数理解析研究所講究録
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ここに掲載された論考は、1997年10月1日逝去された故三井孝美先生の御遺稿です。同じ年の暮、整然と整理された研究室の書棚に、幾度も推敲を重ねたと思われる何編もの下書きと共に、ほぼ完成された原稿として遺されていました。内容は、Tokyo J. Math. に発表されたいくつかの論文の続編であり、病気快癒の後に投稿されるお積もりであったと推察されます。

三井先生は、常に解析数論研究の中心に位置され、この分野の研究者から厚い人望と尊敬を集め、リーダー的存在として研究の発展、後進の育成に尽くされました。先生が、その研究生活を始めた当時は、日本の解析数論の黎明記で研究者の数も少なく、1969年に数理解析研究所で開かれたこの分野で最初の研究集会は、雑誌“数学”誌上に研究代表者 竜沢周雄教授によって『小グループの集会』と紹介される程でした。この集会の講演者の一人であった先生は、以後、日本の解析数論の第一人者として多くの研究集会に関わりました。現在、解析数論はその研究者を増やし活発に研究されるに至りましたが、その発展に先生が果たした役割の大きいことは、たとえば、数理解析研究所での三度の研究集会で研究代表者を務められたことからも窺われます。

今回、山梨大学の中井喜信教授のご尽力により、本稿が同研究所講究録に掲載され先生のご研究が公となり、原稿発見者として、また数論研究者の一人として深い喜びと安堵を感じます。

1999年1月
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註；§3以降，§の No.，Theorem の No.，式 No. の(*.*) に，一部ずれている所がありますが，そのままにしてあります。あるいは，そこを修正される御積りであったのかも知れません。
中井喜信

On Certain Multiple Series with Functional Equation in a Totally Imaginary Number Field II

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§1. Introduction

In the former paper [3] we treated the series of the type

$$\sum_{0\neq(\mu)\subset\mathfrak{a}}\ \sum_{0\neq\nu\in\mathfrak{b}} \frac{1}{N(\mu)^{1/2}} exp\{-2\pi S(|\mu\nu|\tau)+2\pi i S(\mu\nu\xi)\}.$$

In this paper, we shall consider a multiple series of more general type (see (1.6) below) and prove that it satisfies a transformation formula.

Before stating our main result, however, we shall have to explain some definitions and notations.

Let ϕ_a be mappings from N^a to N $(a=1,\ldots,k)$, where N is the set of positive rational integers, and $\phi_0=2l$ an even number. Let $\mathcal{J}$ be the set of $(k+1)$-tuples $J=(j_1,\ldots,j_{k+1})$ of positive integers satisfying the conditions

$$\begin{cases} 1\leq j_1\leq \phi_0=2l, \\ 1\leq j_2\leq \phi_1(j_1), \\ 1\leq j_3\leq \phi_2(j_1,j_2), \\ \quad\cdots \\ 1\leq j_{k+1}\leq \phi_k(j_1,\ldots,j_k). \end{cases}$$

In particular, we denote by I the element of $\mathcal{J}$ such that

$$j_1=j_2=\cdots=j_{k+1}=1.$$

For each $J \in \mathcal{J}$ we put

$$P_J = l \prod_{a=1}^{k} \phi_a(j_1, \ldots, j_a),$$

$$\phi(J) = P_J/(P_I, P_J), \tag{1.1}$$

$$\alpha_J^{(q)} = \sum_{a=0}^{q} \frac{j_{a+1} - 1}{\phi_a(j_1, \ldots, j_a) \cdots \phi_q(j_1, \ldots, j_q)} \qquad (q = 0, \ldots, k) \tag{1.2}$$

and

$$\alpha_J^{*(q)} = \sum_{a=0}^{q} \frac{\phi_a(j_1, \ldots, j_a) - j_{a+1}}{\phi_a(j_1, \ldots, j_a) \cdots \phi_q(j_1, \ldots, j_q)} \qquad (q = 0, \ldots, k). \tag{1.3}$$

For the case $q = k$, we write

$$\alpha_J = \alpha_J^{(k)}, \qquad \alpha_J^* = \alpha_J^{*(k)}.$$

We note that

$$\alpha_I = 0, \qquad \alpha_J,\ \alpha_J^* \geq 0.$$

Let K be a totally imaginary number field of degree $n = 2r, K^{(p)}, K^{(r+p)} = \overline{K^{(p)}}$ $(p = 1, \ldots, r)$ the pairs of the complex conjugates of K. If μ is a number of K, then we denote by $\mu^{(q)}$ the conjugates of μ in $K^{(q)}$ $(q = 1, \ldots, n)$. We define n-dimensional vector $\mu = (\mu^{(1)}, \ldots, \mu^{(n)})$. More generally, we shall use n-dimensional complex vector $\xi = (\xi_1, \ldots, \xi_n)$ such that $\xi_{p+r} = \overline{\xi}_p$ $(p = 1, \ldots, r)$ and write

$$S(\xi) = \sum_{q=1}^{n} \xi_q, \qquad N(\xi) = \prod_{q=1}^{n} \xi_q.$$

Let U be the unit group of K. We denote by U^k the subgroup of U consisting of the k-th powers of the elements of U. The non-zero numbers α, β are said to be associated to each other with respect to U^k if α/β is a number of U^k. Let $\mathfrak{d}$ be the differente ideal of

K, $N(\mathfrak{d}) = D$ the absolute value of the discriminant of K, and R the regulator of K. For non-zero ideal $\mathfrak{a}$ we put $\mathfrak{a}^* = (\mathfrak{a}\mathfrak{d})^{-1}$.

To each J in $\mathcal{J}$ we associate non-zero fractional ideal $\mathfrak{a}_J$. Let $\tau_1, \ldots, \tau_n$ be positive numbers such that $\tau_{p+r} = \tau_p$ $(p = 1, \ldots, r)$, $\xi_1, \ldots, \xi_n$ complex numbers such that $\xi_{p+r} = \overline{\xi}_p$ $(p = 1, \ldots, r)$.

We shall often use the multiple summations and products written as follows:

$$\sum_{j_1=1}^{\phi_0} \sum_{j_2=1}^{\phi_1(j_1)} \cdots \sum_{j_{q+1}=1}^{\phi_q(j_1,\ldots,j_q)}, \quad \prod_{j_1=1}^{\phi_0} \prod_{j_2=1}^{\phi_1(j_1)} \cdots \prod_{j_{q+1}=1}^{\phi_q(j_1,\ldots,j_q)} \qquad (q = 0, \ldots, k).$$

For the sake of simplicity, we shall denote them by

$$\sum_J^{(q)}, \quad \prod_J^{(q)},$$

respectively. For the case $q = k$, we write

$$\sum_J = \sum_J^{(k)} \quad \Big(= \sum_{J \in \mathcal{J}}\Big), \qquad \prod_J = \prod_J^{(k)} \quad \Big(= \prod_{J \in \mathcal{J}}\Big).$$

Using the notations above, we put

$$M_a = \prod_J^{(a-1)} (\phi_a)^{1/(\phi_0 \cdots \phi_{a-1})} \qquad (a = 1, \ldots, k) \tag{1.4}$$

and

$$P = l M_1 \cdots M_k. \tag{1.5}$$

Now we define the series as follows:

$$\begin{aligned} M(\tau, \xi; \alpha_J, \mathfrak{a}_J) &= M(\tau_1, \ldots, \tau_r, \xi_1, \ldots, \xi_r; \{\alpha_J\}_J, \{\mathfrak{a}_J\}_J) \\ &= \sum_{\substack{0 \neq \nu_J \in \mathfrak{a}_J \\ \nu_J / U^{\phi(J)} \\ (J \neq I)}} \cdots \sum \sum_{0 \neq \nu_I \in \mathfrak{a}_I} \prod_J N(\nu_J)^{-\alpha_J} \\ &\quad \times \exp\Big\{-2P\pi S\Big(\tau \prod_J |\nu_J|^{1/P_J}\Big) + 2P\pi i S\Big(\xi \prod_J \nu_J^{1/P_J}\Big)\Big\}, \end{aligned} \tag{1.6}$$

where the multiple sum is taken over $\{\nu_J\}_{J\in\mathcal{J}}$; for each J $(\neq I)$, ν_J runs through non-zero numbers in $\mathfrak{a}_J$ not associated to each other with respect to $U^{\phi(J)}$, and ν_I runs through all non-zero numbers in $\mathfrak{a}_I$.

We shall show that the series (1.6) is well-defined. We replace each ν_J $(J \neq I)$ by $\nu_J\varepsilon_J^{\phi(J)}$ $(\varepsilon_J \in U)$. Then

$$N(\nu_J\varepsilon_J^{\phi(J)}) = N(\nu_J)$$

and

$$\prod_J (\nu_J\varepsilon_J^{\phi(J)})^{1/P_J} = \nu_I^{1/P_I} \prod_{J\neq I} (\nu_J^{1/P_J}\varepsilon_J^{\phi(J)/P_J}).$$

By (1.1),

$$\frac{\phi(J)}{P_J} = \frac{1}{P_I}\frac{P_I}{(P_I,P_J)},$$

so

$$\prod_J (\nu_J\varepsilon_J^{\phi(J)})^{1/P_J} = \prod_{J\neq I} \nu_J^{1/P_J}(\nu_I \prod_{J\neq I} \varepsilon_J^{P_I/(P_I,P_J)})^{1/P_I}.$$

This shows that

$$\nu_I \prod_{J\neq I} \varepsilon_J^{P_I/(P_I,P_J)}$$

runs through all non-zero elements of $\mathfrak{a}_I$ with ν_I. Hence the sum over ν_I remains unchanged.

Let $\varepsilon_1, \ldots, \varepsilon_{r-1}$ be the fundamental units of K We denote $\rho = e^{2\pi i/w}$.

Let $\omega = e^{2\pi i/L}$, where $L = \mathrm{lcm}_{J\in\mathcal{J}}\{P_J\}$.

We define the sum

$$T(\tau,\xi\,;\alpha_J,\mathfrak{a}_J) = \sum_{h_1=0}^{L-1}\cdots\sum_{h_r=0}^{L-1} M(\tau,\xi_1\omega^{h_1},\ldots,\xi_r\omega^{h_r}\,;\alpha_J,\mathfrak{a}_J).$$

We put

$$(1.7) \qquad N_a = \prod_J^{(a-1)} (\phi_a)^{-1+2\alpha_J^{(a-1)}} \qquad (a = 1,\ldots,k)$$

and

$$N_a^* = \prod_J^{(a-1)} (\phi_a)^{-1+2\alpha_J^{*(a-1)}} \qquad (a = 1, \ldots, k), \tag{1.8}$$

For $\tau_1, \ldots, \tau_n$ and $\xi_1, \ldots, \xi_n$ we define

$$\begin{cases} \tau_p^* = \dfrac{\tau_p}{\tau_p^2 + |\xi_p|^2} & (p = 1, \ldots, r), \\ \xi_p^* = \dfrac{\xi_{p+r}}{\tau_p^2 + |\xi_p|^2}, \qquad \xi_{p+r}^* = \dfrac{\xi_p}{\tau_p^2 + |\xi_p|^2} & (p = 1, \ldots, r). \end{cases} \tag{1.9}$$

Let w be the number of the roots of unity in K. We put

$$d_I = (w, P_I), \qquad d_J = (w, \phi(J)) \qquad (J \neq I\,;\, J \in \mathcal{J}).$$

We put

$$s_J = (1 - \alpha_J) P_J, \qquad s_J^* = (1 - \alpha_J^*) P_J \qquad (J \in \mathcal{J}). \tag{1.10}$$

We introduce the series as follows:

$$\zeta(s, \mathfrak{a}) = \sum_{0 \neq (\mu) \subset \mathfrak{a}} \frac{1}{N(\mu)^s} \qquad (s = \sigma + it, \quad \sigma > 1),$$

where the sum is taken over all non-zero principal ideals contained in $\mathfrak{a}$. This function $\zeta(s, \mathfrak{a})$ of s is continued analytically on the whole s-plane. (See [3] or §2 below.)

