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Game theoretic properties of the club filter on $\mathcal{P}_\kappa\lambda$

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Let κ be a regular cardinal $> \omega$, λ a cardinal $\geq \kappa$ and F a filter on $\mathcal{P}_\kappa\lambda$. Jech and Prikry [JP] called F precipitous when the poset F^+ ordered by inclusion forces that the generic ultrapower is well-founded. Refining an ingenious idea of Foreman, Magidor and Shelah [FMS], Goldring [G] and Shelah independently established that the club filter $\mathcal{C}_{\kappa\lambda}$ on $\mathcal{P}_\kappa\lambda$ can be made precipitous by Levy collapsing a Woodin cardinal to the successor of λ when λ is regular. In [MS] we showed that $\mathcal{C}_{\kappa\lambda}$ cannot be precipitous e.g. when λ is strong limit and of cofinality $< \kappa$. On the other hand Galvin, Jech and Magidor [GJM] observed that precipitousness is also definable in terms of games (see below). This led us to investigate game theoretic properties of $\mathcal{C}_{\kappa\lambda}$ in this paper.

We refer to Kanamori [K] for basic material. We use μ to denote a regular cardinal. By S^ω_κ and $\lim X$ for $X \subset \kappa$ we denote the set $\{\alpha < \kappa : \operatorname{cf} \alpha = \omega\}$ and $\{\alpha < \kappa : \sup(X \cap \alpha) = \alpha > 0\}$ respectively. For $\alpha < \lambda$ we fix a bijection $\pi_\alpha : |\alpha| \to \alpha$. Suppose that two players I and II take in turn $S_n \in F^+$ and $T_n \in F^+$ respectively so that $S_n \supset T_n \supset S_{n+1}$. We define the game $\mathcal{G}_0(F)$ by: I wins iff $\bigcap_{n<\omega} S_n = \emptyset$. Recall from [GJM] that F is precipitous iff I has no winning

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strategy in $\mathcal{G}_0(F)$. Also the game $\mathcal{G}_1(F)$ (resp. $\mathcal{G}_*(F)$) is defined by: I wins iff $|\bigcap_{n<\omega} S_n| \leq 1$ (resp. $\bigcap_{n<\omega} S_n \in F^*$). As observed by Jech [J], F^+ is ω_1-Baire iff I has no winning strategy in $\mathcal{G}_*(F)$. A winning strategy for I in $\mathcal{G}_1(\mathcal{C}_\kappa)$ from [GJM] yields the following

Proposition 1. *$C^+_{\kappa\lambda}$ is not ω_1-Baire.*

Proof. We give a winning strategy for I in $\mathcal{G}_*(\mathcal{C}_{\kappa\lambda})$. We let $S_0 = \{x \in \mathcal{P}_\kappa\lambda : x \cap \kappa \in S^\omega_\kappa\}$ be the first move of I, and for $x \in S_0$ fix an unbounded set $\{\alpha^x_n : n < \omega\} \subset x \cap \kappa$. Next suppose that T is an n-th move of II. Then by the regressive function lemma we have $\gamma_n < \kappa$ such that $\{x \in T : \alpha^x_n = \gamma_n\}$ is stationary, which we let be the next move of I.

Let $\langle S_n : n < \omega\rangle$ be a play of I following the above strategy with $\langle \gamma_n : n < \omega\rangle$ as above. Then $\bigcap_{n<\omega} S_n \subset \{x \in \mathcal{P}_\kappa\lambda : x \cap \kappa = \sup_{n<\omega} \gamma_n\} \in C^*_{\kappa\lambda}$, as desired. $\square$

The following lemma is proved similarly to the original version in [GJM].

Lemma 1. *Let $S \subset \{x \in \mathcal{P}_\kappa\lambda : x \cap \kappa \in S^\omega_\kappa\}$ be stationary and $\{\alpha^x_n : n < \omega\}$ unbounded in $x \cap \kappa$ for $x \in S$. Then for some $m < \omega$ $\{\gamma < \kappa : \{x \in S : \alpha^x_m = \gamma\}$ is stationary$\}$ is unbounded.*

Now we have a $\mathcal{P}_\kappa\lambda$ generalization of the Galvin-Jech-Magidor result [GJM], which shows that Goldring's result is sharp from a game theoretic point of view:

Proposition 2. II *has no winning strategy in $\mathcal{G}_0(\mathcal{C}_{\kappa\lambda})$.*

