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CUSP OPENINGS IN COMPLEX HYPERBOLIC GEOMETRY

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The space of marked n distinct points on the complex projective line $\mathbf{CP}^1$ up to projective transformations is called a configuration space and we denote it by $\mathcal{Q}$. It admits a structure of a complex manifold of dimension $n-3$, and has a long history for attracting many mathematicians, though we focused in the talk only on ones related with complex hyperbolic geometry.

Deligne and Mostow construct a family of equivariant maps of the universal cover of $\mathcal{Q}$ to the $(n-3)$-dimensional complex projective space with respect to the action of $\pi_1(\mathcal{Q})$ and the projective transformations in [3]. It is parameterized by the exponents of an integral representation of a several variable analogue of the hypergeometric function. The main focus of their paper is to discuss when the holonomy representation, which is shown to lie in $\mathrm{PU}(1,n-3) \subset \mathrm{PGL}_{n-2}(\mathbf{C})$ is discrete, and to find many complex hyperbolic lattices.

On the other hand, Thurston provides a different construction of complex hyperbolic structures on $\mathcal{Q}$ in [13] based on euclidean cone structures on $\mathbf{CP}^1$, each of which is assigned to a configuration via a generalized Schwarz-Christofell correspondence. It is parameterized by the cone angles. His approach re-discovers complex hyperbolic lattices found by Deligne and Mostow. Strictly speaking, Thurston constructed structures not on $\mathcal{Q}$ but rather on the quotient of $\mathcal{Q}$ by the action of remarking cone points with the same cone angles, and in fact he found more lattices.

Although the discovery of lattices has been emphasized as a common part of their papers, they both actually constructed the rooted families of incomplete complex hyperbolic structures on $\mathcal{Q}$ which provide lattices in particular cases. The first purpose of the talk was to confirm that their underlying families of complex hyperbolic structures on $\mathcal{Q}$ are the same.

Deligne and Mostow studied the family from a viewpoint of Mumford's compactification in [10]. On the other hand, Thurston studied their completion from a viewpoint of cone manifolds. However, neither papers have deformation theoretic viewpoints, though Kapovich and Millson pointed out such aspect in relation with the study of mechanical linkages in [5, 6]. The second purpose of the talk was to review their families as the deformations of complex hyperbolic cone structures on $\mathcal{Q}$ for small n, in view of the deformation theory for real hyperbolic cone 3-manifolds developed by [12, 2, 11, 4, 7, 1]. The study stays in very primitive stage still, but a few small, and we believe suggestive, observations in contrast with [9, 14] were presented.

We will discuss the details in [8].

REFERENCES

[1] D. Cooper, C. Hodgson and S. Kerckhoff, ———— , to appear in MSJ Memoirs.

[2] M. Culler and P. Shalen, *Varieties of group representations and splittings of 3-manifolds*, Ann. of Math., 117 (1983), 109–146.

[3] P. Deligne and G. D. Mostow, *Monodromy of hypergeometric functions and non-lattice integral monodromy*, Publ. IHES, 63 (1986), 5–89.

[4] C. Hodgson and S. Kerckhoff, *Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery*, J. Differential Geom., 48 (1998), 1–59.

[5] M. Kapovich and J. Millson, *On the moduli space of polygons in the euclidean plane*, J. Differential Geom., 42 (1995), 133–164.

[6] M. Kapovich and J. Millson, *The symplectic geometry of polygons in euclidean space*, J. Differential Geom., 44 (1996), 479–513.

[7] S. Kojima, *Deformations of hyperbolic 3-cone-manifolds*, J. Differential Geom., 49 (1998), 469–516.

[8] S. Kojima, *Complex hyperbolic cone structures on the configuration spaces*, in preparation.

[9] S. Kojima, H. Nishi and Y. Yamashita, *Configuration spaces of points on the circle and hyperbolic Dehn fillings*, Topology, 38 (1999), 497–516.

[10] D. Mumford, *Geometric Invariant Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete 34, Springer-Verlag, 1965.

[11] W. Neumann and D. Zagier, *Volumes of hyperbolic 3-manifolds*, Topology, 24 (1985), 307–332.

[12] W. Thurston, *Geometry and Topology of 3-manifolds*, Lecture Notes, Princeton Univ., 1977/78,

[13] W. Thurston, *Shapes of polyhedra and triangulations of the sphere*, reviseted version of Research Report GCG 7 (1987) available at http://front.math.ucdavis.edu/math.gt/mathGT9801088.

[14] Y. Yamashita, H. Nishi and S. Kojima, *Configuration spaces of points on the circle and hyperbolic Dehn fillings, II*, preprint, 1998.

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