We now state Main Theorem:

MAIN THEOREM. *$T(\tau, \xi\,; \alpha_J, \mathfrak{a}_J)$ satisfies a transformation formula as follows:*

$$\frac{\prod_J N(\mathfrak{a}_J)^{1/2}}{(N_1 \cdots N_k)^{r/2}} \prod_{p=1}^{r} (\tau_p^2 + |\xi_p|^2)^{1/4} \cdot \Big\{ T(\tau, \xi\,; \alpha_J, \mathfrak{a}_J) - S(\tau, \xi\,; \alpha_J, \mathfrak{a}_J) \Big\}$$
$$= \frac{\prod_J N(\mathfrak{a}_J^*)^{1/2}}{(N_1^* \cdots N_k^*)^{r/2}} \prod_{p=1}^{r} (\tau_p^{*2} + |\xi_p^*|^2)^{1/4} \cdot \Big\{ T(\tau^*, \xi^*\,; \alpha_J^*, \mathfrak{a}_J^*) - S(\tau^*, \xi^*\,; \alpha_J^*, \mathfrak{a}_J^*) \Big\},$$

where

(1.11)

$$
\begin{aligned}
&S(\tau,\xi;\alpha_J,\mathfrak{a}_J)\\
&=\frac{(2^2\pi)^r}{d_I\sqrt{D}}\{P_I\prod_J\phi(J)\}^{r-1}L^r\prod_J d_J\\
&\times\sum_J\Bigg\{\frac{P_J\Gamma(2s_J)^r}{N(\mathfrak{a}_J)(2^2P\pi)^{ns_J}}\prod_{p=1}^{r}\frac{1}{(\tau_p^2+|\xi_p|^2)^{s_J}}F\left(s_J,\frac{1}{2}-s_J,1;\frac{|\xi_p|^2}{\tau_p^2+|\xi_p|^2}\right)\\
&\qquad\times\prod_{\substack{J_1\in\mathcal{J}\\(J_1\neq J)}}\zeta\left(1+\frac{s_J-s_{J_1}}{P_{J_1}},\mathfrak{a}_{J_1}\right)\Bigg\}
\end{aligned}
$$

and

$$
\begin{aligned}
&S(\tau^*,\xi^*;\alpha_J^*,\mathfrak{a}_J^*)\\
&=\frac{(2^2\pi)^r}{d_I\sqrt{D}}\{P_I\prod_J\phi(J)\}^{r-1}L^r\prod_J d_J\\
&\times\sum_J\Bigg\{\frac{P_J\Gamma(2s_J^*)^r}{N(\mathfrak{a}_J^*)(2^2P\pi)^{ns_J^*}}\prod_{p=1}^{r}\frac{1}{(\tau_p^{*2}+|\xi_p^*|^2)^{s_J^*}}F\left(s_J^*,\frac{1}{2}-s_J^*,1;\frac{|\xi_p^*|^2}{\tau_p^{*2}+|\xi_p^*|^2}\right)\\
&\qquad\times\prod_{\substack{J_1\in\mathcal{J}\\(J_1\neq J)}}\zeta\left(1+\frac{s_J^*-s_{J_1}^*}{P_{J_1}},\mathfrak{a}_{J_1}^*\right)\Bigg\},
\end{aligned}
$$

$F(\alpha,\beta,\gamma\,;x)$ *being the Gauss hypergeometric function.*

First we shall consider, in §2, the zeta functions $\zeta(s,\lambda\,;\mathfrak{a})$ and summarize some properties of them in Theorems 2.1 and 2.2.

In §3, we shall consider the function $G(s,x\,;l)$ that was first studied by Rademacher [4], and prove two theorems. We shall have to estimate $G(s,x\,;l)$ in wider range (Theorem 3.2).

Next in §4, by applying the transformation formula of Hecke-Rademacher, we shall obtain the representation of $T(\tau,\xi\,;\alpha_J,\mathfrak{a}_J)$ as the series of the complex integrals:

$$T(\tau,\xi\,;\alpha_J,\mathfrak{a}_J)=A\sum_\lambda\frac{1}{2\pi i}\int_{(\sigma_0)}H_\lambda(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\,ds$$

(see (4.17) and (4.18)).

Using theorems in §2 and §3, we shall have the estimate of $H_\lambda(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)$ and the functional equation satisfied by $H_\lambda(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)$ (Theorems 5.1 and 5.2).

These reslults and the residues will give, in §6, the proof of Main Theorem.

§2. Zeta functions with Grössencharacters

We introduce the zeta function with Grössencharacter λ:

$$\zeta(s,\lambda\,;\mathfrak{a}) = \sum_{0\neq(\mu)\subset\mathfrak{a}} \frac{\lambda(\mu)}{N(\mu)^s} \qquad (s=\sigma+it, \quad \sigma>1),$$

where the sum is taken over all non-zero principal ideals contained in $\mathfrak{a}$,

$$\lambda(\mu) = \prod_{p=1}^{r} |\mu^{(p)}|^{-iv_p} \prod_{q=1}^{n} \left(\frac{\mu^{(q)}}{|\mu^{(q)}|}\right)^{a_q}$$

and

$$v_p = \sum_{j=1}^{r-1} e_p^{(j)} \left(2\pi m_j + \sum_{q=1}^{n} a_q \arg \varepsilon_j^{(q)}\right) \qquad (p=1,\ldots,r). \tag{2.1}$$

$\{e_p^{(j)}\}$ are the numbers satisfying the following equations:

$$\sum_{p=1}^{r} e_p^{(j)} = 0 \qquad (j=1,\ldots,r-1),$$

$$\sum_{p=1}^{r} e_p^{(i)} \log|\varepsilon_j^{(p)}| = \begin{cases} 1 & i=j, \\ 0 & i\neq j, \end{cases} \qquad (i,j=1,\ldots,r-1).$$

where $m_1,\ldots,m_{r-1}$ are the rational integers and $a_1,\ldots,a_n$ are non-negative rational integers such that $a_p\cdot a_{p+r}=0$ $(p=1,\ldots,r)$. Further $a_1,\ldots,a_n$ satisfy the additional condition

$$\prod_{q=1}^{n} {\rho^{(q)}}^{a_q} = 1.$$

(See [3, p.63].)

Here we note that

$$\sum_{p=1}^{r} v_p = 0.$$

THEOREM 2.1. (1) *$\zeta(s,\lambda;\mathfrak{a})$ has the analytic continuation over the whole s-plane and satisfies the functional equation as follows;*

$$\zeta(s,\lambda;\mathfrak{a})=\frac{(2\pi)^{n(s-1/2)}}{N(\mathfrak{a})\sqrt{D}}\frac{\Gamma(1-s;\bar{\lambda})}{\Gamma(s;\lambda)}\zeta(1-s,\bar{\lambda};\mathfrak{a}^*), \tag{2.2}$$

where

$$\Gamma(s;\lambda)=\prod_{p=1}^{r}\Gamma\left(s+\frac{iv_p}{2}+\frac{a_p+a_{p+r}}{2}\right).$$

(2) *If $\lambda\neq 1$, then*

$$\Gamma(s;\lambda)\zeta(s,\lambda;\mathfrak{a})$$

is an entire function.

(3) *In the case $\lambda=1$, $\zeta(s,1;\mathfrak{a})=\zeta(s,\mathfrak{a})$ and*

$$\Gamma(s)^r\zeta(s,\mathfrak{a})$$

is a meromorphic function with only two simple poles at $s=0$ and 1.

(4) *$\zeta(s,\mathfrak{a})$ is regular in the whole s-plane except at $s=1$, where $\zeta(s,\mathfrak{a})$ has a simple pole with the residue*

$$\frac{(2\pi)^r R}{wN(\mathfrak{a})\sqrt{D}}.$$

PROOF. (See [3, Lemma 2.1].) □

THEOREM 2.2. *In the strip*

$$-1/2\le\sigma\le 1+\max_{J\in\mathcal{J}}\{P_J\},$$

we have

$$\zeta(s,\lambda;\mathfrak{a})(s-1)^{e(\lambda)}\ll(1+|t|)^{2n}, \tag{2.3}$$

where

$$e(\lambda)=\begin{cases}1 & \text{if } \lambda=1,\\ 0 & \text{if } \lambda\neq 1\end{cases}$$

and the constants inplied in this estimation (2.3) depend on λ and $\mathfrak{a}$.

PROOF. If $-1/2 \leq \sigma \leq 3/2$, then (2.3) is true ([4, Hilfssatz 15]). If $3/2 \leq \sigma \leq 1+\max_{J\in\mathcal{J}}\{P_J\}$, then $\zeta(s,\lambda;\mathfrak{a}) = O(1)$. Thus Theorem is proved. □

§3. Hypergeometric functions

Let l be a non-negative number, s a complex number and x a real number such that $0 \leq x < 1$. We put

$$F(s,l\,;x) = F\left(\frac{s+l}{2}, \frac{1-s+l}{2}, l+1\,; x\right), \tag{3.1}$$

where $F(\alpha,\beta,\gamma\,;x)$ is the Gauss hypergeometric function, and

$$G(s,l\,;x) = \frac{\Gamma(s+l)}{2^{s+l}\Gamma(l+1)}F(s,l\,;x). \tag{3.2}$$

It is known that $F(s,l\,;x)$ is an entire function and that $G(s,l\,;x)$ is regular in the half-plane $\sigma > 0$ ([4, p.368]). From (3.1) and (3.2) we easily see that

$$F(1-s,l\,;x) = F(s,l\,;x), \tag{3.3}$$

$$G(1-s,l\,;x) = \frac{\Gamma(1-s+l)}{\Gamma(s+l)}2^{2s-1}G(s,l\,;x), \tag{3.4}$$

which shows that $G(s,l\,;x)$ is meromorphic in the whole s-plane.

THEOREM 4.1. *Let m be non-negative rational integer. If*

$$-\frac{l+1}{2}+m < \sigma < \frac{l+3}{2}+m,$$

then $G(s,l\,;x)$ is represented by an integral as follows:

(3.5)

$$G(s,l\,;x) = \frac{1}{2\sqrt{\pi}}\frac{\Gamma\left(\dfrac{s+l}{2}\right)}{\Gamma\left(\dfrac{l-s+3}{2}+m\right)}\int_0^\infty \frac{v^{(l+s-1)/2-m}g_m(v)}{(1+v)^{(l-s)/2+2}(1+v-x)^{(l+s)/2+m+1}}\,dv,$$

where $g_m(v)$ is a polynomial of degree $\leq m+1$ and $g_m(0) \neq 0$.

PROOF. By [4, (3.322)], we obtain

$$G(s,l;x) = \frac{1}{2\sqrt{\pi}} \frac{\Gamma\left(\dfrac{s+l}{2}\right)}{\Gamma\left(\dfrac{l-s+3}{2}\right)} \int_0^\infty \frac{v^{(l+s-1)/2} g_0(v)}{(1+v)^{(l-s)/2+2}(1+v-x)^{(l+s)/2+1}}\, dv, \tag{3.6}$$

where

$$g_0(v) = (l+1)(1+v-x) - (l+s)xv/2. \tag{3.7}$$

The integral in the right-hand side of (3.6) is convergent for

$$-(l+1)/2 < \sigma < (l+3)/2$$

and we see that

$$g_0(0) = (l+1)(1-x) \neq 0,$$

which shows that Theorem is true for $m = 0$.

Assume that Theorem is true for $m\,(>0)$ and that

$$-(l-1)/2 + m < \sigma < (l+3)/2 + m.$$

By the integration by parts, the integal of the right-hand side of (3.5) is

$$\begin{aligned}
I &= \int_0^\infty \frac{v^{(l+s-1)/2-m} g_m(v)}{(1+v)^{2+(l-s)/2}(1+v-x)^{(l+s)/2+m+1}}\, dv \\
&= \int_0^\infty \left(\frac{1}{1+v}\right)^{(l-s+5)/2+m} \frac{v^{(l+s-1)/2-m}(1+v)^{m+1/2} g_m(v)}{(1+v-x)^{(l+s)/2+m+1}}\, dv \\
&= \left[\frac{-1}{(l-s+3)/2+m}\left(\frac{1}{1+v}\right)^{(l-s+3)/2+m} \frac{v^{(l+s-1)/2-m}(1+v)^{m+1/2} g_m(v)}{(1+v-x)^{(l+s)/2+m+1}}\right]_0^\infty \\
&\quad + \frac{1}{\dfrac{l-s+3}{2}+m} \int_0^\infty \frac{1}{(1+v)^{(l-s+3)/2+m}} \frac{d}{dv}\left\{\frac{v^{(l+s-1)/2-m}(1+v)^{m+1/2} g_m(v)}{(1+v-x)^{(l+s)/2+m+1}}\right\} dv
\end{aligned}$$

$$= \frac{1}{\dfrac{l-s+3}{2}+m} \int_0^\infty \frac{1}{(1+v)^{(l-s+3)/2+m}} \frac{d}{dv} \left\{ \frac{v^{(l+s-1)/2-m}(1+v)^{m+1/2} g_m(v)}{(1+v-x)^{(l+s)/2+m+1}} \right\} dv.$$

The last integrand is

$$\frac{v^{(l+s-3)/2-m} g_{m+1}(v)}{(1+v)^{(l-s)/2+2}(1+v-x)^{(l+s)/2+m+2}},$$

where

$$\begin{aligned} g_{m+1}(v) = \{&-(m+1)v^2 + ((l+s-3)/2 - 2m - x(l+s)/2)v \\ &+ ((l+s-1)/2 - m)(1-x)\} g_m(v) \\ &+ \{v^3 + (2-x)v^2 + (1-x)v\} g'_m(v). \end{aligned} \tag{3.8}$$

If $\deg g_m \le m$, then $\deg g_{m+1} \le m+2$, and if $\deg g_m = m+1$, then $\deg g_{m+1} = m+2$. We further see that

$$g_{m+1}(0) = ((l+s-1)/2 - m)(1-x) g_m(0) \ne 0.$$

The intgral

$$\int_0^\infty \frac{v^{(l+s-3)/2-m} g_{m+1}(v)}{(1+v)^{(l-s)/2+2}(1+v-x)^{(l+s)/2+m+2}} ds$$

is convergent for

$$-(l-1)/2 + m < \sigma < (l+5)/2 + m.$$

Thus Theorem is proved by the induction on m. □

Theorem 4.2. *In the strip $-M \le \sigma \le M$, we have*

$$|\, G(s,l;x) \,| \le c_1 (1 + l + |t|)^{\sigma - 1/2} e^{-c_2 |t|},$$

where the positive constants c_1, c_2 depend only on M and x.

