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A canonical cellular decomposition of the Teichmüller space of compact surfaces with boundary

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Abstract

This article is a summary of [Us1].

Using the Euclidean decomposition of the hyperbolic surface, R. C. Penner gave a canonical cellular decomposition of the decorated Teichmüller space of punctured surfaces, which is invariant by the action of the mapping class group. Adapting his method, we give a canonical cellular decomposition of the Teichmüller space of compact orientable surfaces with non-empty boundary.

1 Introduction

This article is a summary of [Us1].

R. C. Penner introduced in [Pe] a method for dividing the "decorated" Teichmüller space of punctured surfaces by "natural" cells. Here, "decorated" means that each puncture is given some "weight," and "natural" means that the decomposition is invariant by the action of the mapping class group. In his method, the Euclidean decomposition of punctured surfaces with weight plays an important role (see [EP]). Since then, it has been tried to extend his construction to the Teichmüller space of other kinds of surfaces (see Table 1).

S. Kojima introduced in [Ko] a canonical method to decompose compact hyperbolic manifolds with non-empty totally geodesic boundary into truncated polyhedra. In this paper, using this decomposition and Penner's method, we give a canonical cellular decomposition of the Teichmüller space of compact orientable surfaces with non-empty boundary (see Theorem 2.1).

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On the other hand, for the decomposition, M. Näätänen obtained a cellular decomposition of the Teichmüller space of closed surfaces with a distinguished point in [Nä]. In her study, the decomposition of such surfaces introduced in [NP] plays a role of Euclidean decomposition in Penner's work.

Table 1: Cellular decompositions of Teichmüller spaces by the canonical decomposition of the surface

surface	*canon. decomp.*	*Tei. sp.*	*application*
cusped srfc.	[EP]	[Pe]	Penner et. al.
srfc. with a point	[NP]	[Nä]	[NN] etc.
cpt. srfc. with bdry.	[Ko]	[Us1]	...

2 Main theorem

Let $F_{g,r}$ be a compact orientable surface obtained from closed orientable surface of genus g by removing the interior of r disjoint closed disks on the surface. Moreover we assume $2g-2+r>0$. This assumption means that $F_{g,r}$ admits a complete hyperbolic structure. Now we denote by $\mathcal{T}_{g,r}$ the *Teichmüller space* of $F_{g,r}$. By the assumption as above, we regard $\mathcal{T}_{g,r}$ as the set of hyperbolic structures on $F_{g,r}$ (with each boundary component being totally geodesic) up to $\mathrm{Diff}_0 F_{g,r}$, the set of diffeomorphisms acting on $F_{g,r}$, which is homotopic to the identity relative to the boundary. Each element of $\mathcal{T}_{g,r}$ determines a marked discrete subgroup of the group consisting of the orientation-preserving isometries of the hyperbolic plane $\mathbf{H}^2$. So we denote by Γ_m the element of $\mathcal{T}_{g,r}$. We denote by $\mathrm{MC}_{g,r}$ the *mapping class group* of $F_{g,r}$, namely $\mathrm{MC}_{g,r} := \mathrm{Diff} F_{g,r} / \mathrm{Diff}_0 F_{g,r}$, where $\mathrm{Diff} F_{g,r}$ is the set of diffeomorphisms acting on $F_{g,r}$. For detailed definitions, please see, for example, [Ra, Th].

Fix an element Γ_m of $\mathcal{T}_{g,r}$. Then, as we saw the above, it gives a hyperbolic structure on $F_{g,r}$ with each boundary component being totally geodesic. Now the set of points in $F_{g,r}$ each of which admits at least two distinct shortest paths to the boundary consists the graph (see the left of Figure 1). We call this graph the *cut locus*, and the decomposition of $F_{g,r}$ obtained by the dual of the cut locus the *canonical decomposition* of $F_{g,r}$ with respect to Γ_m (see the center of Figure 1). It is known that the decomposition is actually cellular, that is, each piece obtained by the decomposition is homeomorphic to a disk. For detailed definitions, please see [Ko].