Proof. Let τ be a strategy for II. We construct a play in which II follows τ and I wins. Build inductively $\langle S_t : t \in \mathcal{T}\rangle$ with $\mathcal{T}$ a subtree of $\kappa^{<\omega}$ such that for any $t \in \mathcal{T}$ $\mathrm{suc}_\mathcal{T}(t) = \{\gamma < \kappa : t * \langle\gamma\rangle \in \mathcal{T}\}$ is unbounded and $\langle S_{t|i} : i < |t|\rangle$ is a partial play of I with which II follows τ as follows: First set $S_\emptyset = \{x \in \mathcal{P}_\kappa\lambda : x \cap \kappa \in S\}$, where $S \subset S^\omega_\kappa$ and $S^\omega_\kappa - S$ are both stationary, and for $x \in S_\emptyset$ fix an unbounded set $\{\alpha^x_n : n < \omega\} \subset x \cap \kappa$. Next suppose that $\langle S_t : t \in \mathcal{T} \cap \kappa^n\rangle$ is defined. Fix $t \in \mathcal{T} \cap \kappa^n$. Let T be the n-th move of II following τ when I plays $\langle S_{t|i} : i < n\rangle$. Then by the above lemma take $m < \omega$ such that $\{\gamma < \kappa : \{x \in T : \alpha^x_m = \gamma\}$ is stationary$\}$ is unbounded, which we let be $\mathrm{suc}_{\mathcal{T}}(t)$. For $\gamma \in \mathrm{suc}_{\mathcal{T}}(t)$ set $S_{t*\langle\gamma\rangle} = \{x \in T : \alpha^x_m = \gamma\}$. Fix $\gamma \in S^\omega_\kappa - S$ with $\gamma \in \lim \mathrm{suc}_{\mathcal{T}}(t)$ for any $t \in \mathcal{T} \cap \gamma^{<\omega}$, and an unbounded set $\{\gamma_n : n < \omega\} \subset \gamma$. Build inductively a branch $b : \omega \to \gamma$ through $\mathcal{T}$ with $\gamma_n \leq b(n)$ for any $n < \omega$. Then $\bigcap_{n<\omega} S_{b|n} \subset \{x \in S_\emptyset : x \cap \kappa = \gamma\} = \emptyset$, as desired. □

For the rest of the paper we give some cases where I has a winning strategy in the game $\mathcal{G}_1(\mathcal{C}_{\kappa\lambda})$, and show optimality of Goldring's result in a different way. Let us begin by the following observation, which should be folklore like non-ω_1-Baireness of F^+ for a fine filter F on $\mathcal{P}_{\omega_1}\lambda$:

Proposition 3. *Let F be a normal filter on $\mathcal{P}_{\omega_1}\lambda$. Then* I *has a winning strategy in $\mathcal{G}_1(F)$.*

Proof. We let $\mathcal{P}_{\omega_1}\lambda$ be the first move of I, and for $x \in \mathcal{P}_{\omega_1}\lambda$ fix a list $\{\alpha^x_n : n < \omega\}$ of x. Next suppose that $T \in F^+$ is an n-th move of II. Then by the normality of F we have $\gamma_n < \lambda$ with $\{x \in T : \alpha^x_n = \gamma_n\} \in F^+$, which we let be the next move of I.

Let $\langle S_n : n < \omega\rangle$ be a play of I following the above strategy with $\langle \gamma_n : n < \omega\rangle$ as above. Then $\bigcap_{n<\omega} S_n = \{\gamma_n\}$, as desired. □

Our first result, Proposition 4 follows from Proposition 3 and our Theorem below. We nevertheless present it here because of the following lemma, which refines a fact from [BT] and seems of independent interest.

Lemma 2. *Let $\lambda < \kappa^{+\omega}$. Then there is a stationary set $S \subset \{x \in \mathcal{P}_\kappa\lambda : \kappa \le \forall\mu \le \lambda(\text{cf } \sup(x \cap \mu) = \omega)\}$ on which the map $x \mapsto \langle \sup(x \cap \mu) : \kappa \le \mu \le \lambda\rangle$ is injective.*

