数理解析研究所講究録
1102 巻 1999 年 139-147

Scattering Theory for Nonlinear Klein-Gordon Equation with Sobolev Critical Power

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1. Introduction

In this note I would like to report my result on the scattering of large energy solutions to the nonlinear Klein-Gordon equation (NLKG) of the following form:

$$\ddot{u} - \Delta u + m^2 u + |u|^{p-1} u = 0, \tag{NLKG}$$

where $(t,x) \in \mathbb{R}^{1+n}$ with $n \geq 3$ and $m \geq 0$. We will consider the case

$$p = p^* := \frac{n+2}{n-2}.$$

Then $p^* + 1 = 2n/(n+2)$ is the Sobolev critical exponent. This equation has the conserved energy:

$$E(u;t) := \int_{\mathbb{R}^n} \frac{1}{2} \left(\dot{u}^2 + |\nabla u|^2 + (mu)^2 \right) + \frac{|u|^{p+1}}{p+1} \, dx = E(u;0).$$

Denote by X the energy space:

$$X := \left\{ (\varphi, \psi) \;\middle|\; \int \psi^2 + |\nabla \varphi|^2 + |m\varphi|^2 \, dx < \infty \right\}.$$

We consider the asymptotic behavior of solutions to (NLKG) with finite energy, as t tends to ∞, compared with solutions to the linear Klein-Gordon:

$$\ddot{v} - \Delta v + m^2 v = 0. \tag{KG}$$

We find v from u or u from v, where u is a solution of (NLKG) and v is a solution of (KG), such that

$$\|(u(t), \dot{u}(t)) - (v(t), \dot{v}(t))\|_X \to 0 \quad \text{as } t \to \infty.$$

Then, the aim of this study is to show that

$$(v(0), \dot{v}(0)) \mapsto (u(0), \dot{u}(0)) : X \to X \text{ homeo.} \tag{S}$$

This means the asymptotic completeness of the wave operators.

Now I mention the known results on the scattering for (NLKG). First, for the subcritical case, if $1+4/n < p < p^*$ and $m > 0$, then (S) was obtained by Brenner [3] and Ginibre and Velo [5]. In the critical case $p = p^*$, there were 2 results available. If the solutions are radially symmetric, the scattering (S) can be obtained easily from the a priori estimate derived by Ginibre, Soffer and Velo [4]. If $m = 0$, namely for the nonlinear wave, (S) easily follows from the decay property of the solutions obtained by Bahouri and Shatah [1].

But, in the case where p is the critical power, the data is nonsymmetric and $m > 0$, none of the arguments in the above results can be applied, so the scattering (S) in this case was left open. I have proved the scattering in that case, which is the main result in this note:

Theorem. *Let $p = p^*$ and $m \geq 0$. Then we have* (S).

In the rest of this note, I describe the outline of the proof of this theorem. For a more detailed, rigorous and general proof, see [7].

2. Outline of the proof

For simplicity, assume that $n = 3$ and $m = 1$. Then we have $p = 5$. It is known that if we have global a priori estimates for certain space-time norms depending only on the energy:

$$\|u\|_{ST(\mathbb{R})} < C(E), \tag{G}$$

then we obtain the desired result (S). Hereafter, 'ST' denotes a certain appropriate space-time norm, which is, in our context, $L_t^8(L_x^8)$ norm, and 'E' denotes the energy of the solution u. So, our aim is to derive the global a priori estimate (G). It is known that there is a unique global solution of

(NLKG) for any initial data with finite energy such that the ST-norm of the solution on any bounded time interval is finite (see, e.g., [6, 8]). But, it was not known that there are estimates for the ST-norm depending only on the energy. It was unknown even on a finite time interval. This is a typical difficulty in the critical case.

In order to prove the a priori estimate (G), first I use Bourgain's idea, which he used to solve the nonlinear Schrödinger equation with the critical exponent in the radial case [2]. Roughly speaking, his idea is to relate the distribution of the ST-norm in time with the distribution of the energy in space-time. Remark that the ST-norm is a Lebesgue norm for t. At first, we do not know how large it is. But we can divide the time interval into small subintervals such that each subinterval contains the same small ST-norm, say, ε. Then Bourgain's lemma below tells us that in each subinterval, somewhere in the space, there is a certain amount of localized energy. See Figure 1.