Let $\triangle$ be a set of arcs in $F_{g,r}$ with the following two conditions: each arc is a disjointly embedded simple arc connecting boundaries (maybe the same boundary), and the closure of each complementary region of arcs is a hexagon. We call such region a *truncated triangle*, and $\triangle$ a *truncated triangulation* of

$F_{g,r}$. Euler characteristic considerations show that there are $6g-6+3r$ arcs in $\triangle$. We call the cellular decomposition of $F_{g,r}$ obtained by deleting several arcs (maybe empty) from a truncated triangulation a *truncated cellular decomposition* of $F_{g,r}$ (see the right of Figure 1). Of course if we delete much arcs from $\triangle$, then the decomposition is not even cellular. For a truncated cellular decomposition $\triangle$ of $F_{g,r}$, we denote by $\overset{\circ}{\mathcal{C}}(\triangle)$ the set of points in $\mathcal{T}_{g,r}$ each of whose canonical decomposition coincides (topologically) to $\triangle$. By the definition of the canonical decomposition, it is easy to see that the union of $\overset{\circ}{\mathcal{C}}(\triangle)$ through all truncated cellular decompositions gives an $\mathrm{MC}_{g,r}$-invariant decomposition of $\mathcal{T}_{g,r}$. Furthermore we can prove the following theorem, the main theorem of this article:

Theorem 2.1 ([Us1, Theorem 6.6]) *If $\triangle$ is a truncated cellular decomposition of $F_{g,r}$, $\overset{\circ}{\mathcal{C}}(\triangle)$ is an open cell of dimension $\#\triangle$. The set $\left\{\overset{\circ}{\mathcal{C}}(\triangle)\,\middle|\,\triangle \text{ is a truncated cellular decomposition of } F_{g,r}\right\}$ is a $\mathrm{MC}_{g,r}$-invariant cellular decomposition of $\mathcal{T}_{g,r}$ itself. Furthermore, the isotropy group of $\mathcal{C}(\triangle)$ in $\mathrm{MC}_{g,r}$ is isomorphic to the (finite) group of mapping classes of $F_{g,r}$ leaving $\triangle$ invariant.*

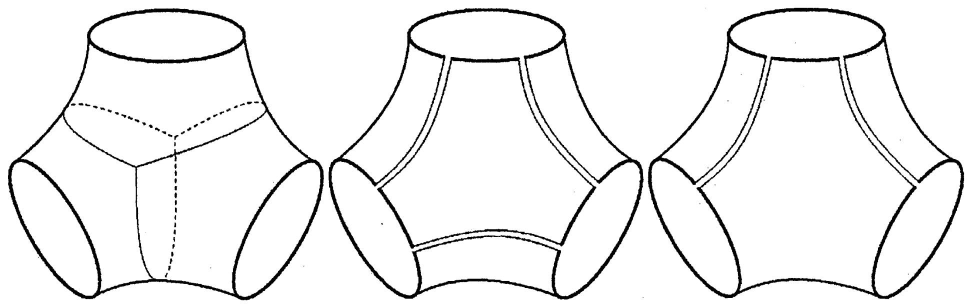


Figure 1: A canonical decomposition of $F_{0,3}$

3 Proof of the main theorem

In this section, under the assumption that $\triangle$ is a truncated triangulation, not a truncated cellular decomposition, we explain the proof that $\overset{\circ}{\mathcal{C}}(\triangle)$ is homeomorphic to an open ball of dimension $q := 6g-6+3r$ by the following four steps.

