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The asymptotic behavior of Eisenstein series and a comparison of the Weil-Petersson and the Zograf-Takhtajan metrics

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ABSTRACT. Of interest to us is the asymptotic behavior of Eisenstein series for degenerating hyperbolic surfaces with cusps. In order to investigate it we use integral representations of eigenfunctions of the Laplacian, the collar lemma, the interior Schauder estimates, the maximum principles for harmonic functions and the unique extention theorem of solutions to elliptic equations. As an application, we will compare the Weil-Petersson and the Zograf-Takhtajan metrics near the boundary of moduli spaces.

§0. INTRODUCTION

The Quillen metric defined for the determinant line bundle of Laplacian over the Teichmüller space $T_{g,n}$ of compact hyperbolic surfaces with genus g has played an important role in moduli theory ([14]). The metric is described as the product of a special value of the Selberg zeta function and the usual L^2-fibre metric with respect to Poincaré metric. The first Chern form of the metric is represented by the Weil-Petersson two-form for T_g, which formula has been shown by various methods.

Zograf and Takhtajan proved the formula by quasiconformal deformation theory. Moreover they defined the regularized metric for the determinant bundle of Laplacian for Teichmüller space $T_{g,n}$ of hyperbolic surfaces with cusps of type (g,n) and calculated its first Chern form, which is described in terms of the Weil-Petersson metric and a new Kähler metric as called Zograf-Takhtajan metric. They showed in [16] that the Zograf-Takhtajan metric is Kählerian and invariant under the action of the mapping class group as the Weil-Petersson metric is. It has been recently shown that the Zograf-Takhtajan metric is incomplete for $T_{g,n}$ as the Weil-Petersson metric is ([13]). The proof has been accomplished by showing that the length of a curve approaching the boundary of $T_{g,n}$ with respect to the Zograf-Takhtajan metric is finite. The construction of the curve is due to Wolpert ([25] II).

On the other hand, recently Fujiki and Weng shed new light on the geometry of moduli space of punctured Riemann surfaces and the Zograf-Takhtajan metric ([4],[19]). From Arakelov geometric points of view, Weng found arithmetic Riemann-Roch theorem for singular metrics, and established a generalization of Mumford type isometries. Fujiki and Weng have observed that the Zograf-Takhtajan metric is algebraic, and Weng proposed a general arithmetic factorization in terms of Weil-Petersson metric and Zograf-Takhtajan metric and Selberg zeta functions.

Therefore the asymptotic behavior of the Zograf-Takhtajan metric near the boundary of moduli space is of importance and of interest for studying compactification of moduli space of punctured Riemann surfaces. In the previous paper ([13]), we have observed that the metric is incomplete. In that proof we have obtained an estimate of Eisenstein series of index 2 just around pinching geodesics, which is regrettably very rough and far from the precise asymptotic behavior, and a rough estimate of the Zograf-Takhtajan metric ([13]).

In this paper we find the asymptotic behavior of Eisenstein series of index s with Re $s > 1$ in Theorem 1 and 2. As a simple application, we improve estimate of the Zograf-Takhtajan metric near the boundary of moduli space (Theorem 3). As a result it turns out that the magnitude of the Zograf-Takhtajan norm are less than or equal to the one of the Weil-Petersson norm.

We close this chapter with surveying and proposing some approaches to the asymptotic behavior of Eisenstein series. Wolpert has investigated Eisenstein series $E(z,s)$ with $\{s \in \mathbb{C} | \text{Re } s = \frac{1}{2},\ s \neq \frac{1}{2}\}$, while we shall investigate for Re $s > 1$ ([25] I). He has shown that a subsequence of $\hat{E}(z,s)$, the normalized Eisenstein series so that L^2-norm of that on a thick part of S_l could be constant, converges to a non-trivial sum of the Eisenstein series for S_0. Its beautiful proof has been accomplished by investigating Legendre functions and showing his original Schauder-type inequalities. It seems hard to apply our method directly to the case where Re $s \leq 1$. The reason for difficulty is that we can not use the maximum principles because $E(z,s)$ with Re $s \leq 1$ is not subharmonic, and can not extend Lemma 1 to the case of Re $s \leq 1$, and $E(z,s)$ has poles on $\{s \in [0,1]\}$ (For example, we have observed that the constants $M_1(\text{Re } s, a)$ takes infinity at Re $s = 1$). What we have to study seems to investigate the behavior of the scattering matrix $\Phi(s)$ in (1.4).

