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Homeomorphism groups of 2-manifolds and spaces of embeddings into 2-manifolds

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1. Introduction

The purpose of this article is to survey main results on the topology of homeomorphism groups of 2-manifolds and spaces of embeddings of compact polyhedra into 2-manifolds with the compact-open topology. Homeomorphism groups of topological manifolds draw our interest in two aspects: group structures and topological structures. In this article we are mainly concerned with topology of homeomorphism groups (homotopy types, geometry as infinite-dimensional manifolds, etc).

There is a long history on the study of homeomorphism groups of topological manifolds (cf. [22, §5.6]). In the 2-dimensional case, in the series of papers [11] M. E. Hamstrom studied the homotopy types of the identity components of the homeomorphism groups of compact 2-manifolds (with finite punctures). The PL-case is studied in [25] in the context of semisimplicial complex, and the homotopy types of diffeomorphism groups of compact smooth 2-manifolds are investigated in [4]. On the other hand, R. Luke - W. K. Mason [17, 18] showed that the homeomorphism groups of compact 2-manifolds are ANR's, and R. Geoghegan - W. E. Haver [10] showed that the pair of the homeomorphism group of any compact PL 2-manifold and the subgroup of PL-homeomorphisms forms an (ℓ_2, ℓ_2^f)-manifold. The subgroups of Lipschitz homeomorphisms were studied by K. Sakai - R. Y. Wong [24], and in [28] we showed that the triple of the homeomorphism group of any compact Euclidean PL 2-manifold and the subgroups of Lipschitz and PL-homeomorphisms forms an (s, Σ, σ)-manifolds. Since the topological types of these infinite-dimensional topological manifolds are determined by their homotopy types, these results enable us to determine the topological types of these homeomorphism groups and subgroups.

In the noncompact case, the whole homeomorphism groups of noncompact 2-manifolds are not necessarily even locally connected [29]. However, the identity components are always ℓ_2-manifolds and are contractible except only several cases [32]. Therefore, we can also determine the topological types of the identity components of these homeomorphism groups and subgroups [32, 33].

For Riemann surfaces we can consider the subgroups of quasiconformal homeomorphisms. Quasiconformality is a sort of boundedness condition like Lipschitz condition, and in [30] we showed that these groups are also Σ-manifolds.

Spaces of embeddings into manifolds are closely related to the study of homeomorphism groups. In [31] we showed that the restriction maps from homeomorphism groups of 2-manifolds to spaces of proper embeddings of compact subpolyhedra are principal bundles. These bundles were used in [31, 32, 33] to derive some conclusions on homeomorphism groups of noncompact 2-manifolds and embedding spaces into 2-manifolds from the corresponding results on homeomorphism groups of compact 2-manifolds. In particular, in [31] we showed that the triple of the space of embeddings of any compact subpolyhedron into a Euclidean PL 2-manifold and the subspaces of Lipschitz and PL-embeddings is also an (s, Σ, σ)-manifold, and determined the topological types of the components of spaces of embbeddings of an arc, a disk and a circle into 2-manifolds.

In Section 2 we provide two background materials: topological characterization of infinite-dimensional manifolds and basic facts on homeomorphism groups of n-manifolds. The main part, a survey on homeomorphism groups of 2-manifolds and embedding spaces into 2-manifolds is included in Sections 3 and 4. The final section 5 contains some results about extension of embeddings into 2-manifold to homeomorphisms and principal bundles.

2. Backgrounds

2.1. **Basic facts on infinite dimensional manifolds.** First we recall some basic facts on infinite-dimensional manifolds. A metrizable space X is called an ANR (absolute neighborhood retract) if any map $f : B \to X$ from a closed subset of a metrizable space Y has an extension to a neighborhood U of B. By ℓ^2 we denote the separable Hilbert space $\{(x_n) \in \mathbb{R}^\infty : \sum_n x_n^2 < \infty\}$. The following is the simplest form of topological characterization of ℓ^2-manifolds:

Theorem 2.1. ([26]) *A space X is an ℓ^2-manifold iff X is a separable completely metrizable ANR and $X \times \ell^2 \cong X$.*

It is known that the topological types of any ℓ^2-manifold is determined by its homotopy type.