3.1 s-length coordinate

Let $\Delta = (c_1, c_2, \ldots, c_q)$ be a truncated triangulation of $F_{g,r}$. For any element Γ_m of $\mathcal{T}_{g,r}$, we define the *s-length of c_i relative to Γ_m* as follows:

$$s(c_i; \Gamma_m) := \sqrt{2} \cosh \frac{d_i}{2} \in \mathbf{R}_s ,$$

where d_i means the hyperbolic distance of the geodesic c_i, and $\mathbf{R}_s := \{t \in \mathbf{R} \mid t > \sqrt{2}\}$. Using s-lengths, we define the mapping S_Δ from $\mathcal{T}_{g,r}$ to $\mathbf{R}_s^q$ as follows:

$$S_\Delta (\Gamma_m) := \left(s(c_1; \Gamma_m), s(c_2; \Gamma_m), \ldots, s(c_q; \Gamma_m)\right) \in \mathbf{R}_s^q .$$

For this mapping we have the following theorem:

Theorem 3.1 ([Us1, Theorem 4.1]) *If Δ is a truncated triangulation of $F_{g,r}$, then S_Δ is a homeomorphism.* □

By this theorem, for a fixed $\Sigma \in \mathbf{R}_s^q$, the pair (Δ, Σ) is regarded as a point of $\mathcal{T}_{g,r}$.

Definition 3.2 (short) Fix a point (Δ, Σ) of $\mathcal{T}_{g,r}$.

(1) For any arc e in Δ, we denote by $\Sigma(e)$ the s-length of e. Then we say that $(e, \Sigma(e))$ is *short* if $\Sigma(e) < \mathrm{dist}_{\mathbf{H}}(\partial_1, \partial_2)$ holds, where $\mathrm{dist}_{\mathbf{H}}(\partial_1, \partial_2)$ means the hyperbolic distance between ∂_1 and ∂_2 (see Figure 2).

(2) We say that (Δ, Σ) is *short* if $(e, \Sigma(e))$ is short for all $e \in \Delta$.

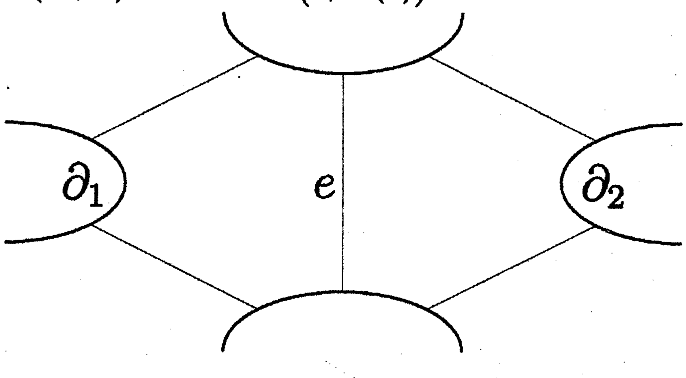


Figure 2: $(e, \Sigma(e))$ is short

By the definition of the canonical decomposition, we have the following proposition:

Proposition 3.3 (cf. [Us1, Theorem 6.1]) *For an element $(\Delta, \Sigma) \in \mathcal{T}_{g,r}$, (Δ, Σ) is short if and only if $(\Delta, \Sigma) \in \mathring{\mathcal{C}}(\Delta)$.* □

This proposition implies that it is important to obtain an efficient tool to decide the given arc is short or not, and the following *h-length coordinate* is the one.

3.2 h-length coordinate

For a truncated triangulation $\triangle$, the boundary of $F_{g,r}$ is decomposed into several segments. We denote by $\mathcal{B}_\triangle$ the set of such segments. Let $\mathbf{R}_+ := \{t \in \mathbf{R} \,|\, t > 0\}$. Fix $\Sigma \in \mathbf{R}_s^q$, and we define the *h-length of* $E \in \mathcal{B}_\triangle$ *for* $\Gamma_m = (\triangle, \Sigma)$ as follows (see also Figure 3):