Proof. To polynomial

$$P(u) = \sum_k a_k u^k$$

of complex coefficients, we put

$$\|P\| = \max_k(|a_k|).$$

It is obvious that for polynomials P, Q,

$$\begin{cases} \|P+Q\| \leq \|P\| + \|Q\|, \\ \|PQ\| \leq \{\max(\deg P, \deg Q) + 1\}\|P\|\|Q\|, \\ \|P'\| \leq \|P\| \deg P, \\ |P(u)| \leq (\deg P + 1)\|P\| \max(1, |u|^{\deg P}). \end{cases}$$

Now by (3.7),

$$\|g_0\| = \max\{(l+1)(1-x), |\, l+1-(l+s)x\,|\} \leq 1+l+|s|. \tag{3.9}$$

By (3.8) we have

$$\begin{aligned} \|g_{m+1}\| &\leq (\max(2, m+1)+1)(l+|s|+2m+2)\|g_m\| \\ &\quad + \{(\max(3, m)+1\}2(m+1)\|g_m\| \\ &\leq 2(m+3)(l+|s|+m+1)\|g_m\| + (m+4)2(m+1)\|g_m\| \\ &\leq 4(m+4)(l+|s|+m+1)\|g_m\|. \end{aligned} \tag{3.10}$$

By (3.9), (3.10) and the induction on m,

$$\|g_m\| \leq C(1+l+|s|)^{m+1}. \tag{3.11}$$

We denote by C the constants depending only on m.

We put

$$F(v) = \frac{v^{(l+s-1)/2-m} g_m(v)}{(1+v)^{(l-s)/2+2}(1+v-x)^{(l+s)/2+m+1}},$$

where we assume

$$-(l+1)/2 + m + \epsilon \leq \sigma \leq (l+3)/2 + m - \epsilon \qquad (0 < \epsilon < 1/4). \tag{3.12}$$

We consider $F(z)$ for complex variable z $(-\pi < \arg z < \pi)$. Let R be a large number and

$$\varphi = \begin{cases} \pi/6 & \text{if } \mathrm{Im}(s) = t \geq 0, \\ -\pi/6 & \text{if } \mathrm{Im}(s) = t < 0. \end{cases}$$

In z-plane, we take a closed curve

$$0 \xrightarrow{L_1} R \xrightarrow{L_2} Re^{i\varphi} \xrightarrow{L_3} 0.$$

L_1 and L_3 are the straight lines from 0 to R and from $Re^{i\varphi}$ to 0 respectively, and L_2 is a part of the circle with center $z = 0$. On $L_2 : z = Re^{i\theta}$, we have, for (3.11) and (3.12),

$$|F(Re^{i\theta})| \ll R^{(\sigma-l-5)/2-m} \ll R^{-1-\epsilon}.$$

Hence

$$\int_{L_2} F(z)\,dz = \int_{\theta=0}^{\varphi} F(Re^{i\theta})d(Re^{i\theta}) \ll R^{-\epsilon} \longrightarrow 0 \quad (R \longrightarrow \infty),$$

which gives

$$I = \int_0^\infty F(v)\,dv = \int_V F(z)\,dz,$$

where V is the half line

$$z = ve^{i\varphi} \qquad (0 \leq v < \infty).$$

We take the absolute value of $F(z)$:

$$|F(z)| = \frac{|\,z\,|^{(l+\sigma-1)/2-m}|\,g_m(z)\,|}{|\,1+z\,|^{(l-\sigma)/2+2}|\,1+z-x\,|^{(l+\sigma)/2+m+1}} \exp\left(-\frac{t}{2}\arg\frac{z(1+z)}{1+z-x}\right). \tag{3.13}$$

On the line V

(3.14)

$$\begin{aligned} \arg\frac{z(1+z)}{1+z-x} &= \varphi + \tan^{-1}\left(\frac{v\sin\varphi}{1+v\cos\varphi}\right) - \tan^{-1}\left(\frac{v\sin\varphi}{1-x+v\cos\varphi}\right) \\ &= \varphi - \tan^{-1}\left(\frac{xv\sin\varphi}{1-x+(2-x)v\cos\varphi+v^2}\right), \end{aligned}$$

which shows that

$$\frac{t}{2}\arg\frac{z(1+z)}{1+z-x} = \left|\frac{t}{2}\arg\frac{z(1+z)}{1+z-x}\right|.$$

For $0 \le v < \infty$, the function

$$f(v) = \frac{xv}{1 - x + (2-x)v\cos\varphi + v^2}$$

takes extremal value at $v = \sqrt{1-x}$. Hence

$$\begin{aligned} f(v) &\le \frac{x\sqrt{1-x}}{2 - 2x + (2-x)\sqrt{1-x}\cos\varphi} = \frac{x}{2\sqrt{1-x} + (2-x)\cos\varphi} \\ &\le \frac{1}{\sqrt{1-x} + \cos\varphi} \end{aligned}$$

and by (3.14)

(3.15)
$$\begin{aligned} \left| \arg \frac{z(1+z)}{1+z-x} \right| &\ge \left| \varphi - \tan^{-1}\left(\frac{\sin\varphi}{\sqrt{1-x} + \cos\varphi} \right) \right| = \left| \tan^{-1}\left(\frac{\sqrt{1-x}\sin\varphi}{1 + \sqrt{1-x}\cos\varphi} \right) \right| \\ &\ge \frac{1}{2}\sqrt{1-x}\,|\sin\varphi|. \end{aligned}$$

From (3.11), (3.13) and (3.15) we obtain

(3.16)
$$\begin{aligned} |I| &\le \int_0^\infty \frac{|z|^{(l+\sigma-1)/2-m} |g_m(z)|}{|1+z|^{(l-\sigma)/2+2} |1+z-x|^{(l+\sigma)/2+m+1}} e^{-c|t|}\,|dz| \\ &\le C(1 + l + |s|)^{m+1} e^{-c|t|} \int_0^\infty \frac{v^{(l+\sigma-1)/2-m} \max(1,v)^{m+1}}{((1-x)\cos\varphi + v)^{(l+\sigma)/2+m+1}} \\ &\qquad \times \max\left\{ (1 + v\cos\varphi)^{(\sigma-l)/2-2}, (1+v)^{(\sigma-l)/2-2} \right\} dv. \end{aligned}$$

We divide the last integral of (3.16) into two parts:

$$\int_0^\infty = \int_0^1 + \int_1^\infty = I_1 + I_2.$$

Since $(\sigma - l)/2 - 2 \le m - 1/2 - \epsilon$ by (3.12),

$$\begin{aligned} I_1 &\le C \int_0^1 \frac{v^{(l+\sigma-1)/2-m}}{((1-x)\cos\varphi + v)^{(l+\sigma)/2+m+1}}\,dv \\ &= C \int_0^1 \left(\frac{v}{(1-x)\cos\varphi + v} \right)^{(l+\sigma+1)/2-m-\epsilon/2} \frac{dv}{v^{1-\epsilon/2}((1-x)\cos\varphi + v)^{2m+1/2+\epsilon/2}}. \end{aligned}$$

Further (3.12) gives $(l+\sigma+1)/2-m-\epsilon/2 \geq \epsilon/2$ and consequently we have

$$I_1 \leq C\int_0^1 \frac{dv}{v^{1-\epsilon/2}(1+v)^{2m+1/2+\epsilon/2}} \leq C. \tag{3.17}$$

If $(\sigma-l)/2-2 \leq 0$, then

$$\begin{aligned} I_2 &= \int_1^\infty \frac{v^{(l+\sigma+1)/2}}{(1+v\cos\varphi)^{(l-\sigma)/2+2}((1-x)\cos\varphi+v)^{(l+\sigma)/2+m+1}}\,dv \\ &\leq C\int_1^\infty \frac{dv}{(1+v\cos\varphi)^{(l-\sigma)/2+2}((1-x)\cos\varphi+v)^{m+1/2}} \\ &= C\int_1^\infty \left(\frac{1+v\cos\varphi}{(1-x)\cos\varphi+v}\right)^{m+1/2}\frac{dv}{(1+v\cos\varphi)^{(l-\sigma+5)/2+m}} \\ &\leq C\int_1^\infty \frac{dv}{(1+v\cos\varphi)^{(l-\sigma+5)/2+m}} \\ &\leq C\int_1^\infty \frac{dv}{(1+v\cos\varphi)^{1+\epsilon}}\,dv \leq C. \end{aligned} \tag{3.18}$$

If $(\sigma-l)/2-2>0$, then

$$\begin{aligned} I_2 &= \int_1^\infty \frac{v^{(l+\sigma+1)/2}(1+v)^{(\sigma-l)/2-2}}{((1-x)\cos\varphi+v)^{(l+\sigma)/2+m+1}}\,dv \\ &\leq C\int_1^\infty \frac{(1+v)^{m-1/2-\epsilon}}{((1-x)\cos\varphi+v)^{m+1/2}}\,dv \\ &\leq C\int_1^\infty \frac{dv}{(1+v)^{1+\epsilon}}\,dv \leq C \end{aligned} \tag{3.19}$$

because of $(\sigma-l)/2-2 \leq m-1/2+\epsilon$ by (3.12).

Thus we have, by (3.16), (3.17), (3.18) and (3.19),

$$|\,I\,| \leq C(1+l+|s|)^{m+1}e^{-c|t|}$$

and

$$|\,G(s,l;x)\,| \leq C\left|\frac{\Gamma\left(\dfrac{s+l}{2}\right)}{\Gamma\left(\dfrac{l-s+3}{2}+m\right)}\right|(1+l+|s|)^{m+1}e^{-c|t|}. \tag{3.20}$$

Supposing

$$m-1/2+\epsilon \leq \sigma/2 \leq m+3/2+\epsilon, \tag{3.21}$$

we shall show that

$$\left|\frac{\Gamma\left(\frac{s+l}{2}\right)}{\Gamma\left(\frac{l-s+3}{2}+m\right)}\right| \leq C(1+l+|s|)^{\sigma-m-3/2}. \tag{3.22}$$

If $m=0$, then (3.22) is true ([4, (3.325)]). Assume that $m \geq 1$. Using the fact $|\Gamma(z)| = |\Gamma(\bar{z})|$, we write

$$\left|\frac{\Gamma\left(\frac{s+l}{2}\right)}{\Gamma\left(\frac{l-s+3}{2}+m\right)}\right| = \frac{2}{|s+l|}\left|\frac{\Gamma\left(\frac{s+l}{2}+1\right)}{\Gamma\left(\frac{l+s+3}{2}+m-\sigma\right)}\right|. \tag{3.23}$$

The difference of the arguments of Γ-function in the right-hand side is $\{(l+s+3)/2+m-\sigma\} - \{(s+l)/2+1\} = 1/2+m-\sigma$, and by the assumption (3.21),

$$-5/2-m+2\epsilon \leq 1/2+m-\sigma \leq -1/2-m-2\epsilon.$$

so we put

$$1/2+m-\sigma = -m_0-\delta \qquad (m_0 = m+1 \text{ or } m+2\,;\, |\delta| \leq 1/2)$$

and we see

(3.24)

$$\left|\frac{\Gamma\left(\frac{l+s+3}{2}+m-\sigma\right)}{\Gamma\left(\frac{s+l}{2}+1\right)}\right| = \left|\frac{\Gamma(z-m_0-\delta)}{\Gamma(z)}\right|$$

$$= \frac{1}{|(z-m_0-\delta)\cdots(z-1-\delta)|}\left|\frac{\Gamma(z-\delta)}{\Gamma(z)}\right|,$$

where $z=(s+l)/2+1$. Since

$$\mathrm{Re}(z) = (\sigma+l)/2+1 \geq (l-1)/2+m+1 \geq 3/2, \qquad |\delta| \leq 1/2,$$

we have

$$C_1|z|^{-\delta} \leq \left|\frac{\Gamma(z-\delta)}{\Gamma(z)}\right| \leq C_2|z|^{-\delta}$$

([4, Hilfssatz 4.17]). Hence it follows from (3.24) that

$$\left|\frac{\Gamma\left(\frac{s+l}{2}+1\right)}{\Gamma\left(\frac{l+s+3}{2}+m-\sigma\right)}\right| \leq C(1+l+|s|)^{m_0}|z|^{\delta} = C(1+l+|s|)^{m_0+\delta}$$

and by (3.23)

$$\left|\frac{\Gamma\left(\frac{s+l}{2}\right)}{\Gamma\left(\frac{l-s+3}{2}+m\right)}\right| \leq C(1+l+|s|)^{m_0+\delta-1} = C(1+l+|s|)^{\sigma-m-3/2}. \tag{3.25}$$

By this result (3.25) and (3.20),

$$|\,G(s,l;x)\,| \leq C(1+l+|s|)^{\sigma-1/2}e^{-c|t|} \tag{3.26}$$

for

$$2m+\epsilon \leq \sigma \leq 2m+3-\epsilon.$$

If

$$-2-2m+\epsilon \leq \sigma \leq 1-2m-\epsilon,$$

then, using (3.4) and above result (3.26),

$$\begin{aligned}|\,G(s,l;x)\,| &= \left|\frac{\Gamma(s+l)}{\Gamma(1-s+l)}\right| 2^{1-2\sigma}|\,G(1-s,l;x)\,| \\ &\leq C\left|\frac{\Gamma(s+l)}{\Gamma(1-s+l)}\right|(1+l+|s|)^{1/2-\sigma}e^{-c|t|}.\end{aligned}$$

In the same way as is obtained (3.25), we have

$$\left|\frac{\Gamma(s+l)}{\Gamma(1-s+l)}\right| \leq C(1+l+|s|)^{2\sigma-1}.$$

Thus we have

$$|\,G(s,l;x)\,| \leq C(1+l+|s|)^{\sigma-1/2}e^{-c|t|}$$

and the proof is completed $\square$

§5. Representations by integrals

We observe the sum over ν_I in (1.6). This sum becomes double sum:

$$\sum_{0\neq\nu_I\in\mathfrak{a}_I} = \sum_{\substack{\nu_I/U^{P_I}\\ 0\neq\nu_I\in\mathfrak{a}_I}} \sum_{\varepsilon\in U^{P_I}} . \tag{4.1}$$

In this right-hand side, the outer sum is taken over all non-zero numbers ν_I in $\mathfrak{a}_I$ not associated to each other with respect to U^{P_I} and the inner sum is taken over all units ε of U^{P_I}.

