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VARIATIONS OF MCSHANE'S IDENTITY FOR THE RILEY SLICE AND 2-BRIDGE LINKS

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Dedicated to the memory of Professor Katsuo Kawakubo

1. Introduction

G. McShane [8] described a remarkable identity concerning the lengths of simple closed geodesics on a hyperbolic once punctured torus. This identity was extended by B. Bowditch [5] to the following identity for quasifuchsian punctured torus groups.

Theorem 1.1. *Let T be a once-punctured torus and $\mathcal{S}$ the set of the homotopy classes of the essential simple closed curves on T. Then for any quasifuchsian representation $\rho : \pi_1(T) \to \mathrm{PSL}(2, \boldsymbol{C})$, the following identity holds;*

$$\sum_{\gamma\in\mathcal{S}} \frac{1}{1+e^{l(\rho(\gamma))}} = \frac{1}{2},$$

where $l(\rho(\gamma)) \in \boldsymbol{C}/2\pi i\boldsymbol{Z}$ denotes the complex translation length of $\rho(\gamma)$.

Further, B. Bowditch [4] proved the following variation of the identity for the punctured torus bundles over the circle:

Theorem 1.2. *Let M be an orientable complete finite-volume hyperbolic manifold which fibres over the circle with fibre a once-punctured torus. Let C be the set of the homotopy classes of the essential simple closed curves on the fiber. Then the following identity holds:*

$$\sum_{\gamma\in\mathcal{C}} \frac{1}{1+e^{l(\rho(\gamma))}} = 0.$$

Further, there is a natural partition of $\mathcal{C}$ into two subsets $\mathcal{C}_L$ and $\mathcal{C}_R$, such that the following identity holds;

$$\sum_{\gamma\in\mathcal{C}_L} \frac{1}{1+e^{l(\rho(\gamma))}} = \pm\lambda(\partial M) = -\sum_{\gamma\in\mathcal{C}_R} \frac{1}{1+e^{l(\rho(\gamma))}},$$

where $\lambda(\partial M)$ denotes the mudulus of the cusp with respect to a suitably chosen basis.

In this preliminary report, we will point out that there is a variation of McShane's identity which applies to the groups in the Riley slice (Theorem 3.1). We will also show that there is a variation of McShane's identity for some 2-bridge links, and propose a conjectural variation for every hyperbolic 2-bridge link (Conjecture 4.1). We will also discuss the relation with the conjecture and a certain problem for 2-bridge link groups.

This study arose as a byproduct of the author's joint work on punctured torus groups and 2-bridge knot groups with Hirotaka Akiyoshi, Masaaki Wada, and Yasushi Yamashita ([2], [3]). The author would like to express his deepest thanks to B. H. Bowditch, G. Burde and K. Oshika for their stimulating suggestions and T. Ohtsuki for his explanation of his unpublished result with R. Riley [9].

2. Rational tangles and 2-bridge links

Let S be a 4-times punctured sphere. We identify S with the quotient space $(\boldsymbol{R}^2-\boldsymbol{Z}^2)/\Gamma$, where Γ is the group of transformations on $\boldsymbol{R}^2-\boldsymbol{Z}^2$ generated by π-rotations about points in $\boldsymbol{Z}^2$. For each $r \in \hat{\boldsymbol{Q}} := \boldsymbol{Q}\cup\{\infty\}$, let α_r be the simple loop in S obtained as the projection of the line in $\boldsymbol{R}^2-\boldsymbol{Z}^2$ of slope r. Then α_r is *essential*, i.e., it does not bound a disk in S and is not homotopic to a loop around a puncture. Conversely, any essential simple loop α in S is isotopic to α_r for a unique $r \in \hat{\boldsymbol{Q}}$. Then r is called the *slope* of α, and is denoted $s(\alpha)$.

A *trivial tangle* is a pair (B^3, t), where B^3 is a 3-ball and t is a union of two arcs properly embedded in B^3 which is parallel to a union of two mutually disjoint arcs in ∂B^3. A *meridian* m of (B^3,t) is an essential simple loop on $\partial B^3 - t$ which bounds a disk in B^3 separating the components of t. A *rational tangle* is a trivial tangle (B^3,t) endowed with a homeomorphism from $\partial B^3 - t$ to S. The *slope* of a rational tangle is defined to be the slope of the meridian. We denote a rational tangle of slope r by $(B^3, t(r))$.

