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Almost n-dimensional spaces

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We consider only separable metric spaces. A space X is said to be almost n-dimensional if it has a basis $\{U_i\}$ such that if $\mathrm{cl}U_i \cap \mathrm{cl}U_j = \emptyset$ then $X = G \cup H$ where G and H are closed sets, $U_i \subset G \backslash H$, $U_j \subset H \backslash G$ and $\dim G \cap H \leq n-1$ and n is the smallest natural number such that such a basis exists for n. It is clear that n-dimensional spaces are at most almost n-dimensional.

Oversteegen and Tymchatyn [9] proved that almost 0-dimensional spaces are at most 1-dimensional. The Erdös space of irrational sequences in Hilbert space is known to be a universal almost 0-dimensional space [5]. Erdös space is 1-dimensional. Homeomorphism groups of positive dimensional Menger compacta are almost 0-dimensional [9] and at least 1-dimensional by classical results of Brechner [2] and Bestvina [1].

Almost 0-dimensional spaces are at most 1-dimensional and the 1-dimensionality cannot be improved. Our first result shows that this interesting behaviour does not occur in higher dimensions and the following one points out an interesting property of almost 0-dimensional spaces.

Theorem 1 (Levin-Tymchatyn [7]) *If X is almost n-dimensional, $n \geq 1$ then X is n-dimensional.*

Theorem 2 (Levin-Tymchatyn [7]) *Let $X = X_1 \cup X_2$ where X_1 is almost 0-dimensional and X_2 is 0-dimensional. Then $\dim X \leq 1$.*

The proof of these theorems employs so-called L-embeddings. A subset X of a compactum K is L-embedded in K if for every open cover $\mathcal{U}$ of K there is a neighbourhood U of X in K such that the continua in U refine $\mathcal{U}$. An almost 0-dimensional space is L-embeddable in a compactum [6] and

Theorem 3 (Levin-Pol [6]) *If a space X is L-embeddable in a compactum K then $\dim X \leq 1$.*

As an application of almost 1-dimensional spaces we will consider an old question of R. Duda about the dimension of a hereditarily locally connected, non-degenerate space X. Nishiura and Tymchatyn [8] showed that each pair of disjoint, closed, connected subsets of X can be separated by a closed countable subset of X. Hence each basis for X of open connected sets witnesses the almost 1-dimensionality of X. Then Theorem 1 implies:

Theorem 4 (Levin-Tymchatyn [7]) *If X is a hereditarily locally connected, non-degenerate space then* $\dim X = 1$.

A partial solution to the question of R. Duda was given in [9] where it was proved that hereditarily locally connected spaces are at most 2-dimensional.

Finally let us note that Theorem 2 does not hold if X_2 is almost 0-dimensional. Indeed, let Y be 1-dimensional and almost 0-dimensional, let M be a 1-dimensional compactum and let $M = M_1 \cup M_2$, $\dim M_1 = \dim M_2 = 0$. Then $X_1 = Y \times M_1$ and $X_2 = Y \times M_2$ are almost 0-dimensional, and by a theorem of Hurewicz [4] (see also [3], p. 78, 1.9.E(b)) $X = X_1 \cup X_2 = Y \times M$ is 2-dimensional.

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