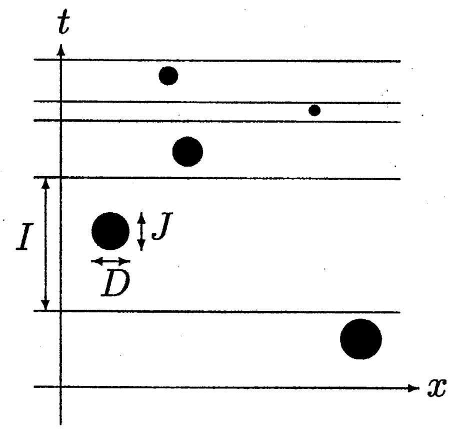

FIGURE 1. Distribution of energy lumps

Lemma 1 (Bourgain). *Let* $\|u\|_{ST(I)} = \varepsilon < C(E)$ *is sufficiently small. Then we have some subinterval* $J \subset I$ *and some ball* $D \subset \mathbb{R}^n$ *such that* $\operatorname{diam} D < C(E,\varepsilon)|J|$ *and for any* $t \in J$,

$$\int_D |\nabla u|^2 + u^2 \, dx > \varepsilon^\alpha, \quad \int_D u^6 \, dx > \varepsilon^\alpha,$$

where $\alpha > 1$ *is a certain constant.*

So, we obtain such an energy lump in $J \times D$ for each subintervals. If the ST-norm is large, then the number of the subintervals is also large, and we obtain a lot of energy lumps correspondingly . Thus, to estimate the ST-norm, it suffices to estimate the number of such energy lumps. If the ST-norm in a finite interval is very large, then many energy lumps are crowding there, and so gathering at some point in space-time, which means concentration of the energy. I have obtained a new estimate to bound the number of such gathering energy lumps.

Lemma 2. *For any finite energy solution u of* (NLKG) *and any* $\lambda > 0$, *we have*

$$\sum_{k \in \mathbb{N}} \sup_{2^{-k} < t < 2^{-k+1}} \int_{|x| < \lambda t} u^6 dx < C(E, \lambda). \tag{N}$$

From this, the energy lumps in a fat cone $\{|x| < \lambda t\}$ can not be contained in so many of dyadic intervals $(2^{-k}, 2^{-k+1})$ (see Figure 2). Combinig this

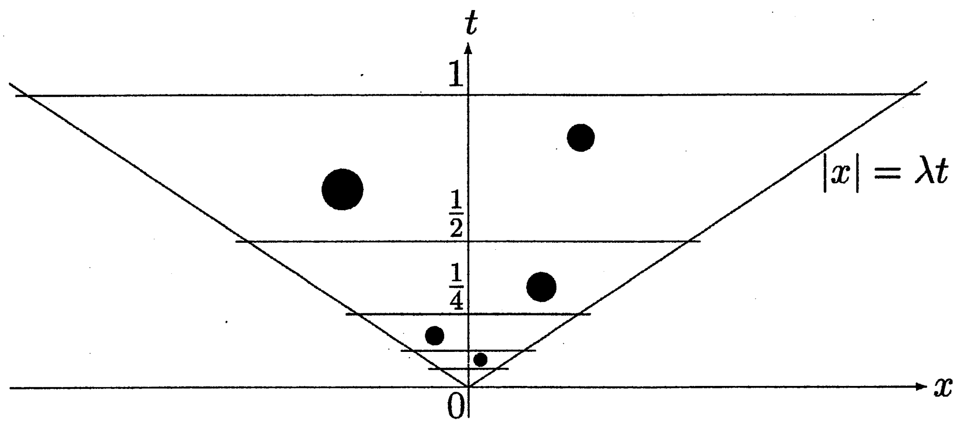

FIGURE 2. Counting the energy concentration

estimate with the finite propagation property and the well-known Morawetz estimate

$$\int_{\mathbb{R}^{n+1}} \frac{u^6}{|x|} dx dt < C(E), \tag{M}$$

we can bound the number of the energy lumps in the time interval $(0, 1)$ so that we obtain the time-local a priori estimate for the ST-norm by the energy:

$$\|u\|_{ST(0,1)} < C(E). \tag{L}$$

In the massless case $m = 0$, we obtain the global estimate (G) from this local estimate (L) by a simple scaling argument. However, in the massive case $m > 0$, there is no scaling which preserves the equation or the energy space. Moreover, we do not know whether the global version of (N) holds in the massive case. So, for the global estimate in the massive case, we can use only the Morawetz estimate (M), the finite propagation property and the local estimate (L). Again consider the distributed energy lumps given by Bourgain's lemma.

$$0 = T_0 < T_1 < \cdots, \quad I_j := (T_j, T_{j+1}), \quad \|u\|_{ST(I_j)} = \varepsilon,$$
$$J_j \subset I_j, \quad D_j : \text{ ball } \subset \mathbb{R}^n,$$
$$\int_{D_j} u^6 dx > \varepsilon^\alpha, \quad (t \in J_j).$$