The fundamental group $\pi_1(B^3 - t(r))$ is identified with the quotient $\pi_1(S)/ < \alpha_r >$, where $<>$ denotes the normal closure, and is a free group of rank two freely generated by meridians m_1 and m_2 of the components of $t(r)$. Here, a *meridian* of a component of $t(r)$ is an element of $\pi_1(B^3 - t(r))$ which is represented by a based simple loop bounding a disk intersecting $t(r)$ transvesely in one point in the component.

Let $\mathcal{D}$ be the *modular diagram*, that is the tesselation of the upper half space $\boldsymbol{H}^2$ by ideal triangles which is obtained from the ideal simplex with the ideal vertex set $\{0/1, 1/1, 1/0\}$ by repeated reflection in the edges. We identify $\hat{\boldsymbol{Q}}$ with the ideal vertices of $\mathcal{D}$. For each $r \in \hat{\boldsymbol{Q}}$, let $\Lambda(r)$ be the group of automorphisms of $\mathcal{D}$ generated by reflections in the edges of $\mathcal{D}$ with an endpoint r. Then Theorem 1.2 of Komori and Series [7] can be paraphrased as follows:

Proposition 2.1. *(1) For each $s \in \hat{\boldsymbol{Q}}$, α_s is null-homotopic in $B^3 - t(r)$ if and only if $s = r$.*

(2) Let s and s' be elements of $\hat{\boldsymbol{Q}} - \{r\}$. Then α_s and $\alpha_{s'}$ are homotopic in $B^3 - t(r)$ if and only if s and s' lies the same orbit of $\Lambda(r)$.

If we choose $r = \infty$, then the above proposition implies a bijective correspondence between $\boldsymbol{Q} \cap [0,1]$ and the set of the homotopy classes in $B^3 - t(\infty)$ of essential simple loops in $\partial B^3 - t(\infty)$ which are not null-homotopic in $B^3 - t(\infty)$.

For each $r \in \hat{\boldsymbol{Q}}$, let $L(r)$ be the *2-bridge link of slope* r, i.e., $(S^3, L(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$ is obtained from the rational tangles of slopes ∞ and r by identifying their boundaries through the identity map. [It should be noted that since the boundaries of the rational tangle complements are identified with S, the term "identity map" has a well-defined meaning.] $L(r)$ has one or two components according as the denominator of r is odd or even. Then the link group $G(L(r)) := \pi_1(S^3 - L(r))$ is identified with $\pi_1(S)/ < \alpha_\infty, \alpha_r >$. Let $\Lambda(\infty, r)$ be the group of automorphisms of $\mathcal{D}$ generated by the reflections in the edges of $\mathcal{D}$ which has ∞ or r as an endpoint. Then there are two rational numbers r_1 and r_2 with $0 < r_1 < r < r_2 < 1$ such that the region bounded by the four edges $< \infty, 0 >$, $< \infty, 1 >$, $< r, r_1 >$, and $< r, r_2 >$ is the canonical fundamental domain of $\Lambda(\infty, r)$. We can obtain the following result:

Proposition 2.2. *Let s and s' be elements of $\hat{\boldsymbol{Q}}$ which lies in the same orbit under $\Lambda(\infty, r)$. Then α_s and $\alpha_{s'}$ are homotopic in $S^3 - L(r)$.*

Corollary 2.3. *Suppose s belongs to the orbit of ∞ or r under $\Lambda(\infty, r)$. Then α_s represent the trivial element of $G(L(r))$. In particular, there is an epimorphism from $G(L(s))$ to $G(L(r))$ sending the meridian generators of $G(L(s))$ to that of $G(L(r))$.*

The above corollary is essentially equivalent to an unpublished result of Ohtsuki and Riley [9]. By studying the "Markoff maps" associated with 2-bridge knots (see [5] and [2]), we can prove that the converse to the first assertion of the above corollary holds when r is $2/5$, $2/7$, or $1/p$ for some integer p. Therefore, we would like to propose the following conjecture:

Conjecture 2.4. *(1) (Strong version) α_s and $\alpha_{s'}$ are homotopic in $S^3 - L(r)$ if and only if they belong to the same orbit under $\Lambda(\infty, r)$.*

(2) (Weak version) α_s represents the trivial element of $G(L(r))$ if and only if s belongs to the orbit of ∞ or r under $\Lambda(\infty, r)$.