$$h(E, \Gamma_m) := \frac{\Sigma(e)}{\Sigma(a)\,\Sigma(b)}\,.$$

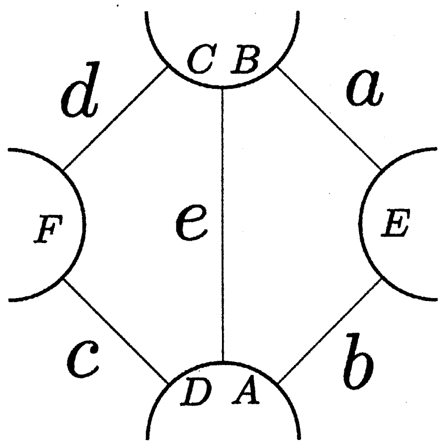


Figure 3: h-length of E

Now we define the mapping $I_\triangle$ from $\mathbf{R}_s^q$ to $\mathbf{R}_+^{\mathcal{B}_\triangle} \approx \mathbf{R}_+^{2q}$ by transforming the s-length coordinate into the h-length coordinate.

We here observe the image of $\mathcal{T}_{g,r}$ by $I_\triangle$. It is easy to see that $\Sigma(e)^{-2} = h(A, \Gamma_m)\,h(B, \Gamma_m)$ holds under the situation of Figure 3. But it also holds that $\Sigma(e)^{-2} = h(C, \Gamma_m)\,h(D, \Gamma_m)$. So the element of $I_\triangle(\mathcal{T}_{g,r})$ is demanded the following condition at every edge of the truncated triangulation:

$$h(A, \Gamma_m)\,h(B, \Gamma_m) = h(C, \Gamma_m)\,h(D, \Gamma_m)\,.$$

We call this equation the *coupling equation*. Furthermore, since s-lengths are greater than $\sqrt{2}$, we also demand the following condition:

$$(0 <)\,h(A, \Gamma_m)\,h(B, \Gamma_m) < \frac{1}{2}\,.$$

We call this inequality the *coupling inequality*. On the other hand, we can easily see that elements in $\mathbf{R}_+^{2q}$ satisfying the two conditions denoted above are also elements in $I_\triangle(\mathcal{T}_{g,r})$. Thus we obtain the following theorem:

Theorem 3.4 ([Us1, Proposition 4.4]) *The mapping $I_\triangle$ is an embedding of $\mathcal{T}_{g,r}$ into $\mathbf{R}_+^{\mathcal{B}_\triangle}$. Explicitly, $I_\triangle(\mathcal{T}_{g,r}) \subset \mathbf{R}_+^{\mathcal{B}_\triangle}$ is characterized by the coupling equations and the coupling inequalities.* $\square$

Using the h-length coordinate, we can easily see whether the given edge is short or not.

Proposition 3.5 ([Us1, Theorem 6.1]) *Under the situation of Figure 3, $(e, \Sigma(e))$ is short if and only if the inequality $h(A, \Gamma_m) + h(B, \Gamma_m) + h(C, \Gamma_m) + h(D, \Gamma_m) > h(E, \Gamma_m) + h(F, \Gamma_m)$ holds.* □

Note

We can extend Proposition 3.5 to the following one:

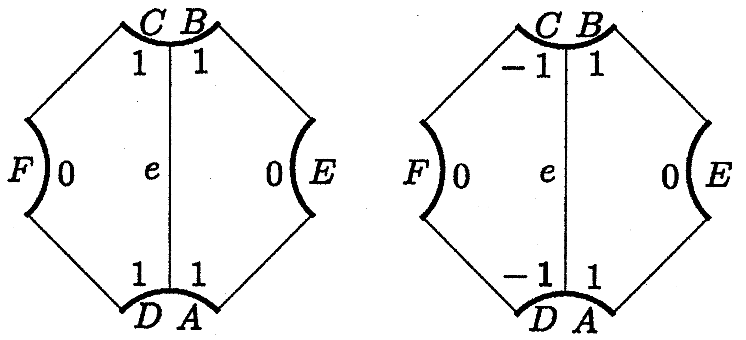
Proposition 3.6 *Under the situation of Figure 3, $\Sigma(e) = \mathrm{dist}_{\mathbf{H}}(E, F)$ (resp. $>$) if and only if $h(A, \Gamma_m) + h(B, \Gamma_m) + h(C, \Gamma_m) + h(D, \Gamma_m) = h(E, \Gamma_m) + h(F, \Gamma_m)$ (resp. $<$) holds.* □

These two propositions are kinds of so-called "tilt proposition." In [Us2], we study a generalization of the tilt proposition.