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§1. Preliminaries

1.1 Eisenstein series.

Let S be a punctured hyperbolic surface of type (g,n) $(n > 0)$. It can be represented as a quotient H/Γ of the upper half plane $H = \{z \in \mathbb{C} | \text{Im} z > 0\}$ by the action of a torsion free finitely generated Fuchsian group $\Gamma \in \mathrm{PSL}_2(\mathbb{R})$. The group is generated by $2g$ hyperbolic transformations $A_1, B_1, \dots, A_g, B_g$ and parabolic transformations $P_1, \dots P_n$ satisfying the relation

$$A_1 B_1 A_1^{-1} B_1^{-1} \dots A_g B_g A_g^{-1} B_g^{-1} P_1 \dots P_n = 1.$$

The fixed points of the parabolic elements $P_1, \dots P_n$ will be denoted by $z_1, z_2, \dots, z_n \in \mathbb{R} \cup \{\infty\}$ respectively and called inequivalent cusps. The projection of the cusps $z_1, z_2, \dots, z_n$ are the punctures $p_1, p_2, \dots, p_n$ of S. For each $i = 1, \dots, n$, denote by Γ_i the stabilizer of z_i in Γ that is the cyclic subgroup of Γ generated by P_i. Pick $\sigma_i \in \mathrm{PSL}_2(\mathbb{R})$

such that $\sigma_i\infty = z_i$ and $\langle\sigma_i^{-1}P_i\sigma_i\rangle = \langle z \mapsto z+1\rangle$. Then, for $a > 1$, the a−cusp region $C_i(a)$ associated to p_i is represented as a quotient $\langle\sigma_i^{-1}P_i\sigma_i\rangle \setminus \{z \in H|\mathrm{Im}z > a\} \simeq \Gamma \setminus \{z \in H|\mathrm{Im}z > a\}$,

$$C_i(a) \simeq [a,\infty) \times S^1, \text{ equipped with the metric } ds^2 = (dy^2 + dx^2)/y^2.$$

Let $\Delta : C^\infty(S) \to C^\infty(S)$ be the negative hyperbolic Laplacian of S. Regarded as an operator in $L^2(S)$ with domain $C_0^\infty(S)$, Δ is essentially self-adjoint. Denote by $\overline{\Delta}$ the unique self-adjoint extension (that is, Friedrichs extension). Then the continuous spectrum of $\overline{\Delta}$ can be described in terms of Eisenstein series ([6]Chap.Seven, [10]Chap.V, [17]§3.2).

The Eisenstein series attached to z_i is defined by

$$E_i(z,s) = \sum_{\gamma\in\langle P_i\rangle\setminus\Gamma} \mathrm{Im}(\sigma_i^{-1}\gamma z)^s, \qquad \mathrm{Re}\ s > 1.$$

The series is absolutely convergent in the half-plane Re $s > 1$ and in the upper half-plane, it satisfies

$$\Delta E_i(z,s) = s(s-1)E_i(z,s). \tag{1.1}$$

A. Selberg originally showed that the series admits meromorphic continuation to the whole complex s-plane, holomorphic on $\{\mathrm{Re}\ s = \frac{1}{2}\}$and satisfies a system of functional equations ([15]§7). Several mathematicians also verified it by the various methods ([3], [6] Th.11.6, [10]$pp.23-46$, [12]). $E_i(z,s)$ has Fourier expansions at punctures p_j, ([6]Prop.8.6 , [10]§2.2, [11]§8, [17]§3.1)

$$E_i(\sigma_j z,s) = \delta_{ij}y^s + \phi_{ij}(s)y^{1-s} + \sum_{m\neq 0} c_m(s)y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|m|y)e^{2\pi\sqrt{-1}mx}, \tag{1.2}$$

$K_{s-\frac{1}{2}}$ the MacDonald-Bessel function ([18], p.78), that has the following asymptotics ([18], p.202)

$$y^{\frac{1}{2}}K_{s-\frac{1}{2}}(y) \sim \sqrt{\frac{\pi}{2}}e^{-y}, \text{ as } y\to\infty, \qquad \text{for any complex } s. \tag{1.3}$$

The scattering matrix $\Phi(s) = (\ \phi_{ij}(s)\ )$ enters in the functional equations ([6]Th.11.8, [10]Th.4.4.2, [15](7.36), [17]Th.3.5.1),

$$\mathbb{E}(z,s) = \Phi(s)\mathbb{E}(z,1-s), \qquad \Phi(s)\Phi(1-s) = 1, \tag{1.4}$$

where $\mathbb{E}(z,s)$ is the vector of Eisenstein series.

