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指標付き約数問題
(Divisor Problem with Characters)

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Let χ be a Dirichlet character mod k and let $r_a(n,\chi)$ be the function defined by

$$r_a(n,\chi)=\sum_{d|n}\chi(d)d^a,$$

where a is a fixed real number. When χ is identically 1, this function is a classical divisor function usually written by $\sigma_a(n)$. On the other hand, when $a=0$ and χ is the Kronecker symbol corresponding to $\mathbb{Q}(i)$, then

$$r_a(n,\chi)=\frac{1}{4}r(n).$$

with $r(n)=\sharp\{(x,y)\in\mathbb{Z}^2|\,x^2+y^2=n\}$. Hence we can also consider $r_a(n,\chi)$ as a generalization of the counting function of the lattice points on a circle. We shall consider the sum of $r_a(n,\chi)$ and the mean square of its error term.

Before stating our results, we shall recall some known results about the sum of $\sigma_a(n)$. Put

$$\Delta_0(x)=\sum_{n\le x}{}'\sigma_0(n)-x(\log x+2\gamma-1)-1/4,$$

where γ is Euler's constant and the prime on the summation means that the last term is to be halved if x is an integer. In 1956, Tong proved that

$$\int_1^X\Delta_0(x)^2dx=\frac{1}{6\pi^2}\frac{\zeta^4(3/2)}{\zeta(3)}X^{3/2}+O(X\log^5 X).$$

and the O-term was improved to $O(X\log^4 X)$ by Preissmann in 1988. For $-1<a<0$, we define

$$\Delta_a(x)=\sum_{n\le x}{}'\sigma_a(n)-\zeta(1-a)x-\frac{\zeta(1+a)}{1+a}x^{1+a}+\frac{1}{2}\zeta(-a).$$

The mean square formula of $\Delta_a(x)$ for $-1/2<a<0$ was first considered by Kiuchi, and improved by Meurman. In fact, Meurman [2] proved that

$$\int_1^X|\Delta_a(x)|^2dx$$
$$=\begin{cases}\dfrac{\zeta(3/2-a)\zeta(3/2+a)\zeta^2(3/2)}{(6+4a)\pi^2\zeta(3)}X^{3/2+a}+O(X) & \text{for } -1/2<a<0\\ \dfrac{\zeta^2(3/2)}{24\zeta(3)}X\log X+O(X) & \text{for } a=-1/2\\ O(X) & \text{for } -1<a<-1/2\end{cases}$$

For the case $-1<a<-1/2$, the more precise formula had already been obtained by S. Chowla in 1932. In [1], he showed that

$$\int_1^X|\Delta_a(x)|^2dx=\frac{\zeta^2(1-a)\zeta(-2a)}{12\zeta(2-2a)}X+O(X^{3/2+a}\log X)$$

for $-1<a<-1/2$. Recently, the last formula was obtained independently by Yanagisawa in a somewhat general situation in [3]. He proved that

$$\int_1^X\Delta_a(\frac{h}{k}x)\Delta_b(x)dx=\frac{1}{2\pi^2kh}\Bigl(\sum_{n=1}^{\infty}\frac{\sigma_{1+a}(hn)\sigma_{1+b}(kn)}{n^2}\Bigr)X$$
$$+O(X^{(3+a+b)/2}\log X).$$

for $-1<a<0,-1<b<0,a+b<-1$ and $h>0,k>0,(h,k)=1$.

The aim of this note is to show the corresponding mean square formula for $r_a(n,\chi)$. We assume throughout that χ is a non-trivial Dirichlet character mod k. In our case, the error term of the sum function of $r_a(n,\chi)$ is defined by

$$\Delta_a(x,\chi)=\sum_{n\le x}{}'r_a(n,\chi)-L(1-a,\chi)x+\frac{1}{2}L(-a,\chi).$$

Theorem 1. *Let $-1 < a,\ b < 0$ be real numbers and let $k \ll \sqrt{X}$. Suppose that χ_1 and χ_2 are primitive Dirichlet characters* mod k *with the same parity.*