3. Variation of McShane's identity for the Riley slice

For each $\omega \in \boldsymbol{C}$, let ρ_ω be the representation of $\pi_1(B^3 - t(\infty))$ defined by

$$\rho_\omega(m_1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \rho_\omega(m_2) = \begin{bmatrix} 1 & 0 \\ \omega & 1 \end{bmatrix}.$$

We denote the image of ρ_ω by G_ω. Let $\mathcal{R}$ be the space defined by:

$\mathcal{R} = \{\omega \in \boldsymbol{C} | \Omega(G_\omega)/G_\omega$ is homeomorphic to a four times punctured sphere$\}$.

This has been called the *Riley slice of Schottky groups* [KeS, KoS].

Theorem 3.1. *Let $\rho = \rho_\omega$ be the representation corresponding to a group G_ω in the Riley slice. Then the following identity holds:*

$$2 \sum_{0<r<1} \frac{1}{1 + e^{l(\rho(\alpha_r))}} + \frac{1}{1 + e^{l(\rho(\alpha_0))}} + \frac{1}{1 + e^{l(\rho(\alpha_1))}} = 0.$$

Further, the parameter ω is determined by the following identity;

$$1/\omega = 2 \sum_{0<r<1/2} \frac{1}{1 + e^{l(\rho(\alpha_r))}} + \frac{1}{1 + e^{l(\rho(\alpha_0))}} + \frac{1}{1 + e^{l(\rho(\alpha_{1/2}))}}.$$

Proof. This theorem can be easily proved by using (a refinement of) Proposition 3.13 of Bowditch [5] and the fact that each representation ρ_ω corresponds to a *Markoff map* sending ∞ to 0 (see Section 6 of [2]). □

4. Variation of McShane's identity for 2-bridge links

Hyperbolic 2-bridge links have the following nice characterization modulo the Poincare Conjecture (see [1]): A discete subgroup G of $\mathrm{PSL}(2, \boldsymbol{C})$ generated by two parabolic transformations is of cofinite valume if and only if it is isomorphic to the fundamental group of the complement of a hyperbolic 2-bridge link.

In this section, we propose a conjectural variation of McShane's identity for 2-bridge links. To do this, note that even if $L(r)$ has two components, the Euclidean structures of the boundary of the cusp neighbourhoods of the hyperbolic manifold $S^3 - L(r)$ are unique up to similarity. This follows from the fact that $L(r)$ has a $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$-symmetry, some element of which interchanges the components of $L(r)$ when $L(r)$ has two components. Let ℓ be a longitude of $L(r)$ constructed from a standard alternating diagram of $L(r)$ as illustrated in Figure 4.1. We may assume that the boundary of a cusp neighbourhood of $S^3 - L(r)$ is represented by the quotient of $\boldsymbol{C}$ by the lattice $\boldsymbol{Z} \oplus \lambda \boldsymbol{Z}$, generated by the translations $[z \to z+1]$ and $[z \to z+\lambda]$ corresponding to the meridian and the longitude ℓ. We define $\lambda(L(r))$ to be $\lambda/2$ or $\lambda/4$ according as the denominator of r is odd or even, and call it the *modulus* of $L(r)$. [Explicitely, $\lambda(L(r))$ represents the "modulus" of the boundary of a cusp neighbourhood of the quotient hyperbolic orbifold $(S^3 - L(r))/(\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2)$.]

Conjecture 4.1. *Let ρ be a faithful disctere* $\mathrm{PSL}(2, \boldsymbol{C})$ *representation of a hyperbolic 2-bridge link group $G(L(r))$. Then the following identity holds:*

$$2 \sum_{0<r<r_1} \frac{1}{1+e^{l(\rho(\alpha_r))}} + 2 \sum_{r_2<r<1} \frac{1}{1+e^{l(\rho(\alpha_r))}} + \sum_{r \in \{0,1,r_1,r_2\}} \frac{1}{1+e^{l(\rho(\alpha_r))}} = -1.$$