Thanks to Colin de Verdiére's keen observation, the Eisenstein series turns out to have the following characterization ([3], [12]).

Claim. *For Re $s > \frac{1}{2}, s \notin (\frac{1}{2},1]$, $E_i(z,s)$ is a unique solution of the equation $\Delta E_i(z,s) = s(s-1)E_i(z,s)$ such that $E_i(z,s) - \mathrm{Im}(\sigma_i^{-1}z)^s$ is square integrable on $C_i(1)$.*

Remark. We will use the above Claim just for the case where Re $s > 1$ (Theorem 1 and 2).

1.2 The Weil-Petersson and the Zograf-Takhtajan metrics.

Denote by $T_{g,n}$ Teichmüller space of hyperbolic surfaces of type (g,n). Now we consider the tangent and cotangent spaces at S of $T_{g,n}$. The cotangent space is $Q(S)$, the integrable holomorphic quadratic differentials on S. Let $B(S)$ be the L^∞-closure of Γ-invariant, bounded, $(-1,1)$-forms, i.e. the Beltrami differentials for S. For $\mu \in B(S), \varphi \in Q(S)$, the integral $\int_S \mu\varphi$ defines a paring, let $Q(S)^\perp$ be the annihilator of $Q(S)$. The tangent space at S to $T_{g,n}$ is $B(S)/Q(S)^\perp \simeq HB(S)$, the Serre dual space of $Q(S)$, i.e. the harmonic Beltrami differentials on S. Then for $\mu,\nu \in HB(S)$, the Weil-Petersson and the Zograf-Takhtajan metrics are defined as follows ([16]),

$$\langle \mu,\nu\rangle_{\mathrm{WP}} = \iint_S \mu(z)\overline{\nu(z)}y^{-2}dxdy \tag{1.5}$$

$$\begin{aligned}\langle \mu,\nu\rangle_{(\mathrm{i})} &= \iint_S E_i(z,2)\mu(z)\overline{\nu(z)}y^{-2}dxdy \\ &= \int_0^\infty \int_0^1 \mu(\sigma_i z)\overline{\nu(\sigma_i z)}dxdy\end{aligned} \tag{1.6}$$

$$\langle \mu,\nu\rangle_{\mathrm{ZT}} = \sum_{i=1}^n \langle \mu,\nu\rangle_{(i)}\,.$$

Both Weil-Petersson and Zograf-Takhtajan metric are Kählerian and incomplete ([13], [16]).

1.3 Degenerating parameters, infinite-energy harmonic maps.

In this part, we consider degeneration of hyperbolic surfaces. Denote by $(S_l(l>0), \rho_l(w)|dw|^2)$ a degenerating family of hyperbolic surfaces of type (g,n). We assume that several disjoint simple closed geodesics $l_1, l_2, \dots, l_m$ on S_l will be pinched (We denote their hyperbolic lengths by the same notations). Let Δ_l be the negative Laplacian of S_l. To compare functions on the limit surface $(S_0, \rho(z)|dz|^2)$ and $(S_l, \rho_l(w)|dw|^2)$, we use infinite-energy harmonic maps $w^l : S_0 \to S_l \setminus \{l_1, l_2, \dots, l_m\}$ constructed by M. Wolf ([9], [21], [26]).

Let regular quadratic differentials $\Psi_j dz^2$ $(j = 1 \dots m)$ for a surface with nodes $(S_0, \rho(z)|dz|^2)$ be holomorhic quadratic differentials that at j-th node have second-order poles, with equal residues, and at remaining cusps and nodes have at most simple poles. When we set $\overrightarrow{l} = (l_1, l_2, \dots, l_m)$ and $\Psi(\overrightarrow{l})dz^2 = \sum_{j=1}^m \frac{l_j^2}{4}\Psi_j dz^2$, the precise real-analytic parameterization are obtained by Wolf ([21], [26])

$$(w^l)^*\rho_l|dw|^2 = \Psi(\overrightarrow{l})dz^2 + (H(\overrightarrow{l}) + L(\overrightarrow{l}))\rho|dz|^2 + \overline{\Psi(\overrightarrow{l})dz^2}, \tag{1.7}$$

$\Psi(\overrightarrow{l})dz^2 = \rho_l w_z^l \overline{w_{\bar z}^l}$, $H(\overrightarrow{l}) = [\rho_l(w^l(z)/\rho(z)]|w_z^l|^2$, $L(\overrightarrow{l}) = [\rho_l(w^l(z)/\rho(z)]|w_{\bar z}^l|^2$.