Proof. We go by induction on $\lambda < \kappa^{+\omega}$. For $\lambda = \kappa$ just note that S^ω_κ is stationary in $\mathcal{P}_\kappa\kappa$. Next suppose that $S \subset \mathcal{P}_\kappa\lambda$ is as above. Then for $\gamma \in S^\omega_{\lambda^+} - \lambda$ $S_\gamma = \{x \in \mathcal{P}_\kappa\gamma : \pi_\gamma^{-1}x = x \cap \lambda \in S \wedge \sup x = \gamma\}$ is stationary, and for any $x, y \in S_\gamma$ with $\langle \sup(x \cap \mu) : \kappa \le \mu \le \lambda\rangle = \langle \sup(y \cap \mu) : \kappa \le \mu \le \lambda\rangle$ we have $x = \pi_\gamma``(x \cap \lambda) = \pi_\gamma``(y \cap \lambda) = y$, since $x \cap \lambda = y \cap \lambda \in S$. We claim that $\bigcup\{S_\gamma : \gamma \in S^\omega_{\lambda^+} - \lambda\}$ is stationary in $\mathcal{P}_\kappa\lambda^+$. Fix $f : (\lambda^+)^{<\omega} \to \lambda^+$. Take $\gamma \in S^\omega_{\lambda^+} - \lambda$ with $f``\gamma^{<\omega} \subset \gamma$. Then by stationarity of S_γ some $x \in S_\gamma$ with $x \cap \kappa \in \kappa$ is closed under f, as desired. □

Now the rest of the proof goes as in Proposition 1:

Proposition 4. I *has a winning strategy in $\mathcal{G}_1(\mathcal{C}_{\kappa\lambda})$ when $\lambda < \kappa^{+\omega}$.*

Proof. We let the stationary set as in Lemma 2 be the first move S_0 of I, and for $x \in S_0$ and a cardinal $\kappa \le \mu \le \lambda$ fix an unbounded set $\{\alpha^x_{\mu,n} : n < \omega\} \subset x \cap \mu$. Next suppose that T is an n-th move of II. Then by finite applications of the regressive function lemma we have $\langle \gamma_{\mu,n} : \kappa \le \mu \le \lambda\rangle$ such that $\{x \in T : \kappa \le \forall\mu \le \lambda(\alpha^x_{\mu,n} = \gamma_{\mu,n})\}$ is stationary, which we let be the next move of I.

Let $\langle S_n : n < \omega\rangle$ be a play of I following the above strategy with $\langle \gamma_{\mu,n} : n < \omega\rangle$ as above. Then $\bigcap_{n<\omega} S_n \subset \{x \in S_0 : \kappa \le \forall\mu \le \lambda(\sup(x \cap \mu) = \sup_{n<\omega} \gamma_{\mu,n})\}$, which is at most a singleton, as desired. □

For our main result below, we are indebted to Shelah's notion [S] of internally approachable submodels of H_θ (see also [BM]), as well as Baumgartner's style [B] of purely combinatorial proofs.

Theorem. I *has a winning strategy in* $\mathcal{G}_1(\mathcal{C}_{\kappa\lambda})$ *when* $\kappa > \omega_1$ *and* $\lambda < \kappa^{+\omega_1}$.

Proof. Let $\{\mu_i : i < \omega\}$ list (not necessarily injectively) the set of regular cardinals between κ and λ. First we claim that $S_0 = \{\bigcup_{n<\omega} x_n : \forall n < \omega(x_n \in \mathcal{P}_\kappa\lambda \wedge x_n \cap \kappa \in \kappa \wedge x_n \cup \{\sup(x_n \cap \mu_i) : i < \omega\} \subset x_{n+1} \wedge \forall \alpha \in x_n(x_n \text{ is closed under } \pi_\alpha \text{ and } \pi_\alpha^{-1}) \wedge \forall i < \omega(\mathrm{cf}\ \sup(x_n \cap \mu_i) = \omega_1 \wedge x_n \cap \mu_i \text{ contains a club subset of } \sup(x_n \cap \mu_i)))\}$ is stationary in $\mathcal{P}_\kappa\lambda$.