The unit ε of U^{P_I} is of the form:

$$\varepsilon = (\rho^b \varepsilon_1^{b_1}\cdots\varepsilon_{r-1}^{b_{r-1}})^{P_I},$$

where $b_1,\ldots,b_{r-1}$ are integers, b takes the values $b = 1,\ldots,w/d_I$ ($d_I = (w, P_I)$). We see that

$$[\,U : U^{P_I}\,] = d_I P_I^{r-1}.$$

We consider the sum over ε in (4.1) and write it as follows;

$$\begin{aligned}
&\sum_{\varepsilon\in U^{P_I}} \exp\left\{-2P\pi S\left(\tau\prod_J |\nu_J|^{1/P_J}|\varepsilon|^{1/P_I}\right) + 2P\pi i S\left(\xi\prod_J \nu_J^{1/P_J}\varepsilon^{1/P_I}\right)\right\}\\
&= \sum_{b=1}^{w/d_I}\ \sum_{b_1,\ldots,b_{r-1}=-\infty}^{\infty} \exp\left\{-2P\pi S\left(\tau\prod_J |\nu_J|^{1/P_J}|\varepsilon_1^{b_1}\cdots\varepsilon_{r-1}^{b_{r-1}}|\right)\right.\\
&\qquad\left. +2P\pi i S\left(\xi\prod_J \nu_J^{1/P_J}\rho^b\varepsilon_1^{b_1}\cdots\varepsilon_{r-1}^{b_{r-1}}\right)\right\}.
\end{aligned} \tag{4.2}$$

Now we quote the following theorem from [4, Hilfssatz 14]:

THEOREM 5.1. *Let $W_1\ldots, W_n$ be positive numbers such that $W_{p+r} = W_p$ ($p = 1,\ldots,r$). Let $U_1,\ldots,U_n$ be complex numbers such that $U_{p+r} = \overline{U}_p$ ($p = 1,\ldots,r$). Then* *we have*

$$\sum_{b_1,\dots,b_{r-1}=-\infty}^{\infty} \exp\left\{-2\pi S(W|\varepsilon_1^{b_1}\cdots\varepsilon_{r-1}^{b_{r-1}}|) + 2\pi i S(U\varepsilon_1^{b_1}\cdots\varepsilon_{r-1}^{b_{r-1}})\right\}$$
$$= \frac{2^r}{R} \sum_{m_1,\dots,m_{r-1}=-\infty}^{\infty} \sum_{\substack{a_1,\dots,a_n\geq 0\\ a_p\cdot a_{p+r}=0}}^{\infty} \frac{1}{2\pi i}\int_{(\sigma_0)} (2\pi)^{-ns}$$
$$\times \prod_{p=1}^{r}\left\{\frac{(iU_p)^{a_p}(iU_{p+r})^{a_{p+r}}}{(W_p^2+|U_p|^2)^{(2s+iv_p+l_p)/2}} G\left(2s+iv_p, l_p\,;\frac{|U_p|^2}{W_p^2+|U_p|^2}\right)\right\} ds,$$

where $m_1,\dots,m_{r-1}$ run throught all rational intergers, $a_1,\dots,a_n$ run through non-negative rational integers such that $a_p\cdot a_{p+r}=0$ $(p=1,\dots,r)$. The v_p are the values defined by (2,1) and $G(s,l\,;x)$ is the function defined in §4. We put $l_p=a_p+a_{p+r}$ $(p=1,\dots,r)$ and the integrals are the complex integrals taken along the vertical line $\sigma=\sigma_0$.

Applying this Theorem to the sum in the right-hand side of (4.2), we have

(4.3)
$$\sum_{\varepsilon\in U^{P_I}} \exp\left\{-2P\pi S\left(\tau\prod_J|\nu_J|^{1/P_J}|\varepsilon|^{1/P_I}\right) + 2P\pi i S\left(\xi\prod_J\nu_J^{1/P_J}\varepsilon^{1/P_I}\right)\right\}$$
$$= \frac{2^r}{R}\sum_{b=1}^{w/d_I}\sum_{\{m\}}\sum_{\{a\}}^{*}\prod_{q=1}^{n}\rho^{(q)^{a_q b}}\frac{1}{2\pi i}\int_{(\sigma_0)}(2P\pi)^{-ns}$$
$$\times\prod_{p=1}^{r}\frac{\left(i\xi_p\prod_J\nu_J^{(p)1/P_J}\right)^{a_p}\left(i\xi_{p+r}\prod_J\nu_J^{(p+r)1/P_J}\right)^{a_{p+r}}}{\left\{(\tau_p\prod_J|\nu_J^{(p)}|^{1/P_J})^2+|\xi_p\prod_J\nu_J^{(p)1/P_J}|^2\right\}^{(2s+iv_p+l_p)/2}}$$
$$\times\prod_{p=1}^{r}G\left(2s+iv_p,l_p\,;\frac{|\xi_p|^2}{\tau_p^2+|\xi_p|^2}\right)ds,$$

where we denote by $\sum_{\{m\}}$ and $\sum_{\{a\}}^{*}$ the summations over $m_1,\dots,m_{r-1}$ and $a_1,\dots,a_n$ in the meaning of Theorem 5.1, respctively, and the integrations are taken along the vertical line

(4.4)
$$\sigma_0=\max_{J\in\mathcal{J}}\{(1-\alpha_J)P_J\}+1.$$

Putting this formula (4.3) into (4.2) and then into (1.6), we have

$$
\begin{aligned}
(4.5)\qquad & M(\tau,\xi\,;\alpha_J,\mathfrak{a}_J) \\
&= \sum_{\substack{0\neq\nu_J\in\mathfrak{a}_J \\ \nu_J/U^{\phi(J)} \\ (J\neq I)}}\cdots\sum \sum_{\substack{0\neq\nu_I\in\mathfrak{a}_I \\ \nu_I/U^{P_I}}} \frac{2^r}{R}\sum_{\{m\}}\sum_{\{a\}}{}^{*}\sum_{b=1}^{w/d_I}\prod_{q=1}^{n}\rho^{(q)^{a_q b}}\frac{1}{2\pi i}\int_{(\sigma_0)}(2P\pi)^{-ns} \\
&\times\prod_{p=1}^{r}\left\{\frac{(i\xi_p)^{a_p}(i\xi_{p+r})^{a_{p+r}}}{(\tau_p^2+|\xi_p|^2)^{(2s+iv_p+l_p)/2}}G(2s+iv_p,l_p\,;x_p)\right\} \\
&\times\prod_J\left\{N(\nu_J)^{-\alpha_J}\prod_{p=1}^{r}\frac{\nu_J^{(p)^{a_p/P_J}}\nu_J^{(p+r)^{a_{p+r}/P_J}}}{|\nu_J^{(p)}|^{(2s+iv_p+l_p)/P_J}}\right\}ds,
\end{aligned}
$$

where we put

$$
x_p=\frac{|\xi_p|^2}{\tau_p^2+|\xi_p|^2}\qquad(p=1,\dots,r).
$$

By Theorem 4.2, we can estimate the integrand in the right-hand side of (4.5):

$$
\begin{aligned}
&(2P\pi)^{-ns}\prod_{p=1}^{r}\left\{\frac{(i\xi_p)^{a_p}(i\xi_{p+r})^{a_{p+r}}}{(\tau_p^2+|\xi_p|^2)^{(2s+iv_p+l_p)/2}}G(2s+iv_p,l_p\,;x_p)\right\} \\
&\qquad\times\prod_J\left\{N(\nu_J)^{-\alpha_J}\prod_{p=1}^{r}\frac{\nu_J^{(p)^{a_p/P_J}}\nu_J^{(p+r)^{a_{p+r}/P_J}}}{|\nu_J^{(p)}|^{(2s+iv_p+l_p)/P_J}}\right\} \\
&\ll\prod_J N(\nu_J)^{-\alpha_J-\sigma_0/P_J}\prod_{p=1}^{r}x_p^{l_p/2}(1+l_p+|2t+v_p|)^{2\sigma_0-1/2}e^{-c|2t+v_p|}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
& M(\tau,\xi\ \alpha_J,\mathfrak{a}_J) \\
&\ll\sum_{\substack{0\neq\nu_J\in\mathfrak{a}_J \\ \nu_J/U^{\phi(J)} \\ (J\neq I)}}\cdots\sum \sum_{\substack{0\neq\nu_I\in\mathfrak{a}_I \\ \nu_I/U^{P_I}}}\prod_J N(\nu_J)^{-\alpha_J-\sigma_0/P_J} \\
&\times\sum_{\{m\}}\sum_{\{a\}}{}^{*}\prod_{p=1}^{r}x_p^{l_p/2}\int_{-\infty}^{\infty}\prod_{p=1}^{r}(1+l_p+|2t+v_p|)^{2\sigma_0-1/2}e^{-c|2t+v_p|}\,dt.
\end{aligned}
$$

In this right-hand side,

$$
\sum_{\{m\}}\sum_{\{a\}}{}^{*}\prod_{p=1}^{r}x_p^{l_p/2}\int_{-\infty}^{\infty}\prod_{p=1}^{r}(1+l_p+|2t+v_p|)^{2\sigma_0-1/2}e^{-c|2t+v_p|}\,dt
$$

is convergent ([2]). The convergence of

$$\sum_{\substack{0\neq\nu_J\in\mathfrak{a}_J \\ \nu_J/U^{\phi(J)} \\ (J\neq I)}}\cdots\sum \sum_{\substack{0\neq\nu_I\in\mathfrak{a}_I \\ \nu_I/U^{P_I}}} \prod_J N(\nu_J)^{-\alpha_J-\sigma_0/P_J}$$

is clear, since

$$\alpha_J+\sigma_0/P_J \geq \alpha_J+((1-\alpha_J)P_J+1)/P_J = 1+1/P_J \qquad (J\in\mathcal{J})$$

by (4.4).

Thus the right-hand side of (4.5) is absolutely convergent. The order of the summations and integrations can be exchanged. Consequently we have

$$\begin{aligned}
&M(\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\\
(4.6)\qquad &= \frac{2^r}{R}\sum_{\{m\}}\sum_{\{a\}}{}^{*}\sum_{b=1}^{w/d_I}\prod_{q=1}^{n}\rho^{(q)^{a_q b}}\frac{1}{2\pi i}\int_{(\sigma_0)}(2P\pi)^{-ns}\\
&\times\prod_{p=1}^{r}\left\{\frac{(i\xi_p)^{a_p}(i\xi_{p+r})^{a_{p+r}}}{(\tau_p^2+|\xi_p|^2)^{(2s+iv_p+l_p)/2}}G(2s+iv_p,l_p\,;x_p)\right\}\\
&\times\sum_{\substack{0\neq\nu_I\in\mathfrak{a}_I \\ \nu_I/U^{P_I}}}\sum_{\substack{0\neq\nu_J\in\mathfrak{a}_J \\ \nu_J/U^{\phi(J)} \\ (J\neq I)}}\cdots\sum\prod_J N(\nu_J)^{-\alpha_J}\prod_J\prod_{p=1}^{r}\frac{\nu_J^{(p)^{a_p/P_J}}\nu_J^{(p+r)^{a_{p+r}/P_J}}}{|\nu_J^{(p)}|^{(2s+iv_p+l_p)/P_J}}\,ds.
\end{aligned}$$

In this integrand, the sum over $\{\nu_J\}_J$ is

(4.7)

$$\begin{aligned}
&\sum_{\substack{0\neq\nu_I\in\mathfrak{a}_I \\ \nu_I/U^{P_I}}}\sum_{\substack{0\neq\nu_J\in\mathfrak{a}_J \\ \nu_J/U^{\phi(J)} \\ (J\neq I)}}\cdots\sum\prod_J\left\{N(\nu_J)^{-\alpha_J-s/P_J}\prod_{p=1}^{r}|\nu_J^{(p)}|^{-iv_p/P_J}\prod_{q=1}^{n}\left(\frac{\nu_J^{(q)}}{|\nu_J^{(q)}|}\right)^{a_q/P_J}\right\}\\
&=\sum_{\substack{0\neq\nu_I\in\mathfrak{a}_I \\ \nu_I/U^{P_I}}}\left\{N(\nu_I)^{-s/P_I}\prod_{p=1}^{r}|\nu_I^{(p)}|^{-iv_p/P_I}\prod_{q=1}^{n}\left(\frac{\nu_I^{(q)}}{|\nu_I^{(q)}|}\right)^{a_q/P_I}\right\}\\
&\quad\times\prod_{J\neq I}\sum_{\substack{0\neq\nu_J\in\mathfrak{a}_J \\ \nu_J/U^{\phi(J)}}}\left\{N(\nu_J)^{-s/P_J-\alpha_J}\prod_{p=1}^{r}|\nu_J^{(p)}|^{-iv_p/P_J}\prod_{q=1}^{n}\left(\frac{\nu_J^{(q)}}{|\nu_J^{(q)}|}\right)^{a_q/P_J}\right\}.
\end{aligned}$$

