small vectors  $(\eta', x_n)$ . Then the morphism

$$P : \mathcal{C}_{\mathbb{R}^n} \rightarrow \mathcal{C}_{\mathbb{R}^n}$$

is surjective on  $\Sigma$ .

Indeed, condition (9) implies condition (8).

### A Sketch of Proof of Theorem 2

Any germ  $f$  of  $\mathcal{C}_{x_n} \mathcal{O}_{z'}$  at  $(0, 0; id_{x_n}) \in \Sigma$  is written as a boundary value  $F(z', x_n + i0)$  of a holomorphic function  $F(z)$  in a domain

$$D_r = \{z \in \mathbb{C}^n; |z'| < r, |z_n| < r, \text{Im}z_n > 0\}. \quad (10)$$

Hence, our problem reduces to finding a holomorphic solution  $U$  of the following equation for any given  $F(z)$  in a complex domain like  $D_r$ :

$$P(z, \partial_z)U(z) = F(z). \quad (11)$$

**Step 1.** Considering the Szegő kernel for a complex ball, we have a decomposition of  $F(z)$  for a sufficiently small  $r > 0$ :

$$F(z) = \text{Const.} \int_{|w'|=r} \frac{F(w', z_n)}{(r^2 - z' \cdot \bar{w}')^n} dS(w'), \quad (12)$$

where  $z' \cdot \bar{w}' = \sum_{j=1}^{n-1} z_j \bar{w}_j$  and  $dS$  is the surface element. Hence, it is sufficient to solve (11) for any holomorphic function  $F$  with continuous parameter  $w'$  of the following form:

$$F = G(z' \cdot \bar{w}', z_n; w'), \quad (13)$$

where  $G(p, z_n; w')$  is a continuous function on

$$\{(p, z_n, w') \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1}; |p| < r, |z_n| < r, \text{Im}z_n > 0, |w'| = r\} \quad (14)$$

depending holomorphically on  $(p, z_n)$ . Therefore, equation (11) reduces to the following one:

$$C_{0,m} \prod_{j=1}^m (z_n \partial_{z_n} - \varphi_j(w') \partial_p) \cdot U(p, z_n; w') = G(p, z_n; w'), \quad (15)$$

where  $\{\varphi_j(w'); j = 1, \dots, m\}$  are the  $m$ -solutions of the algebraic equation

$$\sum_{|\alpha|+\beta=m} C_{\alpha,\beta} \bar{w}'^\alpha \phi^\beta = 0 \quad (16)$$

in  $\phi$ .

**Step 2.** By solving first order equations in (15) successively, we get the final solution of (15). Hence, our problem is to solve the following first order equation:

$$(z_n \partial_{z_n} - \varphi_j(w') \partial_p) U(p, z_n; w') = G(p, z_n; w'). \quad (17)$$

In fact, if  $\varphi_j(w') \neq 0$ , we have a holomorphic solution of (17) of the form

$$U(p, z_n; w') = -\frac{1}{\varphi_j(w')} \int_{\tau(w')}^p G(s, z_n e^{(p-s)/\varphi_j(w')}; w') ds. \quad (18)$$

However this solution is not holomorphic in a domain like (14). To get a solution defined in a domain like (14), we must decompose  $G$  as

$$G(p, z_n; w') = G_+(p, z_n; w') + G_-(p, z_n; w'). \quad (19)$$

Here, roughly speaking,  $G_+$  is holomorphic in

$$0 < \arg z_n < \pi + \epsilon$$

and  $G_-$  is holomorphic in

$$-\epsilon < \arg z_n < \pi$$

for some  $\epsilon > 0$ . Indeed taking  $\tau_{\pm}(w')$  in the formula (18) as

$$0 < \mp \operatorname{Im} \left( \frac{\tau_{\pm}(w')}{\varphi_j(w')} \right) < \epsilon \quad (20)$$

respectively, we can show that the corresponding solutions  $U_{\pm}$  are holomorphic in a domain like (14). The most difficult point of our problem is how to treat the case  $\varphi_j(w') = 0$ . This is not an exceptional problem because for almost all operators  $P$  the sets

$$\begin{aligned} & \{w' \in \mathbb{C}^{n-1}; |w'| = 1, \varphi_j(w') = 0 \text{ for some } j\} \\ & = \{w' \in \mathbb{C}^{n-1}; |w'| = 1, \sum_{|\alpha|=m} C_{\alpha,0} \bar{w}'^{\alpha} = 0\} \end{aligned}$$

are not void (but usually of real codimension  $\geq 1$ ). To overcome this difficulty, we use a good decomposition of  $G$  in (19) based on Hörmander's solution with  $L^2$ -growth order for a  $\bar{\partial}$ -equation in the whole  $\mathbb{C}$ . Before making such a decomposition we choose a better defining function  $G(p, z_n : w')$ . That is, by solving a Cousin problem on  $\mathbb{C} \times \mathbb{P}^1$  with parameter  $w'$ , we can choose a better defining function  $G(p, z_n : w')$ , which is holomorphic on

$$\{p \in \mathbb{C}; |p| < r\} \times \{z_n \in \mathbb{P}^1; \text{Im}z_n > 0 \text{ or } |z_n| > r\} \quad (21)$$

satisfying

$$G(p, \infty; w') = 0. \quad (22)$$

Here neglecting variables  $p, w'$  we consider a holomorphic function

$$H(\tau) = G(p, e^\tau; w') \quad (23)$$

in  $\tau$  defined on

$$\{\tau \in \mathbb{C}; 0 < \text{Im}\tau < \pi\}$$

with a growth order

$$|H(\tau)| < Ce^{-\text{Re}\tau}$$

as  $\text{Re}\tau$  goes to  $+\infty$ . Now we apply Hörmander's Theorem to the decomposition of  $H$ .

### Hörmander's Theorem.

*Let  $\varphi(\tau)$  be a subharmonic function on  $\mathbb{C}$ . Then for any measurable function  $h(\tau)$  satisfying*

$$\iint_{\mathbb{C}} |h(\tau)|^2 e^{-\varphi(\tau)} dv(\tau) < \infty \quad (24)$$

*we have a weak solution  $f(\tau)$  of*

$$\frac{\partial}{\partial \bar{\tau}} f(\tau) = h(\tau) \quad (25)$$

*satisfying*

$$\iint_{\mathbb{C}} |f(\tau)|^2 e^{-\varphi(\tau) - 2 \log(|\tau|^2 + 1)} dv(\tau) < \infty. \quad (26)$$

The detailed proof will be published elsewhere.

## REFERENCES

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