(i) *For $a+b > -1$, we have*

$$\int_1^X \Delta_a(x,\chi_1)\Delta_b(x,\chi_2)dx$$
$$= C_1 X^{(3+a+b)/2} + O(\min(k^2X, kX(\log X)^2)) + O(k^{\max(\frac{5}{4}+\frac{a+b}{2},1)}X),$$

where

$$C_1 = \frac{\tau(\chi_1)\tau(\chi_2)k^{(-1+a+b)/2}}{2\pi^2(3+a+b)} \times \frac{\zeta(\frac{3-a-b}{2})L(\frac{3+a-b}{2},\bar{\chi}_1)L(\frac{3-a+b}{2},\bar{\chi}_2)L(\frac{3+a+b}{2},\bar{\chi}_1\bar{\chi}_2)}{L(3,\bar{\chi}_1\bar{\chi}_2)},$$

and

$$\tau(\chi) = \sum_{n=1}^{k-1}\chi(n)e^{2\pi in/k}.$$

(ii) *For $a+b=-1$, we have*

$$\int_1^X \Delta_a(x,\chi_1)\Delta_{-1-a}(x,\chi_2)dx$$
$$= \begin{cases} C_2 X\log X + O(\min(k^2X, kX(\log X)^2)) & \text{if } \chi_2 = \bar{\chi}_1 \\ O(\min(k^2X, kX(\log X)^2)) & \text{otherwise} \end{cases}$$

with

$$C_2 = \frac{\chi_1(-1)L(2+a,\bar{\chi}_1)L(1-a,\chi_1)}{24\zeta(3)}\prod_{p|k}\frac{p^2}{p^2+p+1}.$$

Theorem 2. *Let $-1 < a,\ b < 0,\ a+b \geq -1$ and $k \ll \sqrt{X}$. Suppose that χ_1 and χ_2 are primitive Dirichlet characters* mod k *with the opposite parity. Then we have*

$$\int_1^X \Delta_a(x,\chi_1)\Delta_b(x,\chi_2)dx$$
$$= O(\min(k^2X, kX(\log X)^2)) + O(k^{\max(\frac{5}{4}+\frac{a+b}{2},1)}X).$$

For the case $a+b<-1$, we have the following theorem.

Theorem 3. *Let $-1<a,\ b<0,\ a+b<-1$ and $k \ll \sqrt{X}$. Let χ_1 and χ_2 be non-trivial Dirichlet characters* mod k. *Then we have*

$$\int_1^X \Delta_a(x,\chi_1)\Delta_b(x,\chi_2)dx = C_3X + O(k^2\log kX^{(3+a+b)/2}\log X)$$

where

$$C_3 = \frac{L(1-a,\chi_1)L(1-b,\chi_2)L(-a-b,\chi_1\chi_2)}{12L(2-a-b,\chi_1\chi_2)}.$$

In these theorems, the O-constants do not depend on the modulus k.

Outline of the proof of Theorems 1 and 2.

For the proof of Theorems 1 and 2, we need the Voronoï formula for $\Delta_a(x,\chi)$. Let $\phi(s)$ be the generating function of $r_a(n,\chi)$. It is easily seen that $\phi(s)=\zeta(s)L(s-a,\chi)$. Let

$$\delta_\chi = \begin{cases} 0 & \text{if } \chi \text{ is an even character,} \\ 1 & \text{if } \chi \text{ is an odd character,} \end{cases}$$

and $W(\chi) = (-1)^{\delta_\chi}\tau(\chi)/\sqrt{k}$. Then the functional equation of $\phi(s)$ is given by

$$\phi(s) = W(\chi)k^{1/2+a-s}\pi^{2s-(1+a)}\frac{\Gamma((1-s)/2)\Gamma((1+a-s+\delta_\chi)/2)}{\Gamma(s/2)\Gamma((s-a+\delta_\chi)/2)}\tilde{\phi}(1+a-s)$$

where

$$\tilde{\phi}(s) = \zeta(s-a)L(s,\bar{\chi}).$$

The coefficients of $\tilde{\phi}(s)$ are given by

$$\tilde{r}_a(n,\chi) = \sum_{d|n}\bar{\chi}(\frac{n}{d})d^a.$$

This function plays an important role in Voronoï formula.