First we consider the sum over ν_I and write it as follows:

$$\sum_{\substack{0\neq\nu_I\in\mathfrak{a}_I\\ \nu_I/U^{P_I}}} = \sum_{\substack{0\neq\nu_I\in\mathfrak{a}_I\\ \nu_I/U}} \Big\{ N(\nu_I)^{-s/P_I} \sum_{\varepsilon\in U/U^{P_I}} \prod_{p=1}^{r} |\varepsilon^{(p)}\nu_I^{(p)}|^{-iv_p/P_I} \times \prod_{q=1}^{n} \left(\frac{\varepsilon^{(q)}\nu_I^{(q)}}{|\varepsilon^{(q)}\nu_I^{(q)}|} \right)^{a_q/P_I} \Big\}. \tag{4.8}$$

In the inner sum, ε runs through the representatives of U/U^{P_I}. Hence we can put

$$\varepsilon = \rho^b \varepsilon_1^{b_1} \cdots \varepsilon_{r-1}^{b_{r-1}},$$

where

$$b = 0, \dots, d_I - 1, \quad b_j = 0, \dots, P_I - 1 \quad (j = 1, \dots, r-1)$$

and the sum over ε in the right-hand side of (4.8) is

(4.9)

$$\sum_{\varepsilon\in U/U^{P_I}} \prod_{p=1}^{r} |\varepsilon^{(p)}|^{-iv_p/P_I} \prod_{q=1}^{n} \left(\frac{\varepsilon^{(q)}}{|\varepsilon^{(q)}|} \right)^{a_q/P_I}$$
$$= \sum_{b=0}^{d_I-1} \prod_{q=1}^{n} \rho^{(q)\,ba_q/P_I} \sum_{b_1=0}^{P_I-1} \cdots \sum_{b_{r-1}=0}^{P_I-1} \prod_{p=1}^{r} |\varepsilon_1^{(p)\,b_1} \cdots \varepsilon_{r-1}^{(p)\,b_{r-1}}|^{-iv_p/P_I} \times \prod_{q=1}^{n} \left\{ \left(\frac{\varepsilon_1^{(q)}}{|\varepsilon_1^{(q)}|} \right)^{b_1} \cdots \left(\frac{\varepsilon_{r-1}^{(q)}}{|\varepsilon_{r-1}^{(q)}|} \right)^{b_{r-1}} \right\}^{a_q/P_I}.$$

Since

$$v_p = \sum_{j=1}^{r-1} e_p^{(j)} \left(2\pi m_j + \sum_{q=1}^{n} a_q \arg \varepsilon_j^{(q)} \right) \qquad (p = 1, \dots, r),$$

we have

$$\sum_{p=1}^{r} v_p \log |\varepsilon_j^{(p)}| = 2\pi m_j + \sum_{q=1}^{n} a_q \arg \varepsilon_j^{(q)} \qquad (j = 1, \dots, r-1)$$

and the sum over $b_1, \dots, b_{r-1}$ in the right-hand side of (4.9) is

$$\sum_{b_1=0}^{P_I-1} \cdots \sum_{b_{r-1}=0}^{P_I-1} \prod_{p=1}^{r} |\varepsilon_1^{(p)\,b_1} \cdots \varepsilon_{r-1}^{(p)\,b_{r-1}}|^{-iv_p/P_I}$$

$$\times \prod_{q=1}^{n} \left\{ \left(\frac{\varepsilon_1^{(q)}}{|\varepsilon_1^{(q)}|} \right)^{b_1} \cdots \left(\frac{\varepsilon_{r-1}^{(q)}}{|\varepsilon_{r-1}^{(q)}|} \right)^{b_{r-1}} \right\}^{a_q/P_I}$$

$$= \sum_{b_1=0}^{P_I-1} \cdots \sum_{b_{r-1}=0}^{P_I-1} \exp\left\{ -\frac{i}{P_I} \sum_{p=1}^{r} v_p \sum_{j=1}^{r-1} b_j \log |\varepsilon_j^{(p)}| + \frac{i}{P_I} \sum_{q=1}^{n} \sum_{j=1}^{r-1} a_q b_j \arg \varepsilon_j^{(q)} \right\}$$

$$= \sum_{b_1=0}^{P_I-1} \cdots \sum_{b_{r-1}=0}^{P_I-1} \exp\left\{ -\frac{2\pi i}{P_I} \sum_{j=1}^{r-1} m_j b_j \right\}$$

$$= \begin{cases} P_I^{r-1} & \text{if } P_I \mid m_1, \ldots, P_I \mid m_{r-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Putting this result into (4.9) and then into (4.6), we have

(4.10)

$$M(\tau, \xi; \alpha_J, \mathfrak{a}_J)$$

$$= \frac{2^r}{R} P_I^{r-1} \sum_{\{m\}}{}' \sum_{\{a\}}{}^* \sum_{b=1}^{w/d_I} \prod_{q=1}^{n} \rho^{(q) a_q b} \sum_{b=0}^{d_I-1} \prod_{q-1}^{n} \rho^{(q) b a_q/P_I}$$

$$\times \frac{1}{2\pi i} \int_{(\sigma_0)} (2P\pi)^{-ns} \prod_{p=1}^{r} \left\{ \frac{(i\xi_p)^{a_p} (i\xi_{p+r})^{a_{p+r}}}{(\tau_p^2 + |\xi_p|^2)^{(2s+iv_p+l_p)/2}} G(2s + iv_p, l_p; x_p) \right\}$$

$$\times \sum_{0 \neq (\nu_I) \subset \mathfrak{a}_I} \left\{ N(\nu_I)^{-s/P_I} \prod_{p=1}^{r} |\nu_I^{(p)}|^{-iv_p/P_I} \prod_{q=1}^{n} \left(\frac{\nu_I^{(q)}}{|\nu_I^{(q)}|} \right)^{a_q/P_I} \right\}$$

$$\times \sum_{\substack{0 \neq \nu_J \in \mathfrak{a}_J \\ \nu_J / U^{\phi(J)} \\ (J \neq I)}} \cdots \sum \left\{ N(\nu_J)^{-\alpha_J - s/P_J} \prod_{p=1}^{r} |\nu_J^{(p)}|^{-iv_p/P_J} \prod_{q=1}^{n} \left(\frac{\nu_J^{(q)}}{|\nu_J^{(q)}|} \right)^{a_q/P_J} \right\} ds,$$

where the sum $\sum'_{\{m\}}$ means that the summation variables $m_1, \ldots, m_{r-1}$ run through all multiples of P_I.

Next we consider the sum over ν_J $(J \neq I)$ in the right-hand side of (4.7). In the same way as (4.8), we write

$$\sum_{\substack{0\neq\nu_J\in\mathfrak{a}_J\\ \nu_J/U^{\phi(J)}}}\left\{N(\nu_J)^{-\alpha_J-s/P_J}\prod_{p=1}^{r}|\nu_J^{(p)}|^{-iv_p/P_J}\prod_{q=1}^{n}\left(\frac{\nu_J^{(q)}}{|\nu_J^{(q)}|}\right)^{a_q/P_J}\right\}$$

$$=\sum_{\substack{0\neq\nu_J\in\mathfrak{a}_J\\ \nu_J/U}}\left\{N(\nu_J)^{-\alpha_J-s/P_J}\sum_{\varepsilon}\prod_{p=1}^{r}|\varepsilon^{(p)}\nu_J^{(p)}|^{-iv_p/P_J}\right.$$

$$\left.\times\prod_{q=1}^{n}\left(\frac{\varepsilon^{(q)}\nu_J^{(q)}}{|\varepsilon^{(q)}\nu_J^{(q)}|}\right)^{a_q/P_J}\right\},$$

where ε runs through the representatives of $U/U^{\phi(J)}$. Hence we can put

$$\varepsilon=\rho^b\varepsilon_1^{b_1}\cdots\varepsilon_{r-1}^{b_{r-1}},$$

where

$$b=0,\dots,d_J-1,\qquad d_J=(w,\phi(J)),$$

$$b_j=0,\dots,\phi(J)-1\quad(j=1,\dots,r-1).$$

The sum over ε is

$$\sum_{\varepsilon\in U/U^{\phi(J)}}\prod_{p=1}^{r}|\varepsilon^{(p)}|^{-iv_p/P_J}\prod_{q=1}^{n}\left(\frac{\varepsilon^{(q)}}{|\varepsilon^{(q)}|}\right)^{a_q/P_J}$$

$$=\sum_{b=0}^{d_J-1}\prod_{q=1}^{n}\rho^{(q)^{ba_q/P_J}}\sum_{b_1=0}^{\phi(J)-1}\cdots\sum_{b_{r-1}=0}^{\phi(J)-1}\prod_{p=1}^{r}|\varepsilon_1^{(p)^{b_1}}\cdots\varepsilon_{r-1}^{(p)^{b_{r-1}}}|^{-iv_p/P_J}$$

$$\times\prod_{q=1}^{n}\left\{\left(\frac{\varepsilon_1^{(q)}}{|\varepsilon_1^{(q)}|}\right)^{b_1}\cdots\left(\frac{\varepsilon_{r-1}^{(q)}}{|\varepsilon_{r-1}^{(q)}|}\right)^{b_{r-1}}\right\}^{a_q/P_J}.$$

In this right-hand side, the sum over $b_1,\dots,b_{r-1}$ is

$$\sum_{b_1=0}^{\phi(J)-1}\cdots\sum_{b_{r-1}=0}^{\phi(J)-1}\exp\left\{-\frac{2\pi i}{P_J}\sum_{j=1}^{r-1}m_jb_j\right\}.$$

Since we can replace all m_j by $P_I m_j$, this sum is

$$\sum_{b_1=0}^{\phi(J)-1}\cdots\sum_{b_{r-1}=0}^{\phi(J)-1}\exp\left\{-2\pi i\frac{P_I}{P_J}\sum_{j=1}^{r-1}m_jb_j\right\} \tag{4.11}$$
$$=\begin{cases}\phi(J)^{r-1} & \text{if } \phi(J)\mid m_1,\dots,\phi(J)\mid m_{r-1},\\ 0 & \text{otherwise,}\end{cases}$$

because of $\phi(J)=P_J/(P_I,P_J)$ ((1.1)).

We see that

$$P_I\operatorname*{lcm}_{J\neq I}\{\phi(J)\}=\operatorname*{lcm}_{J\in\mathcal{J}}\{P_J\}=L.$$

Hence (4.10) and (4.11) show that

$$\begin{aligned}
&M(\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\\
&=\frac{2^r}{R}P_I^{r-1}\prod_J\phi(J)^{r-1}\sum_{\{m\}}''\sum_{\{a\}}^{*}\sum_{b=1}^{w/d_I}\prod_{q=1}^{n}\rho^{(q)^{a_qb}}\prod_J\sum_{b=0}^{d_J-1}\prod_{q=1}^{n}\rho^{(q)^{ba_q/P_J}}\\
&\times\frac{1}{2\pi i}\int_{(\sigma_0)}(2P\pi)^{-ns}\prod_{p=1}^{r}\left\{\frac{(i\xi_p)^{a_p}(i\xi_{p+r})^{a_{p+r}}}{(\tau_p^2+|\xi_p|^2)^{(2s+iv_p+l_p)/2}}G(2s+iv_p,l_p\,;x_p)\right\}\\
&\times\prod_J\sum_{0\neq(\nu_J)\subset\mathfrak{a}_J}\left\{N(\nu_J)^{-s/P_J-\alpha_J}\prod_{p=1}^{r}|\nu_J^{(p)}|^{-iv_p/P_J}\prod_{q=1}^{n}\left(\frac{\nu_J^{(q)}}{|\nu_J^{(q)}|}\right)^{a_q/P_J}\right\}ds,
\end{aligned}$$

where the sum $\sum_{\{m\}}''$ means that the summation variables $m_1,\dots,m_{r-1}$ run through all multiples of L.