Let c_j and R_j be the center and the radius of D_j, and let $t_j := \inf J_j$. Consider the truncated cone $K_j := \{(t, x) \mid t > t_j, |x - c_j| < t - t_j + R_j\}$ for each j. Now, in order to employ the Morawetz estimate most effectively for the ST-norm estimate, we choose some of the truncated cones, such that the bottom of any chosen K_j does not intersect with the other chosen cones, and at the same time, every energy lump intersects with some chosen cones (see Figure 3). Then, by the finite propagation property, we can bound the

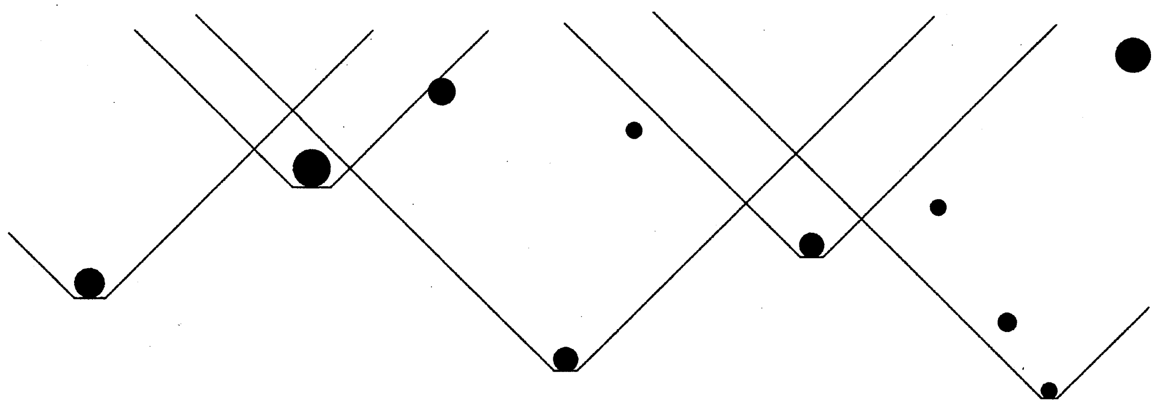

FIGURE 3. Chosen cones

number N of the chosen cones by the total energy. Applying the Morawetz estimate (M) in each chosen cone, we obtain N inequalities. Then, summing them up, we obtain an estimate as follows.

$$\begin{aligned}
C(E,\varepsilon) &\geq C(E)N \\
&\geq \sum_{\text{chosen } j} \iint \frac{u^6}{|x-c_j|} dxdt \\
&\geq \sum_{\text{chosen } j} \sum_{J_k \times D_k \cap K_j \neq \varnothing} \int_{J_k} \int \frac{u^6}{|x-c_j|} dxdt \\
&\geq \sum_{\text{chosen } j} \sum_{J_k \times D_k \cap K_j \neq \varnothing} \frac{\varepsilon^\alpha |J_k|}{|t_k - t_j| + R_k + R_j} \geq C(E,\varepsilon) \sum_{\text{all } k} \frac{R_k}{t_k + R_k}.
\end{aligned}$$

If the ST-norm is very large, then there are plenty of energy lumps. So, to make the above quantity $\sum_k R_k/(t_k + R_k)$ bounded, either R_k becomes very small or I_k becomes very long. Thus, there are two possibilities. The first case is that very highly concentrated energy lumps apper ($\underline{\lim}\, R_k \to 0$). Otherwise, very long intervals with small ST-norm appear ($\overline{\lim}\, |I_k| \to \infty$). In the latter case, we consider the interval I_{k+1} just after such a long interval I_k. Let v_0 be the solution to (KG) with the same initial data as u. Since we have global ST-estimate for v_0 by the Strichartz estimate, if there are many such long intervals I_k, we may choose appropriate I_k such that $\|v_0\|_{ST(I_{k+1})} \ll \varepsilon$. Then we use the property of the Klein-Gordon that the lower frequency part decays faster, to obtain the following lemma.

Lemma 3. *Let $0 < T < U < V$ and $\|u\|_{ST(T,U)} \leq \|u\|_{ST(U,V)} = \varepsilon < C(E)$ sufficiently small. Let v_0 be the solution to* (KG) *with the same initial data as u, and assume $\|v\|_{ST(U,V)} < \varepsilon/9$. Then for any $N > 1$, there exists $L > 0$ depending on E, ε and N, such that if $|T - U| > L$, then we have*

$$\|\psi_N * u\|_{ST(U,V)} < \frac{\varepsilon}{3},$$

where ψ_N is a cut-off function in the frequency, defined as follows. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a function such that its Fourier transform $\widetilde{\psi}$ satisfies

$$\widetilde{\psi}(\xi) = \begin{cases} 1, & |\xi| < 1, \\ 0, & |\xi| > 2. \end{cases}$$

Then we define $\widetilde{\psi_N}(\xi) := \widetilde{\psi}(\xi/N)$.