First we assume that $-1/2 < a < 0$. We have

$$
\begin{aligned}
&\Delta_a(x,\chi)\\
&= -W(\chi)k^{a/2}x^{(a+1)/2}\sum_{n=1}^{\infty}\frac{\tilde{r}_a(n,\chi)}{n^{(a+1)/2}}\Big\{\cos\frac{\pi(a-\delta_\chi)}{2}Y_{1+a}(4\pi\sqrt{nx/k})\\
&\quad +\sin\frac{\pi(a-\delta_\chi)}{2}J_{1+a}(4\pi\sqrt{nx/k})+\frac{2}{\pi}\cos\frac{\pi(a-\delta_\chi)}{2}K_{1+a}(4\pi\sqrt{nx/k})\Big\}\\
&= \frac{W(\chi)}{\sqrt{2}\pi}k^{a/2+1/4}x^{a/2+1/4}\sum_{n=1}^{\infty}\frac{\tilde{r}_a(n,\chi)}{n^{3/4+a/2}}\cos(4\pi\sqrt{nx/k}-(1/4+\delta_\chi/2)\pi)\\
&\quad -\frac{4a^2+8a+3}{32\sqrt{2}\pi^2}W(\chi)k^{a/2+3/4}x^{a/2-1/4}\sum_{n=1}^{\infty}\frac{\tilde{r}_a(n,\chi)}{n^{5/4+a/2}}\\
&\quad \times\cos(4\pi\sqrt{nx/k}-(3/4+\delta_\chi/2)\pi)+O(k^{a/2+5/4}x^{a/2-3/4}).
\end{aligned}
$$

Here, $Y_{1+a}(x)$, $J_{1+a}(x)$ and $K_{1+a}(x)$ denote the standard Bessel functions of order $1+a$. We also need the sum formula of $\tilde{r}_a(n,\chi)$. It is given by

$$
\sum_{n\le x}{}'\,\tilde{r}_a(n,\chi)=\frac{L(1+a,\bar{\chi})}{1+a}x^{1+a}+\zeta(-a)L(0,\bar{\chi})+\tilde{\Delta}_a(n,\chi)
$$

with

$$
\begin{aligned}
&\tilde{\Delta}_a(x,\chi)\\
&= -\overline{W(\chi)}k^{-a/2}x^{(a+1)/2}\sum_{n=1}^{\infty}\frac{r_a(n,\chi)}{n^{(a+1)/2}}\Big\{\cos\frac{\pi(a-\delta_\chi)}{2}Y_{1+a}(4\pi\sqrt{nx/k})\\
&\quad +\sin\frac{\pi(a-\delta_\chi)}{2}J_{1+a}(4\pi\sqrt{nx/k})+\frac{2}{\pi}\cos\frac{\pi(a+\delta_\chi)}{2}K_{1+a}(4\pi\sqrt{nx/k})\Big\}\\
&= \frac{\overline{W(\chi)}}{\sqrt{2}\pi}k^{-a/2+1/4}x^{a/2+1/4}\sum_{n=1}^{\infty}\frac{r_a(n,\chi)}{n^{3/4+a/2}}\cos(4\pi\sqrt{nx/k}-(1/4+\delta_\chi/2)\pi)\\
&\quad -\frac{4a^2+8a+3}{32\sqrt{2}\pi^2}\overline{W(\chi)}k^{-a/2+3/4}x^{a/2-1/4}\sum_{n=1}^{\infty}\frac{r_a(n,\chi)}{n^{5/4+a/2}}\\
&\quad \times\cos(4\pi\sqrt{nx/k}-(3/4+\delta_\chi/2)\pi)+O(k^{-a/2+5/4}x^{a/2-3/4}).
\end{aligned}
$$

Following Meurman, we get the truncated expression of $\Delta_a(x,\chi)$ from the above formulas valid for $-1 < a < 0$.