From this expression we easily obtain

$$\begin{aligned}
T(\tau,\xi\,;\alpha_J,\mathfrak{a}_J)&=\sum_{h_1=0}^{L-1}\cdots\sum_{h_r=0}^{L-1}M(\tau,\xi_1\omega^{h_1},\dots,\xi_r\omega^{h_r}\,;\alpha_J,\mathfrak{a}_J)\\
&=\frac{2^r}{R}P_I^{r-1}\prod_J\phi(J)^{r-1}\sum_{\{m\}}''\sum_{\{a\}}^{*}\sum_{b=1}^{w/d_I}\prod_{q=1}^{n}\rho^{(q)^{a_qb}}\prod_J\sum_{b=0}^{d_J-1}\prod_{q=1}^{n}\rho^{(q)^{ba_q/P_J}}\\
&\times\frac{1}{2\pi i}\int_{(\sigma_0)}(2P\pi)^{-ns}\\
&\times\sum_{h_1=0}^{L-1}\cdots\sum_{h_r=0}^{L-1}\prod_{p=1}^{r}\left\{\frac{(i\xi_p\omega^{h_p})^{a_p}(i\xi_{p+r}\bar{\omega}^{h_p})^{a_{p+r}}}{(\tau_p^2+|\xi_p|^2)^{(2s+iv_p+l_p)/2}}G(2s+iv_p,l_p\,;x_p)\right\}\\
&\times\prod_J\sum_{0\neq(\nu_J)\subset\mathfrak{a}_J}\left\{N(\nu_J)^{-s/P_J-\alpha_J}\prod_{p=1}^{r}|\nu_J^{(p)}|^{-iv_p/P_J}\prod_{q=1}^{n}\left(\frac{\nu_J^{(q)}}{|\nu_J^{(q)}|}\right)^{a_q/P_J}\right\}ds,
\end{aligned}$$

Since

$$\sum_{h_1=0}^{L-1}\cdots\sum_{h_r=0}^{L-1}\prod_{p=1}^{r}\omega^{h_p(a_p-a_{p+r})}=\sum_{h_1=0}^{L-1}\cdots\sum_{h_r=0}^{L-1}\prod_{p=1}^{r}\exp\left(2\pi i\frac{h_p}{L}(a_p-a_{p+r})\right)$$
$$=\begin{cases} L^r & \text{if } L\mid a_1,\dots,L\mid a_n,\\ 0 & \text{otherwise},\end{cases}$$

we have, replacing all $m_1,\dots,m_{r-1},a_1,\dots,a_n$ by $Lm_1,\dots,Lm_{r-1},La_1,\dots,La_n$,

(4.12)

$$T(\tau,\xi;\alpha_J,\mathfrak{a}_J)$$
$$=\frac{2^r}{R}\{P_I\prod_J\phi(J)\}^{r-1}L^r\sum_{\{m\}}\sum_{\{a\}}{}^{*}\sum_{b=0}^{w/d_I}\prod_{q=1}^{n}\rho^{(q)^{La_qb}}\prod_J\sum_{b=0}^{d_J-1}\prod_{q=1}^{n}\rho^{(q)^{bLa_q/P_J}}$$
$$\times\frac{1}{2\pi i}\int_{(\sigma_0)}(2P\pi)^{-ns}\prod_{p=1}^{r}\frac{(i\xi_p)^{La_p}(i\xi_{p+r})^{La_{p+r}}}{(\tau_p^2+|\xi_p|^2)^{(2s+iLv_p+Ll_p)/2}}G(2s+iLv_p,Ll_p\,;x_p)$$
$$\times\prod_J\sum_{0\neq(\nu_J)\subset\mathfrak{a}_J}\left\{N(\nu_J)^{-\alpha_J-s/P_J}\prod_{p=1}^{r}|\nu_j^{(p)}|^{-iLv_p/P_J}\prod_{q=1}^{n}\left(\frac{\nu_J^{(q)}}{|\nu_J^{(q)}|}\right)^{La_q/P_J}\right\}ds.$$

Now consider

$$\sum_{b=1}^{w/d_I}\prod_{q=1}^{n}\rho^{(q)^{La_qb}}\prod_J\sum_{b=0}^{d_J-1}\prod_{q=1}^{n}\rho^{(q)^{bLa_q/P_J}}.$$

Since $d_I\mid L$, so

$$\left(\prod_{q=1}^{n}\rho^{(q)^{La_q}}\right)^{w/d_I}=1.$$

Hence

$$\sum_{b=1}^{w/d_I}\prod_{q=1}^{n}\rho^{(q)^{La_qb}}=\begin{cases} w/d_I & \text{if } \prod_{q=1}^{n}\rho^{(q)^{La_q}}=1,\\ 0 & \text{if not.}\end{cases}\tag{4.13}$$

This first case occurs if and only if

$$\left(\prod_{q=1}^{n}\rho^{(q)^{a_q}}\right)^{(w,L)}=1.$$

Assuming this condition, we consider the sum

$$\sum_{b=0}^{d_I-1}\prod_{q=1}^{n}\rho^{(q)^{La_qb/P_I}}.$$

Since (w, L) divides $d_I L/P_I$, so

$$\left(\prod_{q=1}^{n} \rho^{(q)^{La_q/P_I}}\right)^{d_I} = 1.$$

Hence

$$\sum_{b=0}^{d_I-1} \prod_{q=1}^{n} \rho^{(q)^{La_q b/P_I}} = \begin{cases} d_I & \text{if } \prod_{q=1}^{n} \rho^{(q)^{La_q/P_I}} = 1, \\ 0 & \text{if not.} \end{cases} \tag{4.14}$$

This first case occurs if and only if

$$\left(\prod_{q=1}^{n} \rho^{(q)^{a_q}}\right)^{(w, L/P_I)} = 1.$$

Under this condition we consider

$$\sum_{b=0}^{d_J-1} \prod_{q=1}^{n} \rho^{(q)^{ba_q L/P_J}} \qquad (J \neq I).$$

Since

$$\frac{L}{P_J} \frac{d_J}{(w, L/P_I)} = \frac{L(w, \phi(J))}{P_J(w, L/P_I)} = \frac{(wL, L\phi(J))}{(wP_J, LP_J/P_I)}$$

is an integer, $(w, L/P_I)$ divides Ld_J/P_J and

$$\left(\prod_{q=1}^{n} \rho^{(q)^{La_q/P_J}}\right)^{d_J} = 1.$$

Hence

$$\sum_{b=0}^{d_J-1} \prod_{q=1}^{n} \rho^{(q)^{Lba_q/P_J}} = \begin{cases} d_J & \text{if } \prod_{q=1}^{n} \rho^{(q)^{La_q/P_J}} = 1, \\ 0 & \text{if not.} \end{cases} \tag{4.15}$$

By (4.14) and (4.15), we have, under the condition

$$\prod_{q=1}^{n} \rho^{(q)^{La_q}} = 1,$$

that

$$\prod_{J} \sum_{b=0}^{d_J-1} \prod_{q=1}^{n} \rho^{(q)^{bLa_q/P_J}} \tag{4.16}$$

$$= \begin{cases} \prod_J d_J & \text{if } \prod_{q=1}^{n} \rho^{(q)^{La_q/P_J}} = 1 \quad \text{for all } J \in \mathcal{J}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $L = \mathrm{lcm}_J\{P_J\}$, we see that $\gcd_J\{L/P_J\} = 1$. Hence there exist integers $\{n_J\}_J$ such that

$$\sum_J n_J \frac{L}{P_J} = 1,$$

which shows that the first conditon of (4.16) is equivalent to a single form

$$\prod_{q=1}^{n} \rho^{(q)^{a_q}} = 1.$$

Consequently the products

$$\lambda(\nu_J) = \prod_{p=1}^{r} |\nu_J^{(p)}|^{-iv_p} \prod_{q=1}^{n} \left(\frac{\nu_J^{(q)}}{|\nu_J^{(q)}|} \right)^{a_q}$$

in the integrands of (4.12) define the Grössencharacters λ.

Thus we have from (4.12), (4.13) and (4.16)

(4.17)

$$\begin{aligned} T(\tau, \xi\,; \alpha_J, \mathfrak{a}_J) =& \frac{2^r w}{R d_I} \{P_I \prod_J \phi(J)\}^{r-1} L^r \prod_J d_J \\ & \times \sum_\lambda \frac{1}{2\pi i} \int_{(\sigma_0)} (2P\pi)^{-ns} \\ & \times \prod_{p=1}^{r} \left\{ \frac{(i\xi_p)^{La_p} (i\xi_{p+r})^{La_{p+r}}}{(\tau_p^2 + |\xi_p|^2)^{(2s+iLv_p+Ll_p)/2}} G(2s + iLv_p, Ll_p\,; x_p) \right\} \\ & \times \prod_J \zeta\left(\frac{s}{P_J} + \alpha_J, \lambda^{L/P_J}\,; \mathfrak{a}_J \right) ds, \end{aligned}$$

where the sum $\sum_\lambda$ is taken over all Grössencharacters λ.

We denote by $H_\lambda(s, \tau, \xi\,; \alpha_J, \mathfrak{a}_J)$ the integrand of (4.17):

(4.18)

$$\begin{aligned} & H_\lambda(s, \tau, \xi\,; \alpha_J, \mathfrak{a}_J) \\ & = (2P\pi)^{-ns} \prod_{p=1}^{r} \left\{ \frac{(i\xi_p)^{La_p} (i\xi_{p+r})^{La_{p+r}}}{(\tau_p^2 + |\xi_p|^2)^{(2s+iLv_p+Ll_p)/2}} G(2s + iLv_p, Ll_p\,; x_p) \right\} \\ & \qquad \times \prod_J \zeta\left(\frac{s}{P_J} + \alpha_J, \lambda^{L/P_J}\,; \mathfrak{a}_J \right) \\ & = (2P\pi)^{-ns} \prod_{p=1}^{r} \left\{ \frac{(i\xi_p)^{La_p} (i\xi_{p+r})^{La_{p+r}}}{(\tau_p^2 + |\xi_p|^2)^{(2s+iLv_p+Ll_p)/2}} \frac{\Gamma(2s + iLv_p + Ll_p)}{2^{2s+iLv_p+Ll_p} \Gamma(Ll_p + 1)} \right. \end{aligned}$$

$$\left.\times F(2s+iLv_p, Ll_p\,; x_p)\right\}$$

$$\times \prod_J \zeta\left(\frac{s}{P_J}+\alpha_J, \lambda^{L/P_J}\,; \mathfrak{a}_J\right)$$

and put

$$A = \frac{2^r w}{Rd_I}\{P_I \prod_J \phi(J)\}^{r-1} L^r \prod_J d_J.$$

In particular, we have

(4.19)

$$\begin{aligned} &H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J) \\ &= \frac{\Gamma(2s)^r}{(2^2 P\pi)^{ns}} \prod_{p=1}^{r}\left\{\frac{1}{(\tau_p^2+|\xi_p|^2)^s} F\left(s,\frac{1}{2}-s,1\,;x_p\right)\right\} \prod_J \zeta\left(\frac{s}{P_J}+\alpha_J,\mathfrak{a}_J\right). \end{aligned}$$

§6. Properties of integrands

THEOREM 6.1. (1) *If $\lambda \neq 1$, then $H_\lambda(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)$ is an entire function of s.*

(2) *If $\lambda = 1$, then $H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)$ has simple poles at*

$$s = s_J^0 = -\alpha_J P_J \qquad (J \in \mathcal{J})$$

and

$$s = s_J = (1-\alpha_J)P_J \qquad (J \in \mathcal{J}).$$

PROOF. First we shall show that

$$\prod_{p=1}^{r} \Gamma(2s+iLv_p+Ll_p) = g(s,\lambda)\prod_J \Gamma\left(\frac{s}{P_J}+\alpha_J;\lambda^{L/P_J}\right), \tag{5.1}$$

where $g(s,\lambda)$ is an entire function of s.

We write

$$\prod_J \Gamma\left(\frac{s}{P_J}+\alpha_J\,;\lambda^{L/P_J}\right) = \prod_{p=1}^{r}\prod_J \Gamma\left(\alpha_J^{(k)}+\frac{2s+iLv_p+Ll_p}{\phi_0\cdots\phi_k}\right).$$

We use a well-known formula of the gamma function;

$$\prod_{j=1}^{m} \Gamma\left(z + \frac{j-1}{m}\right) = \frac{(2\pi)^{(m-1)/2}}{m^{mz-1/2}} \Gamma(mz). \tag{5.2}$$

Since

$$\alpha_J^{(q)} = \frac{1}{\phi_q}\alpha_J^{(q-1)} + \frac{j_{q+1}-1}{\phi_q} \qquad (q = 1, \ldots, k),$$

we have, by (5.2),

$$\begin{aligned}
&\prod_J \Gamma\left(\alpha_J^{(k)} + \frac{2s + iLv_p + Ll_p}{\phi_0 \cdots \phi_k}\right) \\
&= \prod_J^{(k-1)} \prod_{j_{k+1}=1}^{\phi_k} \Gamma\left(\frac{\alpha_J^{(k-1)}}{\phi_k} + \frac{j_{k+1}-1}{\phi_k} + \frac{2s + iLv_p + Ll_p}{\phi_0 \cdots \phi_k}\right) \\
&= \prod_J^{(k-1)} \Gamma\left(\alpha_J^{(k-1)} + \frac{2s + iLv_p + Ll_p}{\phi_0 \cdots \phi_{k-1}}\right) \\
&\times \prod_J^{(k-1)} \left\{(2\pi)^{(\phi_k-1)/2}(\phi_k)^{1/2-\alpha_J^{(k-1)}-(2s+iLv_p+Ll_p)/(\phi_0\cdots\phi_{k-1})}\right\}.
\end{aligned}$$