Every ℓ^2-manifold contains various submanifolds modeled on incomplete infinite-dimensional spaces. We use the following standard notations:

(1) $s = \mathbb{R}^\infty$ $(\cong \ell_2)$, $\Sigma = \{(x_n) \in s : \sup_n |x_n| < \infty\}$, $\sigma = \{(x_n) \in s : x_n = 0 \text{ (almost all } n)\}$,

(2) $s^\infty \cong s$, Σ^∞, σ^∞ (with the product topology),

(3) $\Sigma_f^\infty = \{(x_n) \in \Sigma^\infty : x_n = 0 \text{ (almost all } n)\}$, $\sigma_f^\infty = \{(x_n) \in \sigma^\infty : x_n = 0 \text{ (almost all } n)\}$.

To treat these submanifolds systematically, we need the notion of infinite-dimensional manifold tuples: A triple (X, X_1, X_2) is called an (E, E_1, E_2)-manifold if each point of X has a neighborhood U such that $(U, U \cap X_1, U \cap X_2) \cong (E, E_1, E_2)$. As typical examples we consider the following triples:

$(E, E_1, E_2) = (s, \Sigma, \sigma)$, $(s^2, s \times \sigma, \sigma^2)$, $(s^\infty, \sigma^\infty, \sigma_f^\infty)$ and $(s^\infty, \Sigma^\infty, \Sigma_f^\infty)$.

Theorem 2.1 extends to a characterizations of manifolds modeled on these triples. To state the precise statement we need some terminology: (X, X_1, X_2) is (E, E_1, E_2)-stable if $(X \times E, X_1 \times E_1, X_2 \times E_2) \cong (X, X_1, X_2)$. A subset Y has the homotopy negligible (h.n.) complement in X if there exists a homotopy $\varphi_t : X \to X$ $(0 \le t \le 1)$ such that $\varphi_0 = id_X$ and $\varphi_t(X) \subset Y$ $(0 < t \le 1)$. A space is σ-(fd-) compact if it is a countable union of (finite dimensional) compact subsets. For each case $\mathcal{M}(E, E_1, E_2)$ denotes the class of triples (X, X_1, X_2) satisfying the following conditions:

(E, E_1, E_2)	X_1	X_2
(s, Σ, σ)	σ-compact	σ-fd-compact
$(s^2, s \times \sigma, \sigma^2)$	F_σ in X	σ-fd-compact
$(s^\infty, \sigma^\infty, \sigma_f^\infty)$	$F_{\sigma\delta}$ in X	σ-fd-compact
$(s^\infty, \Sigma^\infty, \Sigma_f^\infty)$	$F_{\sigma\delta}$ in X	σ-compact

Theorem 2.2. ([28])
A triple (X, X_1, X_2) is an (E, E_1, E_2)-manifold iff
(i) *X is a separable completely metrizable ANR,* (ii) *X_2 has the h.n. complement in X,*
(iii) *$(X, X_1, X_2) \in \mathcal{M}(E, E_1, E_2)$ and* (iv) *(X, X_1, X_2) is (E, E_1, E_2)-stable.*

The topological types of these manifolds are detected by their homotopy types.

Proposition 2.1. (Homotopy invariance [28]) *Suppose (X, X_1, X_2) and (Y, Y_1, Y_2) are (E, E_1, E_2)-manifolds.*
(i) *$(X, X_1, X_2) \cong (Y, Y_1, Y_2)$ iff $X \simeq Y$ (homotopy equivalent).*
(ii) *If X has the homotopy type of a locally compact polyhedron P, then $(X, X_1, X_2) \cong P \times (E, E_1, E_2)$.*

We refer to [3, 19] for other basic results in infinite-dimensional topology.

2.2. **Basic facts on homeomorphism groups of n-manifolds.** Next we list basic properties of homeomorphism groups and embedding spaces. Suppose M is a topological n-manifold and X is a closed subset of M. We denote by $\mathcal{H}_X(M)$ the group of the homeomorphisms h of M onto itself with $h|_X = id_X$, equipped with the compact-open topology. When M has a prefered metric, $\mathcal{H}_X^{\text{(L)LIP}}(M)$ denotes the subgroup of (locally) LIP-homeomorphisms of M, and when M is a PL-manifold and X is a subpolyhedron of M, $\mathcal{H}_X^{\text{PL}}(M)$ denotes the subgroup of PL-homeomorphisms of M. The superscript "c" denotes "compact supports", the subscript "+" means "orientation preserving", and "0" denotes "the identity connected components" of the corresponding groups. A Euclidean PL-manifold means a PL-manifold which is a subpolyhedron of some Euclidean space $\mathbb{R}^n$ and has the standard metric induced from $\mathbb{R}^n$.