The Beltrami differential of w^l is $\mu_l = w_{\bar z}^l / w_z^l = \overline{\Psi(\overrightarrow{l})}/\rho H$. Furthermore Wolf finds the following expansions ([21], Cor. 5.4),

$$H(\overrightarrow{l}) = 1 + \sum_{j=1}^{2m} \frac{l_j^2}{6} E_{q_j}(z,2) + O(\sum_{j=1}^{m} l_j^3), \tag{1.8}$$

$$L(\overrightarrow{l}) = O(\sum_{j=1}^{m} l_j^3),$$

for $q_1, \dots, q_{2m}$ the new cusps that come from pinching geodesics, where $E_{q_j}(z,2)$ are Eisenstein series attached to q_j for the component containing q_j.

Moreover the Beltrami differential μ_l converges uniformly to zero on compact subsets of S_0. And the harmonic map w^l converges uniformly to id on compact subsets of S_0.

Of interest to us now is the behavior of Δ_l. We review the discussion of Wolpert ([24], [25], [26]). There exists a basic map $\sigma_l : L^2(S_l) \to L^2(S_0)$, for $f \in L^2(S_l)$, $\sigma_l(f) = f(w^l)$. Define $(w^l)^*\Delta_l = \sigma_l \Delta_l \sigma_l^{-1}$. Then $(w^l)^*\Delta_l$ is real-analytic family ([24], *p*.450, [25], *p*.98, [26], *pp*.254 – 258). Especially from [24] Lemma 5.3, we easily see that for any $k \in \mathbb{N}$in C^k-norm on compact subsets of components of S_0, $(w^l)^*\Delta_l$ converges uniformly to Δ_0 that is defined to be the formal sum of the hyperbolic Laplacians for components of S_0.

Many mathematicians investigated, by various parameterizations, degeneration of hyperbolic surfaces and the asymptotic behavior of several functions; for example, Green's functions ([7], [8], [9], [20]), eigenfunctions of the Laplacian and eigenvalues ([7], [8], [9], [25], [26]), Riemannian matrix and Faltings invariant ([20]).

Remark. Basically those various parameterizations turn out to be almost the same powerful tools. But what we should pay attention to is that the infinite-energy parameters are independent of twist-angles around the pinching geodesics. Nevertheless, the family S_l with pairs of opened collars glued by adequate twist-angles agrees with R_l constructed by Wolpert [25], *pp*.103 – 104 ([23], *Appendix*, [26], *pp*.251 – 252).

§ 2. Some elementary estimates of Eisenstein series

We give key lemmas which play important roles in the proof of the main theorem (cf. [13], Lemma 4).

Lemma 1. *We set the same notations as in § 1. Let the index of Eisenstein series Re $s > 1$. For any $i = 1, 2, \dots, n$ and sufficiently large $a > 0$,*

$$|E_i(z,s)| < M_1(Re\ s, a)\ a^{\mathrm{Re} s - 1}, \qquad on\ \partial C_i(a).$$

Here $M_1(Re\ s, a)$ is a constant depending only on s, a, independent of complex structure and topological type of the surface.

Let $l_1, \dots, l_m$ be pinching geodesics on S_l. For $0 < k \le 1$ and $j = 1, \dots, m$, set

$$N_{l_j}(k) = \{p \in S_l | d(p, l_j) < k \sinh^{-1}(1/\sinh\frac{l_j}{2})\}$$

, the collar neighborhood around l_j in S_l ([2], 4.1). Here we quote an important claim due to S. Wolpert ([25] II, Lemma 2.1).

Claim. *Let $\rho(z)$ be the injectivity radius of S_l at z. There is an absolute positive constant C_0 such that for $l < 2\sinh^{-1}1$, and any $z \in N_l(1)$, then $\rho(z)e^{d(z,\partial N_l(1))} \geq C_0$.*

We improve the estimate of the Eisenstein series in [13], Lemma 4 just for the case where there exist separating pinching geodesics on S_l. Let $E_i^l(z,s)$ be the Eisenstein series attached to p_i for S_l.

Lemma 2. *Let the index Re $s > 1$. Assume that there is only one pinching geodesic $l = l_1$ on S_l, separating S_l into two parts; $S_{l,1}$ containing the puncture p_i and the other component $S_{l,2}$. Then for $l < 2\sinh^{-1}1$,*

$$|E_i^l(z,s)| < M_2(k, \mathrm{Re}\ s)\ l^{\mathrm{Re}s(1+k)-2}, \qquad \text{on } \partial N_l(k) \cap S_{l,2}.$$

Here $M_2(k, \mathrm{Re}\ s)$ is an absolute constant depending only on k, Re s.