Thus, we may deduce that the ST-norm in I_{k+1} comes mainly from the higher frequency part. Since high frequency means high concentration, we get a highly concentrated energy lump in this interval I_{k+1}. So we arrive at the same situation as in the former case.

Now again we use an idea due to Bourgain. We have obtained a very highly concentrated energy lump. Consider the wave component v corresponding to the concentrated energy. Since v is also very highly concentrated, it decays very soon. Then its interaction with the remaining part is small, so that we can separate the concentrated wave v from the solution u, and estimate u by the remaining part. More precisely, we have the following perturbation lemma.

Lemma 4 (Bourgain). *Let u, W be solutions of* (NLKG), *and let v be a solution of* (KG) *with the same initial data as $u - W$. Let $E(u) \leq E$, $E(v) \leq E$ and $\|W\|_{ST(0,\infty)} = M < \infty$. Then, there exists $\kappa > 0$ depending E and M, such that if $\|v\|_{ST(0,\infty)} < \kappa$ we have the estimate*

$$\|u\|_{ST(0,\infty)} < C(E, M),$$

depending on E and M.

Since the energy of the remaining part $E(W)$ is reduced by the separated energy, repeating this argument, the problem comes down to the estimate for small energy data. Since the global ST-estimate (G) is well-known for small energy data, so we obtain the desired estimate by the induction on the energy size.

Finally, I explain how we can extract the concentrated wave v in the above argument. The idea of the separation of the energy is due to Bourgain, but the realization of the idea below is quite different from that in [2]. Bourgain's argument uses essentially the properties of the nonlinear Schrödinger, whereas my argument uses those of (NLKG). Here we use again an estimate similar to (N):

Lemma 5. *For any finite energy solution u of* (NLKG) *and any $\lambda > 0$, we have*

$$\sum_{k\in\mathbb{N}} \sup_{2^{-k}<t<2^{-k+1}} \int_{|x|<\lambda t} Q(u)\,dx < C(E,\lambda), \tag{NQ}$$

where we denote

$$Q(u) := \left(\dot{u} + \frac{r}{t}u_r + \frac{2}{t}u\right)^2 + \left(\frac{r}{t}\dot{u} + u_r\right)^2 + \left(1 + \frac{r^2}{t^2}\right)\left(|u_\theta|^2 + u^2\right) + \frac{u^2}{r^2},$$
$$r = |x|, \quad \theta = \frac{x}{r}, \quad u_r = \theta\cdot\nabla u, \quad u_\theta = \nabla u - \theta u_r.$$

In fact, the estimate (N) can be derived from (NQ) and the following Hardy-Sobolev type inequality:

$$\int_{|x|<\lambda t} u^6 dx < C(\lambda)\|\nabla u\|_{L^2}^4 \int_{|x|<\lambda t} Q(u)\,dx. \tag{H}$$

Now we set the space-time origin such that we have the concentrated energy in $\{(R,x) \mid |x| < R\}$. We may assume that the radius of the energy lump $R > 0$ is very small. By (NQ), for any $\kappa > 0$ we have some time $T \in (R, C(\kappa,E)R)$ when $Q(u)$ in the fat cone $\{|x| < 5t\}$ becomes small:

$$\int_{|x|<5T} Q(u;T)\,dx < \kappa,$$

provided $C(\kappa,E)R < 1$. On the other hand, the energy in the light cone does not decrease. So, at this time T, we cut off the data by a smooth cut-off function $\chi(x)$ which satisfies $\chi = 1$ in the light cone and $\chi = 0$ out of the fat cone. And we define v as the solution of (KG) with the data $\chi(x)(u(T),\dot{u}(T))$. Then the concentrated energy at $t = R$ is inherited by

v. Meanwhile, using the fact that $\operatorname{supp} v$ is contained in the fat cone and $\operatorname{supp} v(T)$ is very small, we can deduce that $Q(v)$ remains small forever:

Lemma 6. *Let v be a solution to* (KG) *satisfying*

$$\operatorname{supp}(v(T), \dot{v}(T)) \subset \{x \mid |x| < \widetilde{T}\}$$

for some $\widetilde{T} \geq T > 0$. Then we have for any $t > T$,

$$\int Q(v;t)\,dx \leq \int Q(v;T)\,dx + C\widetilde{T}^2 E_0(v),$$

where $E_0(v)$ denotes the linear energy of v.

By this lemma and the inequality (H), $\|v(t)\|_{L^6_x}$ remains small for $t > T$. Then, by the interpolation with the Strichartz estimate, it follows that $\|v\|_{ST(T,\infty)}$ is also small. Thus, we have succeeded in extracting the concentrated wave v as desired.

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