Putting

$$P_a = \prod_J^{(a-1)} (2\pi)^{(\phi_a-1)/2} \qquad (a = 1, \ldots, k),$$

$$N_a = \prod_J^{(a-1)} (\phi_a)^{-1+2\alpha_J^{(a-1)}} \qquad (a = 1, \ldots, k) \qquad ((1.7))$$

and using the induction and (1.4), we have

$$\begin{aligned}
&\prod_J \Gamma\left(\alpha_J^{(k)} + \frac{2s + iLv_p + Ll_p}{\phi_0 \cdots \phi_k}\right) \\
&= P_k N_k^{-1/2} M_k^{-(2s+iLv_p+Ll_p)} \\
&\quad \times \prod_J^{(k-1)} \Gamma\left(\alpha_J^{(k-1)} + \frac{2s + iLv_p + Ll_p}{\phi_0 \cdots \phi_{k-1}}\right) \\
&= P_k \cdots P_1 (N_k \cdots N_1)^{-1/2} (M_k \cdots M_1)^{-(2s+iLv_p+Ll_p)} \\
&\quad \times \prod_J^{(0)} \Gamma\left(\alpha_J^{(0)} + \frac{2s + iLv_p + Ll_p}{\phi_0}\right)
\end{aligned}$$

$$
\begin{aligned}
&= (2\pi)^{l-1/2}(2l)^{1/2-(2s+iLv_p+Ll_p)} \\
&\quad \times P_k \cdots P_1 (N_k \cdots N_1)^{-1/2}(M_k \cdots M_1)^{-(2s+iLv_p+Ll_p)} \\
&\quad \times \Gamma(2s+iLv_p+Ll_p),
\end{aligned}
$$

which gives (5.1) with

$$
\begin{aligned}
&g(s,\lambda) \\
&= \prod_{p=1}^{r} \Big\{ (2\pi)^{1/2-l}(2l)^{-1/2+2s+iLv_p+Ll_p} \\
&\qquad \times (P_k \cdots P_1)^{-1}(N_k \cdots N_1)^{1/2}(M_k \cdots M_1)^{2s+iLv_p+Ll_p} \Big\} \\
&= (2\pi)^{r/2-lr}(2l)^{-r/2+2rs}(P_k \cdots P_1)^{-r}(N_k \cdots N_1)^{r/2}(M_k \cdots M_1)^{2rs} \\
&\quad \times \prod_{p=1}^{r} (2lM_k \cdots M_1)^{Ll_p}.
\end{aligned}
$$

Consequently we have

$$
\begin{aligned}
(4.3) \qquad & H_\lambda(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J) \\
&= g(s,\lambda)(2P\pi)^{-ns} \\
&\quad \times \prod_{p=1}^{r} \left\{ \frac{(i\xi_p)^{La_p}(i\xi_{p+r})^{La_{p+r}}}{(\tau_p^2+|\xi_p|^2)^{(2s+iLv_p+Ll_p)/2}} \frac{F(2s+iLv_p, Ll_p\,; x_p)}{2^{2s+iLv_p+Ll_p}\Gamma(Ll_p+1)} \right\} \\
&\quad \times \prod_{J} \left\{ \Gamma\left(\frac{s}{P_J}+\alpha_J\,;\lambda^{L/P_J}\right) \zeta\left(\frac{s}{P_J}+\alpha_J, \lambda^{L/P_J}\,;\mathfrak{a}_J\right) \right\}.
\end{aligned}
$$

According to Theorem 3.1, this expression (5.3) shows that, if $\lambda \neq 1$, then $H_\lambda(s,\tau,\xi\,;\alpha_J, \mathfrak{a}_J)$ is an entire function of s, and if $\lambda = 1$, then $H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)$ has simple poles at

$$
s = s_J^0 = -\alpha_J P_J \qquad (J \in \mathcal{J})
$$

and

$$
s = s_J = (1-\alpha_J)P_J \qquad (J \in \mathcal{J}).
$$

Thus the proof is completed. $\square$

Similarly to (5.1), we obtain

$$\prod_J \Gamma\left(1-\alpha_J-\frac{s}{P_J};\bar{\lambda}^{L/P_J}\right)=g^*(s,\lambda)\prod_{p=1}^{r}\Gamma(1-2s-iLv_p+Ll_p),$$

where

$$\begin{aligned}(5.4)\qquad & g^*(s,\lambda)\\ &=\prod_{p=1}^{r}\Big\{(2\pi)^{l-1/2}(2l)^{-1/2+2s+iLv_p-Ll_p}\\ &\qquad\times P_k\cdots P_1(N_k\cdots N_1)^{1/2}(M_k\cdots M_1)^{2s+iLv_p-Ll_p}\Big\}\\ &=(2\pi)^{lr+r/2}(2l)^{-r/2+2rs}(P_k\cdots P_1)^r(N_k\cdots N_1)^{r/2}(M_k\cdots M_1)^{2rs}\\ &\qquad\times\prod_{p=1}^{r}(2lM_k\cdots M_1)^{-Ll_p}.\end{aligned}$$

THEOREM 6.2. *$H_\lambda(s,\tau,\xi;\alpha_J,\mathfrak{a}_J)$ satisfies the functional equation as follows:*

$$\begin{aligned}&H_\lambda(s,\tau,\xi;\alpha_J,\mathfrak{a}_J)\\ &=\frac{(N_1\cdots N_k)^r(M_1\cdots M_k)^r}{D^{N/2}\prod_J N(\mathfrak{a}_J)}\prod_{p=1}^{r}\frac{1}{(\tau_p^2+|\xi_p|^2)^{1/2}}\cdot H_{\bar{\lambda}}(1/2-s,\tau^*,\xi^*;\alpha_J^*,\mathfrak{a}_J^*).\end{aligned}$$

PROOF. By the functional equations (2.2) of zeta functions, we have

$$\begin{aligned}(5.5)\qquad &\prod_J\left\{\Gamma\left(\frac{s}{P_J}+\alpha_J;\lambda^{L/P_J}\right)\zeta\left(\frac{s}{P_J}+\alpha_J,\lambda^{L/P_J};\mathfrak{a}_J\right)\right\}\\ &=\prod_J\frac{(2\pi)^{r\{2(\alpha_J+s/P_J)-1\}}}{N(\mathfrak{a}_J)\sqrt{D}}\\ &\quad\times\prod_J\left\{\Gamma\left(1-\alpha_J-\frac{s}{P_J};\bar{\lambda}^{L/P_J}\right)\zeta\left(1-\alpha_J-\frac{s}{P_J},\bar{\lambda}^{L/P_J};\mathfrak{a}_J^*\right)\right\}.\end{aligned}$$

We shall show that

$$(5.6)\qquad \sum_J\alpha_J=\frac{1}{2}(N-1),$$

where $N = \sum_J 1$. In fact,

$$\begin{aligned}\sum_J \alpha_J &= \sum_J \sum_{a=0}^{k} \frac{j_{a+1}-1}{\phi_a \cdots \phi_k} = \sum_{a=0}^{k} \sum_J \frac{j_{a+1}-1}{\phi_a \cdots \phi_k} \\ &= \sum_{a=0}^{k} \sum_J^{(a)} \frac{j_{a+1}-1}{\phi_a} = \sum_{j_1=1}^{\phi_0} \frac{j_1-1}{\phi_0} + \sum_{a=1}^{k} \sum_J^{(a-1)} \frac{\phi_a - 1}{2} \\ &= \frac{1}{2}(\phi_0 - 1) + \frac{1}{2}\sum_{a=1}^{k}\left(\sum_J^{(a)} 1 - \sum_J^{(a-1)} 1\right) = \frac{1}{2}(N-1).\end{aligned}$$

Further we have

$$\sum_J \frac{1}{P_J} = \sum_J \frac{2}{\phi_0 \cdots \phi_k} = 2. \tag{5.7}$$

By (5.6) and (5.7)

$$\prod_J \frac{(2\pi)^{r\{2(\alpha_J + s/P_J)-1\}}}{N(\mathfrak{a}_J)\sqrt{D}} = \frac{(2\pi)^{4rs-r}}{D^{N/2}\prod_J N(\mathfrak{a}_J)}. \tag{5.8}$$

By (5.8), (5.4) and (5.5),

$$\begin{aligned}&\prod_J \left\{ \Gamma\left(\frac{s}{P_J} + \alpha_J ; \lambda^{L/P_J}\right) \zeta\left(\frac{s}{P_J} + \alpha_J, \lambda^{L/P_J} ; \mathfrak{a}_J\right)\right\} \\ &= \frac{(2\pi)^{4rs-r}}{D^{N/2}\prod_J N(\mathfrak{a}_J)} g^*(s,\lambda) \\ &\times \prod_{p=1}^{r} \Gamma(1 - 2s - iLv_p + Ll_p) \prod_J \zeta\left(1 - \frac{s}{P_J} - \alpha_J, \bar{\lambda}^{L/P_J} ; \mathfrak{a}_J^*\right).\end{aligned} \tag{5.9}$$

Putting (5.9) into (5.3),

$$\begin{aligned}&H_\lambda(s,\tau,\xi ; \alpha_J, \mathfrak{a}_J) \\ &= \frac{(2\pi)^{4rs-r}}{D^{N/2}\prod_J N(\mathfrak{a}_J)} g(s,\lambda) g^*(s,\lambda)(2P\pi)^{-ns} \\ &\quad \times \prod_{p=1}^{r} \left\{ \frac{(i\xi_p)^{La_p}(i\xi_{p+r})^{La_{p+r}}}{(\tau_p^2 + |\xi_p|^2)^{(2s+iLv_p+Ll_p)/2}} \right. \\ &\quad \left. \times \frac{\Gamma(1-2s-iLv_p+Ll_p)}{2^{2s+iLv_p+Ll_p}\Gamma(Ll_p+1)} F(2s+iLv_p, Ll_p ; x_p) \right\}\end{aligned}$$

$$\times \prod_J \zeta\Big(1 - \frac{s}{P_J} - \alpha_J, \bar{\lambda}^{L/P_J}\, ; \mathfrak{a}_J^*\Big)$$
$$= \frac{(N_1 \cdots N_k)^r}{D^{N/2} \prod_J N(\mathfrak{a}_J)} P^r 2^{2ns-r} l^{-r} (2P\pi)^{ns-r}$$
$$\times \prod_{p=1}^r \left\{ \frac{(i\xi_p)^{La_p} (i\xi_{p+r})^{La_{p+r}}}{(\tau_p^2 + |\xi_p|^2)^{(2s+iLv_p+Ll_p)/2}} \right.$$
$$\left. \times \frac{\Gamma(1-2s-iLv_p+Ll_p)}{2^{2s+iLv_p+Ll_p}\Gamma(Ll_p+1)} F(2s+iLv_p, Ll_p\, ; x_p) \right\}$$
$$\times \prod_J \zeta\Big(1 - \frac{s}{P_J} - \alpha_J, \bar{\lambda}^{L/P_J}\, ; \mathfrak{a}_J^*\Big).$$

Let $\tau_1^*, \ldots, \tau_r^*$ and $\xi_1^*, \ldots, \xi_r^*$ be the numbers defined by (1.8), then we easily see that

$$\tau_p = \frac{\tau_p^*}{\tau_p^{*2} + |\xi_p^*|^2} \qquad (p = 1, \ldots, r),$$
$$\xi_p = \frac{\xi_{p+r}^*}{\tau_p^{*2} + |\xi_p^*|^2}, \quad \xi_{p+r} = \frac{\xi_p^*}{\tau_p^{*2} + |\xi_p^*|^2} \quad (p = 1, \ldots, r),$$

$$\tau_p^{*2} + |\xi_p^*|^2 = \frac{1}{\tau_p^2 + |\xi_p|^2} \qquad (p = 1, \ldots, r),$$
$$x_p = \frac{|\xi_p|^2}{\tau_p^2 + |\xi_p|^2} = \frac{|\xi_p^*|^2}{\tau_p^{*2} + |\xi_p^*|^2} = x_p^* \qquad (p = 1, \ldots, r),$$

and

$$\frac{(i\xi_p)^{La_p} (i\xi_{p+r})^{La_{p+r}}}{(\tau_p^2 + |\xi_p|^2)^{(2s+iLv_p+Ll_p)/2}} = \frac{(i\xi_{p+r}^*)^{La_p} (i\xi_p^*)^{La_{p+r}}}{(\tau_p^{*2} + |\xi_p^*|^2)^{(-2s-iLv_p+Ll_p)/2}} \qquad (p = 1, \ldots, r). \tag{5.10}$$

Let α_J^* $(J \in \mathcal{J})$ be the numbers defined by (1.3), then

$$1 - \frac{s}{P_J} - \alpha_J = \frac{1}{P_J}\left(\frac{1}{2} - s\right) + \alpha_J^* \qquad (J \in \mathcal{J}). \tag{5.11}$$