Remark. For any s with Re $s > 1$, there is $0 < k \leq 1$ such that Re s $(1 + k) - 2 > 0$.

§ 3. THE ASYMPTOTIC BEHAVIOR OF EISENSTEIN SERIES

Our aim is to prove one of the main theorems. From now on, the cusps of S_0 that arise from the cusps of S_l are called the *old cusps* and the cusps of S_0 that arise from the pinching geodesics of S_l are called the *new cusps.*

Theorem 1. *We set the same notations as in §1. Let the index Re $s > 1$.*

(1) If $\{l_1, \ldots, l_m\}$ do not separate S_l, then for any $i = 1, \ldots, n$, as $l_1, \ldots, l_m \to 0$,

$$(w^l)^* E_i^l(z,s) \longrightarrow E_i^0(z,s) \tag{3.1}$$

uniformly on any compact subset of S_0. Here $E_i^0(z,s)$ is the Eisenstein series attached to the old puncture p_i for S_0.

(2) Assume that $\{l_1, \ldots l_m\}$ separate S_l and all components of S_0 have negative Euler numbers. Denote by $S_{0,1}^i$ and $S_{0,2}^i$ respectively the component of S_0 containing p_i and the union of the components of S_0 not containing p_i. Let q_j ($j = 1, \ldots, m$) be the new cusp arising from l_j. Denote by $C_j(b)$ ($b > 1$) be the cusp region around q_j in S_0, each composed of usual two $b-$cusp regions. Then

(i) For any $i = 1, \ldots, n$, as $l_1, \ldots, l_m \to 0$,

$$(w^l)^* E_i^l(z,s) \longrightarrow E_i^0(z,s) \tag{3.2}$$

uniformly on any compact subset of $S_{0,1}^i$. Here $E_i^0(z,s)$ is the Eisenstein series attached to p_i for $S_{0,1}^i$.

(ii) For any $i = 1, \ldots, n$ and any $b > 1$, as $l_1, \ldots, l_m \to 0$,

$$(w^l)^* E_i^l(z,s) \longrightarrow 0 \tag{3.3}$$

uniformly on $S_{0,2}^i - \bigcup_{j=1}^m C_j(b)$.

Theorem 2. *We set the same assumption and notations as in* Theorem 1 (2). *Pick R one of the components of $S^i_{0,2}$, which have the new cusps $q_1, q_2, \dots, q_t$, arising from the pinching geodesics $l_1, \dots, l_m$ on S_l, and have the old cusps which may be denoted by $p_1, \dots, p_u$, differing from p_i, replacing the enumeration if necessary. Denote by $E_{q_j}(z,s)$ the Eisenstein series attached to q_j for R ($j = 1, \dots, t$).*

Then there exist some constants $K_l \to \infty$ and a subsequence $l_1^{(h)} = \dots = l_m^{(h)} = l^{(h)} \to 0$,

$$K_{l^{(h)}}(w^{l^{(h)}})^* E_i^{l^{(h)}}(z,s) \quad \longrightarrow \quad G_0(z,s)$$

on any compact subset of R, where $G_0(z,s)$ is a non-trivial smooth function on R satisfying

$$(\Delta_0 - s(s-1))\; G_0(z,s) = 0 \qquad \text{on } R.$$

And $\lim\limits_{l \to 0} K_l \; l^{2(\mathrm{Re}\, s-1)-\delta} = \infty$, for any $\delta > 0$.

Moreover $G_0(z,s)$ is of the form,

$$G_0(z,s) = \sum_{j=1}^{t} B_j E_{q_j}(z,s), \tag{3.9}$$

where B_j ($j = 1, \dots, t$) are some constants.

§ 4. A COMPARISON OF THE W-P AND THE Z-T METRICS

Theorem 3. *The Weil-Petersson and the Zograf-Takhtajan metrics have the following behavior near the boundary of $T_{g,n}$, along the degenerating family of hyperbolic surfaces constructed by Wolpert. That is, let τ be the vector field formed by the degenerating family with only one pinching geodesic for simplicity. Then the norms of τ with respect to the Weil-Petersson and Zograf-Takhtajan metrics satisfy*

$$\|\tau\|_{ZT} \;\leq\; n\tilde{c}\|\tau\|_{WP} \qquad \text{as } l \to 0.$$

where $\tilde{c}$ is an absolute constant.

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