Here r_1 and r_2 are the rational numbers such that $0 < r_1 < r < r_2 < 1$ and that the region bounded by the four edges $< \infty, 0 >$, $< \infty, 1 >$, $< r, r_1 >$, and $< r, r_2 >$ is the canonical fundamental domain of $\Lambda(\infty, r)$. Further the modulus $\lambda(L(r))$ of the cusp of the hyperbolic manifold $S^3 - L(r)$ is given by the following formula:

$$\lambda(L(r)) = 2 \sum_{0<r<r_1} \frac{1}{1+e^{l(\rho(\alpha_r))}} + \sum_{r \in \{0,r_1\}} \frac{1}{1+e^{l(\rho(\alpha_r))}}.$$

By using the results and methods of Bowditch [4], [5], together with the recent affirmative solution [3] of the conjecture that the topological ideal triangulation of the hyperbolic 2-bridge link complemets constructed by [10] are the canonical geometric decompositions, we can see that the above conjecture holds for 2-bridge knots of slopes 2/5 and 2/7. Further, we can see that Conjecture 4.1 is valid if and only if the following two assertions hold:

(1) Conjecture 2.4 (2) holds.

(2) There are only finitely many rational numbers $r \in [0, r_1] \cup [r_2, 1]$ such that α_r is peripheral.

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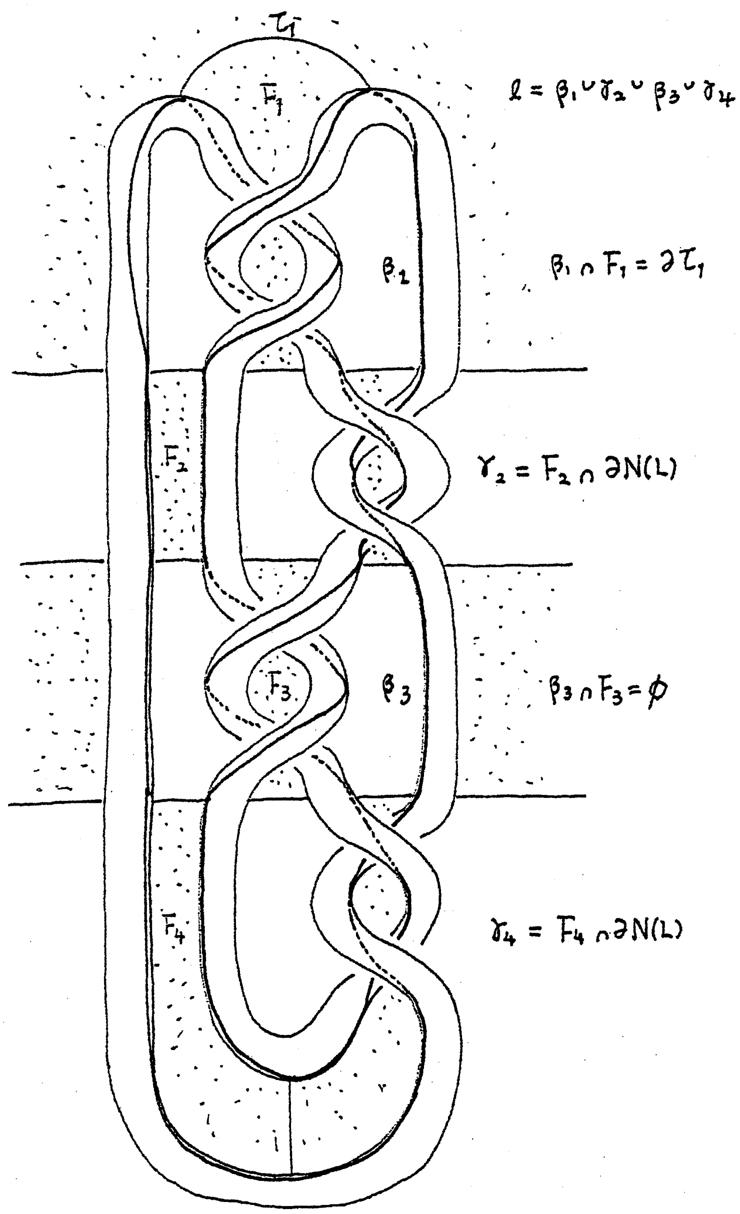

Fig. 4.1