3.3 Changing bases

As we saw the preceding subsection, Though the h-length coordinate gives an effective formula to decide whether the given edge is short or not. But the space is twice the dimension of $\mathcal{T}_{g,r}$. So we extract the information of the short from the half-dimensional space of $\mathbf{R}^{2q}$.

For each edge $e \in \Delta$, we define a pair of vectors B_e and C_e in $\mathbf{R}^{\mathcal{B}_\Delta} \approx \mathbf{R}^{2q}$ as the following figure:



B_e $\quad$ C_e

Figure 4: vectors B_e and C_e

Since C_e does not give any effects to the inequality in Proposition 3.5, we can easily obtain the following proposition:

Proposition 3.7 ([Us1, Lemma 6.3]) *(1) The set of vectors $\{B_e, C_e\}_{e \in \Delta}$ is a basis of $\mathbf{R}^{\mathcal{B}_\Delta} \approx \mathbf{R}^{2q}$. Namely $\mathbf{R}^{\mathcal{B}_\Delta} \cong \langle B_e \rangle_{e \in \Delta} \oplus \langle C_e \rangle_{e \in \Delta}$.*

(2) Suppose $I_\triangle((\triangle,\Sigma)) = \sum_{e\in\triangle} x_e\,B_e + \sum_{e\in\triangle} y_e\,C_e$ *for some* $x_e, y_e, \in \mathbf{R}$. *Then* $(\triangle,\Sigma)$ *is short if and only if* $x_e > 0$ *for every* $e \in \triangle$. $\square$

3.4 The core of the proof

We define subsets of $\mathbf{R}_+^{\mathcal{B}_\triangle}$ as follows:

$$\begin{aligned}
\overset{\circ}{X} &:= \left\{ \sum_{e\in\triangle} x_e\,B_e \in \mathbf{R}_+^{\mathcal{B}_\triangle} \,\middle|\, x_e > 0 \right\}, \\
\overset{\circ}{\mathcal{D}}_\triangle(\triangle) &:= \left\{ z \in \mathbf{R}_+^{\mathcal{B}_\triangle} \,\middle|\, \begin{array}{l} z \text{ satisfies the coupling equations,} \\ \text{and } (\triangle, I_\triangle^{-1}(z)) \text{ is short} \end{array} \right\}, \\
\overset{\circ}{\mathcal{G}}_\triangle(\triangle) &:= \left\{ z \in \overset{\circ}{\mathcal{D}}_\triangle(\triangle) \,\middle|\, z \text{ satisfies the coupling inequalities} \right\}.
\end{aligned}$$

We note that $I_\triangle \circ S_\triangle(\overset{\circ}{\mathcal{C}}(\triangle)) = \overset{\circ}{\mathcal{G}}_\triangle(\triangle)$ by an immediate consequence of Theorem 3.1, Proposition 3.3 and Theorem 3.4.

Now the following theorem holds:

Theorem 3.8 ([Us1, Theorem 6.4] or [Pe, Theorem 5.4]) *The projection* $\Pi_\triangle$ *induces a homeomorphism from* $\overset{\circ}{\mathcal{D}}_\triangle(\triangle)$ *to* $\overset{\circ}{X}$. $\square$