Lemma 1. *For $-1 < a < 0, y \geq 1, X \geq y, Z \geq 2y$ and $y \notin \mathbb{Z}$, we have*

$$\begin{aligned}\Delta_a(y,\chi) &= \Delta_a(y,X,\chi) + R_a(y,X,Z)\\ &\quad +O(k^{1+\varepsilon}y^{-1/4+a/2} + k^{3/4+a/2}y^{-1/2} + k^{5/4+a/2}y^{-3/4+a/2}),\end{aligned}$$

where

$$\begin{aligned}\Delta_a(y,X,\chi) &= \frac{W(\chi)}{\sqrt{2}\pi}k^{a/2+1/4}y^{a/2+1/4}\\ &\quad \times \int_1^2 \sum_{n\leq uX} \frac{\tilde{r}_a(n,\chi)}{n^{3/4+a/2}}\cos(4\pi\sqrt{ny/k} - (1/4+\delta_\chi/2)\pi)du\end{aligned}$$

and

$$\begin{aligned}&R_a(y,X,Z)\\ &= cW(\chi)\sum_{n\leq Z} r_a(n,\chi)\int_1^2\int_{uX}^{\infty} t^{-1}\sin(\frac{4\pi}{\sqrt{k}}(\sqrt{y}-\sqrt{n})\sqrt{t})dtdu\end{aligned}$$

with some constant c which is independent on k.

From this Lemma , we get Theorems 1 and 2. Note that if χ_1 and χ_2 have the opposite parity, there occurs no main term.

Outline of the proof of Theorem 3

In this case the Voronoï formula does not work well. We use the Chowla-Walum's type formula instead. Namely we have

Lemma 2.

$$\Delta_a(x,\chi) = -\sum_{m\leq\sqrt{x}}\chi(m)m^a\psi(\frac{x}{m}) - x^a\sum_{n\leq\sqrt{x}} n^{-a}P(\frac{x}{m}) + O(k^{3/2}(\log k)x^{a/2})$$

where

$$\begin{aligned}\psi(x) &= x - [x] - 1/2\\ P(x) &= -\sum_{n\leq x}\chi(n) - \frac{1}{k}\sum_{n\leq k}\chi(n)n.\end{aligned}$$

The main term comes from the product of the first sums of $\Delta_a(x,\chi_1)$ and $\Delta_b(x,\chi_2)$. The remaining products give the error term. To show this, we need Yanagisawa's main lemma.

Lemma 3. *Let $f(t)$ and $g(t)$ be piecewise continuous functions of period A and of bounded variations. Suppose that*

$$|f(t)| \leq F, \qquad |g(t)| \leq G,$$

and

$$\int_0^A f(t)dt = 0.$$

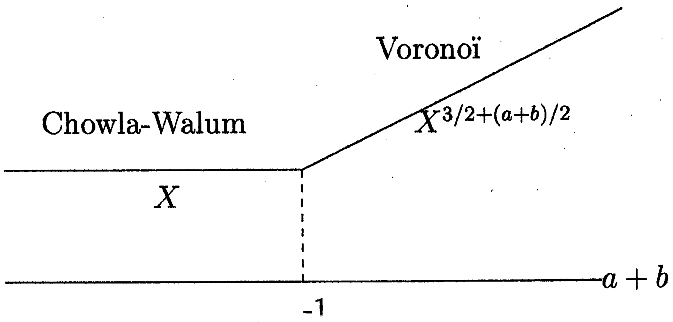
Then, for any $n \leq \sqrt{X}$ and any sequence of points $\{X_m\}$, and $\{Y_m\}$ with $0 < Y_m - X_m \ll X$, we have

$$\sum_{m\leq\sqrt{X}} \left|\int_{X_m}^{Y_m} f(\frac{x}{m})g(\frac{x}{n})dx\right| \ll GX\log X\bigl(F(A+\sigma_{-1}(n))+V_f\bigr)$$

where V_f is the total variation of f in $[0,A]$.

The above summation is estimated as $\sigma_{-1}(n)X\log X$ in [3], but it is not sufficient for our purpose. So we made the dependence on the modulus k explicit in Lemma 3. Note that χ_1 and χ_2 do not need to be primitive characters, because we don't use functional equation in this case.

As in the theorems of Meurman and Chowla-Yanagisawa, the mean square formula is effectively deduced by Voronoï formula when $a+b \geq -1$ and by Chowla-Walum formula when $a+b < -1$. This phenomenon can be expressed symbolically as



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