In an analogy to diffeomorphism groups, these homeomorphism groups are expected to be topological manifold modeled on some typical infinite-dimensional spaces. After R. D. Anderson showed that $\mathcal{H}_+(\mathbb{R}) \cong s$ [1], it was conjectured that $\mathcal{H}(M)$ is always an s-manifold for any compact manifold M. This basic conjecture is still open for $n \geq 3$. By Characterization Theorem 2.2, the following class property and stability property can be used to determine the infinite-dimensional model spaces associated to homeomorphism groups (R. Geoghegan [8, 9], J. Keesling-D. Wilson [15, 16], K. Sakai-R. Y. Wong [24], T. Yagasaki [28]).

2.2.1. **Class Property.**

Lemma 2.1. (i) *$\mathcal{H}(M)$ is a separable completely metrizable topological group.*
(ii) *$\mathcal{H}^{\mathrm{LLIP}}(M)$ is $F_{\sigma\delta}$ in $\mathcal{H}(M)$, and $\mathcal{H}^{\mathrm{LIP}}(M)$ and $\mathcal{H}^{\mathrm{LIP,c}}(M)$ are σ-compact (with respect to any metric on M).*
(iii) *If M is a PL-manifold, then $\mathcal{H}^{\mathrm{PL}}(M)$ is $F_{\sigma\delta}$ in $\mathcal{H}(M)$ and $\mathcal{H}^{\mathrm{PL,c}}(M)$ is σ-fd-compact.*

2.2.2. **Stability Property.** The next lemma is verified by using the Morse length of the image of a fixed segment under the homeomorphisms.

Lemma 2.2. (i) *$\mathcal{H}(M)$ is s-stable for any n-manifold M.*
(2) *Suppose X is a locally compact polyhedron.*
(i) *$(\mathcal{H}(X), \mathcal{H}^{\mathrm{PL}}(X))$ is (s, σ)-stable.*
(ii) *If X is noncompact, then $(\mathcal{H}(X), \mathcal{H}^{\mathrm{PL}}(X), \mathcal{H}^{\mathrm{PL,c}}(X))$ is $(s^\infty, \sigma^\infty, \sigma^\infty_f)$-stable.*
(3) *Suppose X is a Euclidean polyhedron with the standard metric.*
(i) *$(\mathcal{H}(X), \mathcal{H}^{\mathrm{LIP}}(X), \mathcal{H}^{\mathrm{PL}}(X))$ is (s, Σ, σ)-stable.*
(ii) *If X is noncompact, then $(\mathcal{H}(X), \mathcal{H}^{\mathrm{LLIP}}(X), \mathcal{H}^{\mathrm{LIP,c}}(X))$ is $(s^\infty, \Sigma^\infty, \Sigma^\infty_f)$-stable.*

2.2.3. **ANR–Property.** For $n \geq 3$ it is still unknown whether $\mathcal{H}(M)$ is an ANR, and this problem is equivalent to the basic conjecture that $\mathcal{H}(M)$ is an ℓ^2-manifold. It is only known that $\mathcal{H}(M)$ is locally contractible (A. V. Černavskiĭ, R. D. Edwards–R.C. Kirby, D. B. Gauld)

Proposition 2.2. (i) *$\mathcal{H}(M)$ is locally contractible for any compact n-manifold M* [2, 5].
(ii) *$\mathcal{H}^{\mathrm{PL}}(M)$ is locally contractible for any compact PL-manifold M* [7].