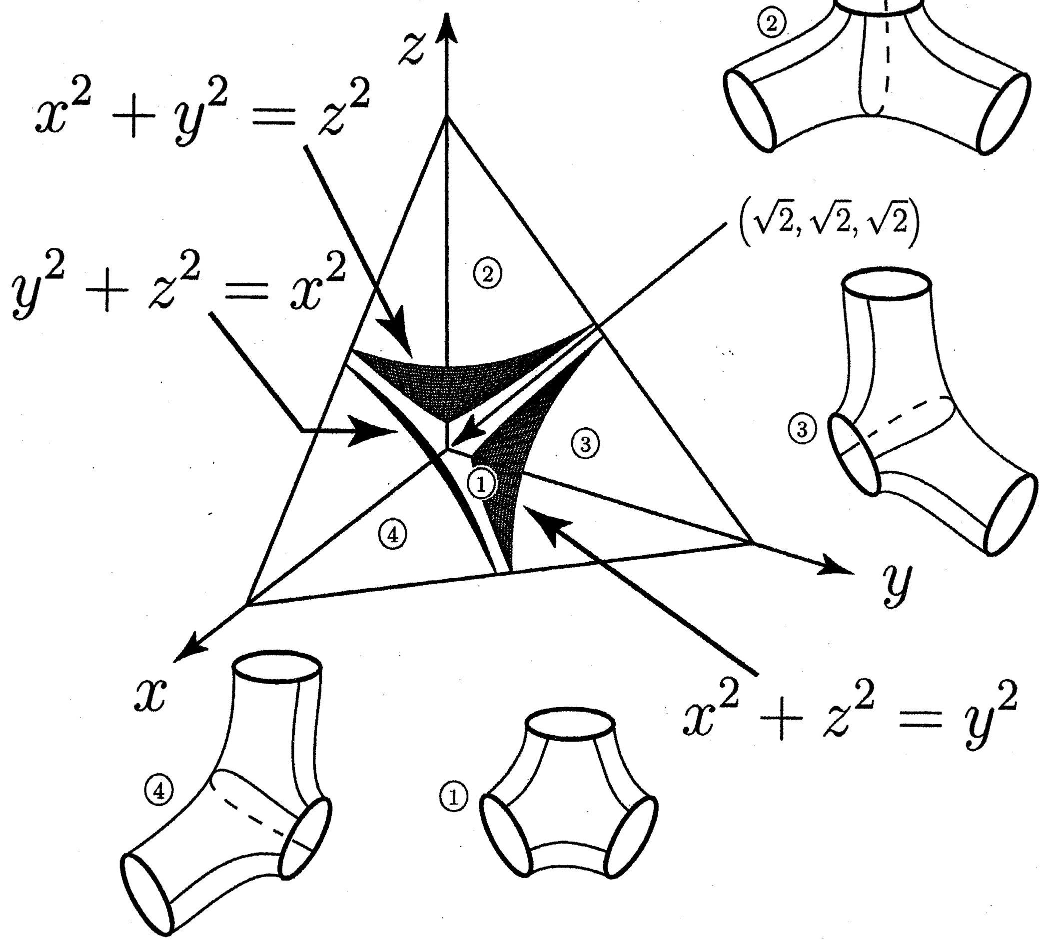
By the definition of the coupling inequality, we can easily prove that $\overset{\circ}{\mathcal{G}}_\triangle(\triangle)$ is homeomorphic to the intersection of $\overset{\circ}{\mathcal{D}}_\triangle(\triangle)$ and the open unit ball in $\mathbf{R}^{\mathcal{B}_\triangle}$ centered at the origin. Thus we obtain the following theorem, which is the goal of this section:

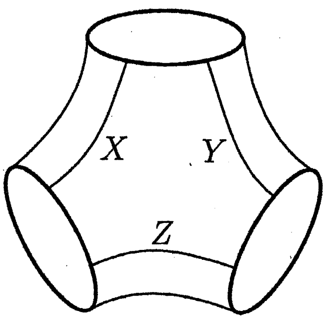
Theorem 3.9 ([Us1, Theorem 6.5]) *The set* $\overset{\circ}{\mathcal{C}}(\triangle)$ *is homeomorphic to an open ball of dimension* $q = 6\,g - 6 + 3\,r$. $\square$

A Examples

We see the decomposition of $\mathcal{T}_{0,3}$ in page 8, and $\mathcal{T}_{1,1}$ in pages 9 and 10.