Using (5.10), (5.11) and (1.5), we write

$$H_\lambda(s, \tau, \xi\, ; \alpha_J, \mathfrak{a}_J) \tag{5.12}$$
$$= \frac{(N_1 \cdots N_k)^r}{D^{N/2} \prod_J N(\mathfrak{a}_J)} P^r 2^{2ns-r} l^{-r} (2P\pi)^{ns-r}$$

$$
\begin{aligned}
&\times \prod_{p=1}^{r}\Big\{\frac{(i\xi_{p+r}^*)^{La_p}(i\xi_p^*)^{La_{p+r}}}{(\tau_p^{*2}+|\xi_p^*|^2)^{(-2s-iLv_p+Ll_p)/2}} \\
&\quad\times \frac{\Gamma(1-2s-iLv_p+Ll_p)}{2^{2s+iLv_p+Ll_p}\Gamma(Ll_p+1)}F(1-2s-iLv_p, Ll_p\,; x_p^*)\Big\} \\
&\times \prod_J \zeta\Big(\frac{1/2-s}{P_J}+\alpha_J^*, \bar{\lambda}^{L/P_J}\,; \mathfrak{a}_J^*\Big) \\
&= \frac{(N_1\cdots N_k)^r(M_1\cdots M_k)^r}{D^{N/2}\prod_J N(\mathfrak{a}_J)}\prod_{p=1}^{r}\frac{1}{(\tau_p^2+|\xi_p|^2)^{1/2}} \\
&\times (2P\pi)^{-n(1/2-s)}\prod_{p=1}^{r}\Big\{\frac{(i\xi_{p+r}^*)^{La_p}(i\xi_p^*)^{La_{p+r}}}{(\tau_p^{*2}+|\xi_p^*|^2)^{(1-2s-iLv_p+Ll_p)/2}} \\
&\times \frac{\Gamma(1-2s-iLv_p+Ll_p)}{2^{1-2s-iLv_p+Ll_p}\Gamma(Ll_p+1)}F(1-2s-iLv_p, Ll_p\,; x_p^*)\Big\} \\
&\times \prod_J \zeta\Big(\frac{1/2-s}{P_J}+\alpha_J^*, \bar{\lambda}^{L/P_J}\,; \mathfrak{a}_J^*\Big).
\end{aligned}
$$

Comparing the last expression of (5.12) with (4.18), we have

$$
\begin{aligned}
&H_\lambda(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J) \\
&\quad= \frac{(N_1\cdots N_k)^r(M_1\cdots M_k)^r}{D^{N/2}\prod_J N(\mathfrak{a}_J)}\prod_{p=1}^{r}\frac{1}{(\tau_p^2+|\xi_p|^2)^{1/2}} \\
&\quad\times H_{\bar{\lambda}}(1/2-s,\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*).
\end{aligned}
\tag{5.13}
$$

□

Let N_a^* $(a=1,\dots,k)$ be the numbers defined by (1.8), then

$$
N_aN_a^*M_a^2=1 \qquad (a=1,\dots,k).
\tag{5.14}
$$

In fact,

$$
N_aN_a^*M_a^2 = \prod_J^{(a-1)} (\phi_a)^{-1+2\alpha_J^{(a-1)}-1+2\alpha_J^{*(a-1)}+2/\phi_0\cdots\phi_{a-1}}
$$

and

$$
\begin{aligned}
&-1+2\alpha_J^{(a-1)}-1+2\alpha_J^{*(a-1)}+\frac{2}{\phi_0\cdots\phi_{a-1}}\\
&=-2+2\left(\frac{j_1-1}{\phi_0\cdots\phi_{a-1}}+\cdots+\frac{j_a-1}{\phi_{a-1}}\right)+2\left(\frac{\phi_0-j_1}{\phi_0\cdots\phi_{a-1}}+\cdots+\frac{\phi_{a-1}-j_a}{\phi_{a-1}}\right)\\
&\quad+\frac{2}{\phi_0\cdots\phi_{a-1}}=0,
\end{aligned}
$$

which shows that

$$N_a N_a^* M_a^2 = 1.$$

We now put

$$K(\tau,\xi\,;\alpha_J,\mathfrak{a}_J)=\frac{D^{N/2}\prod_J N(\mathfrak{a}_J)}{(N_1\cdots N_k)^r(M_1\cdots M_k)^r}\prod_{p=1}^{r}(\tau_p^2+|\xi_p|^2)^{1/2} \tag{5.15}$$

and

$$K(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)=\frac{D^{N/2}\prod_J N(\mathfrak{a}_J^*)}{(N_1^*\cdots N_k^*)^r(M_1\cdots M_k)^r}\prod_{p=1}^{r}(\tau_p^{*2}+|\xi_p^*|^2)^{1/2}. \tag{5.16}$$

Then (5.14) gives that

$$K(\tau,\xi\,;\alpha_J,\mathfrak{a}_J)K(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)=1 \tag{5.17}$$

and (5.13) is written as follows:

$$H_\lambda(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)=K(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)H_{\bar\lambda}(1/2-s,\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*). \tag{5.18}$$

THEOREM 6.3. *In the strip $|\sigma|\le 1+\max_{J\in\mathcal{J}}\{P_J\}$, we have*

$$H_\lambda(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\ll e^{-c|t|}, \tag{5.19}$$

where $c>0$ and the constants in this estimation depend on $\tau,\xi,\alpha_J,\mathfrak{a}_J$ and λ.

PROOF. According to (5.18), it is sufficient to prove Theorem under the condition $0 \le \sigma \le 1 + \max_{J \in \mathcal{J}}\{P_J\}$. From (4.18) we have

$$\begin{aligned} &H_\lambda(s, \tau, \xi\,; \alpha_J, \mathfrak{a}_J) \\ &\ll \left| \prod_{p=1}^{r} G(2s + iLv_p, Ll_p\,; x_p) \prod_J \zeta\left(\frac{s}{P_J} + \alpha_J, \lambda^{L/P_J}\,; \mathfrak{a}_J\right) \right|. \end{aligned}$$

Theorem 4.2 and Theorem 3.2 give

$$\begin{aligned} G(2s + iLv_p, Ll_p\,; x_p) &\ll (1 + Ll_p + |\,2t + Lv_p\,|)^{2\sigma - 1/2} e^{-c|2t + Lv_p|} \\ &\ll e^{-c|t|}. \end{aligned}$$

and

$$\zeta\left(\frac{s}{P_J} + \alpha_J, \lambda^{L/P_J}\,; \mathfrak{a}_J\right) \ll (1 + |t|)^{2n},$$

respectively. Hence (5.19) is obtained at once. □

§7. Functional equation and residues

Theoem 6.3 shows that

$$\int_{\sigma_1 + iT}^{\sigma_0 + iT} H_\lambda(s, \tau, \xi\,; \alpha_J, \mathfrak{a}_J)\, ds \longrightarrow 0 \qquad (|T| \longrightarrow \infty),$$

where

$$\sigma_1 = -\max_J\{(1 - \alpha_J^*)P_J\} - \frac{1}{2}.$$

and the integral is taken along the horizontal line from $\sigma_1 + iT$ to $\sigma_0 + iT$.

Hence, in the expression of $T(\tau, \xi\,; \alpha_J, \mathfrak{a}_J)$ obtained in (4.17), we can shift the path of integraion to the vertical line $\sigma = \sigma_1$.

If $\lambda \ne 1$, then

$$(6.1) \qquad \frac{1}{2\pi i}\int_{(\sigma_0)} H_\lambda(s, \tau, \xi\,; \alpha_J, \mathfrak{a}_J)\, ds = \frac{1}{2\pi i}\int_{(\sigma_1)} H_\lambda(s, \tau, \xi\,; \alpha_J, \mathfrak{a}_J)\, ds.$$

By (5.18),

$$(6.2)\qquad \begin{aligned}&\frac{1}{2\pi i}\int_{(\sigma_1)} H_\lambda(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\,ds\\ &\quad= K(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\frac{1}{2\pi i}\int_{(\sigma_1)} H_{\bar\lambda}(1/2-s,\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\,ds\\ &\quad= K(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\frac{1}{2\pi i}\int_{(\sigma_0^*)} H_{\bar\lambda}(s,\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\,ds,\end{aligned}$$

where

$$\sigma_0^* = \max_J\{(1-\alpha_J^*)P_J\}+1.$$

If $\lambda = 1$, we have similarly

$$(6.3)\qquad \begin{aligned}&\frac{1}{2\pi i}\int_{(\sigma_0)} H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\,ds\\ &\quad= \frac{1}{2\pi i}\int_{(\sigma_1)} H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\,ds + R(\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\\ &\quad= K(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\frac{1}{2\pi i}\int_{(\sigma_0^*)} H_1(s,\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\,ds + R(\tau,\xi\,;\alpha_J,\mathfrak{a}_J),\end{aligned}$$

where

$$(6.4)\qquad R(\tau,\xi\,;\alpha_J,\mathfrak{a}_J) = \sum_{\sigma_1\le\sigma\le\sigma_0}\operatorname*{Res}_{\sigma} H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J).$$

Hence by (6.1), (6.2) and (6.3),

$$(6.5)\qquad \begin{aligned}&T(\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\\ &\quad= A\sum_\lambda \frac{1}{2\pi i}\int_{(\sigma_0)} H_\lambda(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\,ds\\ &\quad= AK(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\sum_\lambda \frac{1}{2\pi i}\int_{(\sigma_0^*)} H_\lambda(s,\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\,ds\\ &\qquad\qquad + AR(\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\\ &\quad= K(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)T(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*) + AR(\tau,\xi\,;\alpha_J,\mathfrak{a}_J).\end{aligned}$$

We divide the sum (6.4) into two parts:

$$(6.6)\qquad R(\tau,\xi\,;\alpha_J,\mathfrak{a}_J) = \sum_J \operatorname*{Res}_{s=s_J} H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J) + \sum_J \operatorname*{Res}_{s=s_J^0} H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J).$$

Here we see

$$\begin{aligned}(6.7)\qquad &\operatorname*{Res}_{s=s_J^0} H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\\ &= \lim_{s\to s_J^0}(s-s_J^0)H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\\ &= K(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\lim_{s\to s_J^0}(s-s_J^0)H_1(1/2-s,\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*).\end{aligned}$$

If we put

$$s_J^* = (1-\alpha_J^*)P_J \qquad ((1.10)),$$

then

$$\frac{1}{2} - s_J^0 = s_J^*$$

and so (6.7) is

$$\begin{aligned}(6.8)\qquad &\operatorname*{Res}_{s=s_J^0} H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\\ &= -K(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\lim_{s\to s_J^*}(s-s_J^*)H_1(s,\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\\ &= -K(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\operatorname*{Res}_{s=s_J^*} H_1(s,\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*).\end{aligned}$$

Thus we have by (6.6) and (6.8)

$$\begin{aligned}(6.9)\qquad R(\tau,\xi\,;\alpha_J,\mathfrak{a}_J) &= \sum_J \operatorname*{Res}_{s=s_J} H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\\ &\quad - K(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\sum_J \operatorname*{Res}_{s=s_J^*} H_1(s,\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*).\end{aligned}$$

From the results (6.5) and (6.10), we have

(6.10)

$$\begin{aligned}T(\tau,\xi\,;\alpha_J,\mathfrak{a}_J) &= K(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)T(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\\ &\quad + A\sum_J \operatorname*{Res}_{s=s_J} H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\\ &\quad - AK(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\sum_J \operatorname*{Res}_{s=s_J^*} H_1(s,\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*).\end{aligned}$$

Further by (5.17), we write (6.10) in a symmetric form:

(6.11)

$$K(\tau,\xi\,;\alpha_J,\mathfrak{a}_J)^{1/2}\left\{T(\tau,\xi\,;\alpha_J,\mathfrak{a}_J)-A\sum_J \operatorname*{Res}_{s=s_J} H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\right\}$$
$$=K(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)^{1/2}\left\{T(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)-A\sum_J \operatorname*{Res}_{s=s_J^*} H_1(s,\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)\right\}.$$

Putting

$$\frac{s}{P_J}+\alpha_J=1+\frac{s-s_J}{P_J},$$

we have

$$\operatorname*{Res}_{s=s_J}\zeta\left(1+\frac{s-s_J}{P_J},\mathfrak{a}_J\right)=\frac{P_J(2\pi)^rR}{wN(\mathfrak{a}_J)\sqrt{D}}.$$

Hence we obtain from (4.19)

$$\begin{aligned}&\operatorname*{Res}_{s=s_J} H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J)\\&=\frac{(2\pi)^rR}{w\sqrt{D}}\frac{P_J\Gamma(2s_J)^r}{N(\mathfrak{a}_J)(2^2P\pi)^{ns_J}}\\&\times\prod_{p=1}^{r}\frac{1}{(\tau_p^2+|\xi_p|^2)^{s_J}}F\left(s_J,\frac{1}{2}-s_J,1\,;\frac{|\xi_p|^2}{\tau_p^2+|\xi_p|^2}\right)\\&\times\prod_{\substack{J_1\in\mathcal{J}\\(J_1\neq J)}}\zeta\left(1+\frac{s_J-s_{J_1}}{P_{J_1}},\mathfrak{a}_{J_1}\right).\end{aligned}$$

and by the definition (1.11) of $S(\tau,\xi\,;\alpha_J,\mathfrak{a}_J)$, we have

$$S(\tau,\xi\,;\alpha_J,\mathfrak{a}_J)=A\sum_J \operatorname*{Res}_{s=s_J} H_1(s,\tau,\xi\,;\alpha_J,\mathfrak{a}_J). \tag{6.12}$$

Similarly

$$S(\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*)=A\sum_J \operatorname*{Res}_{s=s_J^*} H_1(s,\tau^*,\xi^*\,;\alpha_J^*,\mathfrak{a}_J^*). \tag{6.13}$$

Collecting the results (6.11), (6.12) and (6.13) and noting (5.15) and (5.16), we finall obtain Main Theorem.

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