Fix $f : \lambda^{<\omega} \to \lambda$. We build inductively $\langle x_n : n < \omega\rangle$ as above with $\bigcup_{n<\omega} x_n$ closed under f as follows: First set $x_0 = \bigcup_{\xi<\omega_1} x_{0,\xi}$, where $x_{0,\xi}$ is defined inductively by $x_{0,0} = \{\mu_i : i < \omega\}$, $x_{0,\xi+1} = x_{0,\xi} \cup \sup(x_{0,\xi} \cap \kappa) \cup \bigcup\{\pi_\alpha``(x_{0,\xi} \cap |\alpha|) : \alpha \in x_{0,\xi}\} \cup \bigcup\{\pi_\alpha^{-1}(x_{0,\xi} \cap \alpha) : \alpha \in x_{0,\xi}\} \cup \{\sup(x_{0,\xi} \cap \mu_i) : i < \omega\}$ and $x_\xi = \bigcup_{\zeta<\xi} x_{0,\zeta}$ for a limit ξ. Next set $x_{n+1} = \bigcup_{\xi<\omega_1} x_{n+1,\xi}$, where $x_{n+1,\xi}$ is defined inductively by $x_{n+1,0} = x_n \cup f``x_n^{<\omega} \cup \{\sup(x_n \cap \mu_i) : i < \omega\}$, $x_{n+1,\xi+1} = x_{n+1,\xi} \cup \sup(x_{n+1,\xi} \cap \kappa) \cup \bigcup\{\pi_\alpha``(x_{n+1,\xi} \cap |\alpha|) : \alpha \in x_{n+1,\xi}\} \cup \bigcup\{\pi_\alpha^{-1}(x_{n+1,\xi} \cap \alpha) : \alpha \in x_{n+1,\xi}\} \cup \{\sup(x_{n+1,\xi} \cap \mu_i) : i < \omega\}$ and $x_\xi = \bigcup_{\zeta<\xi} x_{n+1,\zeta}$ for a limit ξ. Then $\{\sup(x_{n,\xi} \cap \mu_i) : \xi < \omega_1\} \subset x_n$ is club in $\sup(x_n \cap \mu_i)$ for any $n, i < \omega$.

We let S_0 be the first move of I, and for $x \in S_0$ fix $\langle x_n : n < \omega\rangle$ as above with $x = \bigcup_{n<\omega} x_n$. Next suppose that T is an n-th move of II. Then by $n+1$ applications of the regressive function lemma we have $\langle \gamma_{n,i} : i \le n\rangle$ such that $\{x \in T : \forall i \le n(\sup(x_n \cap \mu_i) = \gamma_{n,i})\}$ is stationary, which we let be the next move of I.

Let $\langle S_n : n < \omega\rangle$ be a play of I following the above strategy with $\langle \gamma_{n,i} : i \le n < \omega\rangle$

as above. To see that $|\bigcap_{n<\omega} S_n| \leq 1$, fix $x, y \in \bigcap_{n<\omega} S_n$. By induction on cardinals $\kappa \leq \mu \leq \lambda$ we show that $x \cap \mu = y \cap \mu$.

First take $i < \omega$ with $\mu_i = \kappa$. Then $x \cap \kappa = \bigcup_{i \leq n < \omega} x_n \cap \kappa = \bigcup_{i \leq n < \omega} \gamma_{n,i} = \bigcup_{i \leq n < \omega} y_n \cap \kappa = y \cap \kappa$, as desired.

Next suppose that $x \cap \mu = y \cap \mu$ with $\mu < \lambda$. Take $i < \omega$ with $\mu_i = \mu^+$. Fix $i \leq n < \omega$. Then $x_n \cap y_n \cap \mu^+$ is unbounded in $\sup(x_n \cap \mu^+) = \sup(y_n \cap \mu^+) = \gamma_{n,i}$. Thus $x \cap \gamma_{n,i} = y \cap \gamma_{n,i}$, since $x \cap \alpha = \bigcup_{n \leq j < \omega} x_j \cap \alpha = \bigcup_{n \leq j < \omega} \pi_\alpha``(x_j \cap \mu) = \pi_\alpha``(x \cap \mu) = \pi_\alpha``(y \cap \mu) = \bigcup_{n \leq j < \omega} \pi_\alpha``(y_j \cap \mu) = \bigcup_{n \leq j < \omega} y_j \cap \alpha = y \cap \alpha$ for any $\alpha \in x_n \cap y_n \cap (\mu^+ - \mu)$. Now we have $x \cap \mu^+ = \bigcup_{i \leq n < \omega} x \cap \gamma_{n,i} = \bigcup_{i \leq n < \omega} y \cap \gamma_{n,i} = y \cap \mu^+$ as desired, since $\sup(x \cap \mu^+) = \sup(y \cap \mu^+) = \sup_{i \leq n < \omega} \gamma_{n,i}$.

The limit case is clear from the inductive hypothesis. □

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