Example 1. $\mathcal{T}_{0,3}$





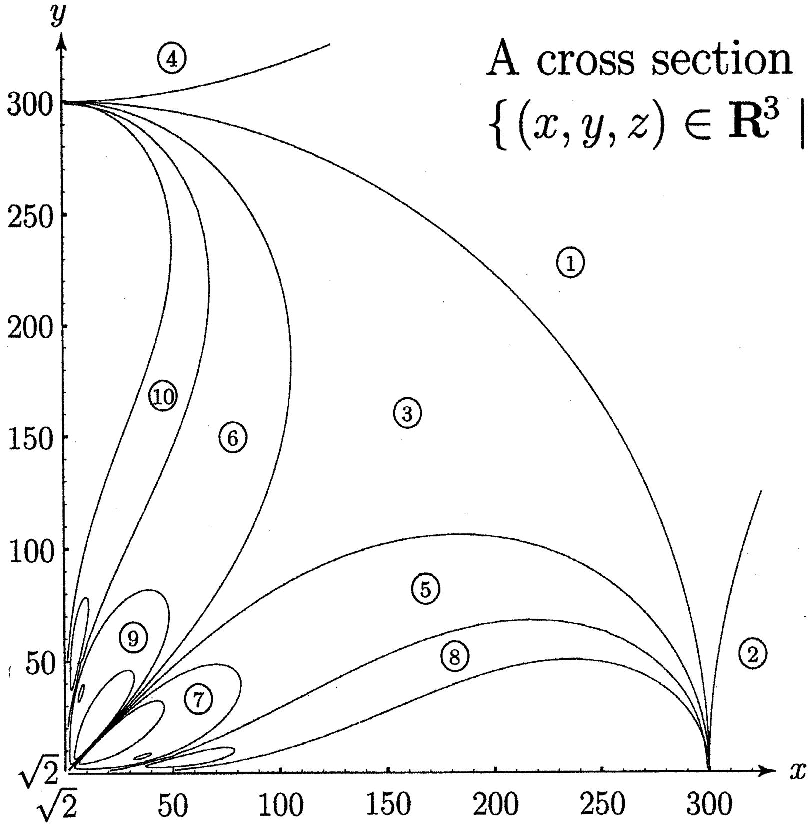
$$x := \sqrt{2}\cosh\frac{d_X}{2},$$

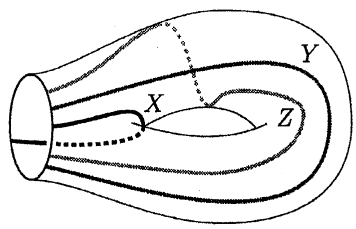
$$y := \sqrt{2}\cosh\frac{d_Y}{2},$$

$$z := \sqrt{2}\cosh\frac{d_Z}{2}.$$

Example 2. $\mathcal{T}_{1,1}$

A cross section by $\{(x,y,z) \in \mathbf{R}^3 \mid z = 300\}$.





$$x := \sqrt{2}\cosh\frac{d_X}{2},$$

$$y := \sqrt{2}\cosh\frac{d_Y}{2},$$

$$z := \sqrt{2}\cosh\frac{d_Z}{2}.$$

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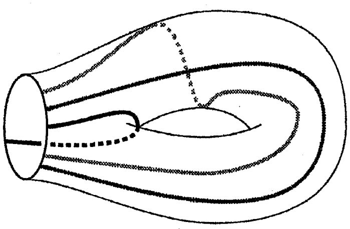

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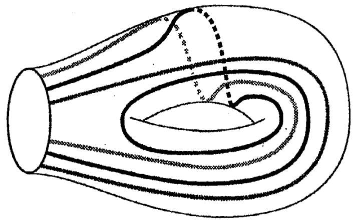

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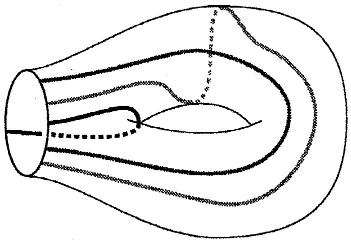

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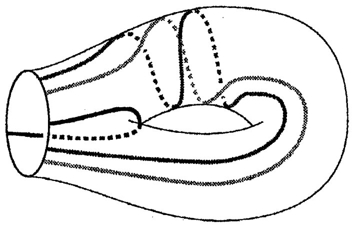

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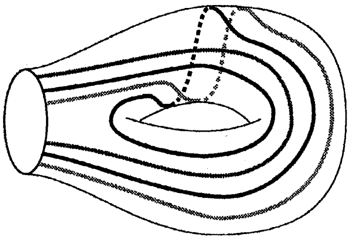

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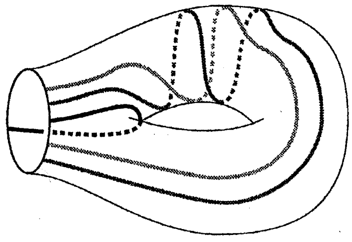

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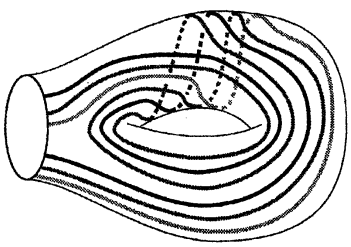

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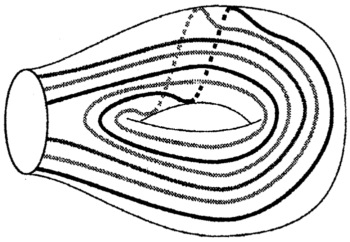

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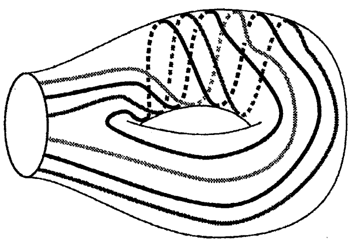

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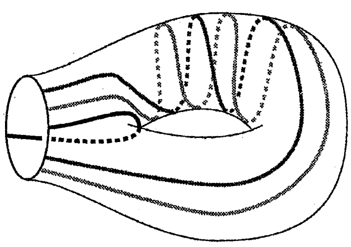

References

[EP] D. B. A. Epstein and R. C. Penner, *Euclidean decompositions of non-compact hyperbolic manifolds*, Journal of Differential Geometry **27** (1988), 67–80.

[Ko] Sadayoshi Kojima, *Polyhedral decomposition of hyperbolic manifolds with boundary*, On the Geometric Structure of Manifolds, edited by Dong Pyo Chi, Proceedings of Workshops in Pure Mathematics, Volume 10, Part III (1990), 37–57.

[Nä] Marjatta Näätänen, *A cellular parameterization for closed surfaces with a distinguished point*, Annales Academiæ Scientiarum Fennicæ Series A. I. Mathematica **18** (1993), 45–64.

[NN] Toshihiro Nakanishi and Marjatta Näätänen, *The Teichmüller space of a punctured surface represented as a real algebraic space*, Michigan Mathematical Journal **42** (1995), 235–258.

[NP] M. Näätänen and R. C. Penner, *The convex hull construction for compact surfaces and the Dirichlet polygon*, Bulletin of the London Mathematical Society **23** (1991), 568–574.

[Pe] R. C. Penner, *The decorated Teichmüller space of punctured surfaces*, Communications in Mathematical Physics **113** (1987), 299–339.

[Ra] John G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Graduate Texts of Mathematics, **149**, Springer-Verlag, 1994.

[Th] William P. Thurston, *Three Dimensional Geometry and Topology*, Princeton Mathematical Series, **35**, Princeton University Press, 1997.

[Us1] Akira Ushijima, *A canonical cellular decomposition of the Teichmüller space of compact surfaces with boundary*, Communications in Mathematical Physics **201** (1999), 305–326.

[Us2] Akira Ushijima, *The tilt formula for weighted simplices in hyperbolic space*, preprint.