Since any countable dimensional locally contractible metric space is an ANR (W.E. Haver [13]), it follows that $\mathcal{H}^{\mathrm{PL}}(M)$ is an ANR, and Characterization of σ-manifolds means the next conclusion (J. Keesling - D. Wilson [16]):

Theorem 2.3. *$\mathcal{H}^{\mathrm{PL}}(M)$ is an σ-manifold for any compact PL-manifold M.*

Once we assume that $\mathcal{H}(M)$ is an ANR, Characterization Theorem 2.2 implies some conclusions on the triples of homeomorphism groups and subgroups. Let $\mathcal{H}(M)^* = cl\,\mathcal{H}^{\mathrm{PL}}(M)$ and let $\mathcal{H}^{\mathrm{LIP}}(M)^* = \mathcal{H}^{\mathrm{LIP}}(M) \cap cl\mathcal{H}^{\mathrm{PL}}(M)$ when M is a Euclidean PL-manifold. Consider the following condition:

$$(*) \quad n \neq 4 \quad \text{and} \quad \partial M = \emptyset \text{ for } n = 5.$$

Under this condition $\mathcal{H}(M)^*$ is the union of some components of $\mathcal{H}(M)$.

Theorem 2.4. *Suppose that M is a compact n-dimensional Euclidean PL-manifold which satisfies $(*)$ and that $\mathcal{H}(M)$ is an ANR.*
(1) $(\mathcal{H}(X), \mathcal{H}^{\mathrm{LIP}}(X))$ *is an (s, Σ)-manifold* (K. Sakai–R. Y. Wong [24]).
(2) $(\mathcal{H}(X)^*, \mathcal{H}^{\mathrm{LIP}}(X)^*, \mathcal{H}^{\mathrm{PL}}(X))$ *is an (s, Σ, σ)-manifold* (R. Geoghegan - W. E. Haver [10], T. Yagasaki [28]).

The 2-dimensional case will be treated in the next section. In the 1-dimensional case we have ([1, 28])

Proposition 2.3. (1) $(\mathcal{H}(G), \mathcal{H}^{\mathrm{LIP}}(G), \mathcal{H}^{\mathrm{PL}}(G))$ *is an (s, Σ, σ)-manifold for any Euclidean graph G.*
(2) $(\mathcal{H}_+(\mathbb{R}), \mathcal{H}_+^{\mathrm{PL}}(\mathbb{R}), \mathcal{H}^{\mathrm{PL},c}(\mathbb{R})) \cong (s^\infty, \sigma^\infty, \sigma_f^\infty)$ *and* $(\mathcal{H}_+(\mathbb{R}), \mathcal{H}_+^{\mathrm{LLIP}}(\mathbb{R}), \mathcal{H}^{\mathrm{LIP},c}(\mathbb{R})) \cong (s^\infty, \Sigma^\infty, \Sigma_f^\infty)$.

2.2.4. **Embedding spaces.** Suppose Y is a Euclidean polyhedron and $K \subset X$ are compact subpolyhedra of Y. Let $\mathcal{E}_K(X, Y)$ denote the spaces of embeddings f of X into Y with $f|_K = id_K$, equipped with the compact-open topology, and let $\mathcal{E}_K^{\mathrm{LIP}}(X, Y)$ and $\mathcal{E}_K^{\mathrm{PL}}(X, Y)$ denote the subspaces of Lipschitz and PL-embeddings respectively. Here, a Lipschitz embedding is a Lipschitz homeomorphism onto its image.

Lemma 2.3. *$\mathcal{E}_K^{\mathrm{LIP}}(X, Y)$ is σ-compact and $\mathcal{E}_K^{\mathrm{PL}}(X, Y)$ is σ-fd-compact* [9].

Lemma 2.4. *If $\dim(X \setminus K) \geq 1$, then $(\mathcal{E}_K(X, Y), \mathcal{E}_K^{\mathrm{LIP}}(X, Y), \mathcal{E}_K^{\mathrm{PL}}(X, Y))$ is (s, Σ, σ)-stable* [24].

3. Homeomorphism groups of 2-manifolds

3.1. The compact case. Suppose M is a compact connected PL 2-manifold and X is a compact subpolyhedron of M.

Theorem 3.1. *$\mathcal{H}_X(M)$ is an ANR and hence it has the homotopy type of a CW-complex* (R. Luke–W. K. Mason [17]).

Theorem 3.2. *$\mathcal{H}_X(M)$ is an ℓ_2-manifold.*

Homotopy types of the identity component $\mathcal{H}_X(M)_0$ was studied by M. E. Hamstrom and it was shown that $\mathcal{H}_X(M)_0$ is contractible in most cases. In the PL-case, G. P. Scott studied the weak homotopy type of $\mathcal{H}_X^{\mathrm{PL}}(M)_0$ in the context of semisimplicial complex. These results are summarized in the next statements: The notations $\mathbb{S}^2$, $\mathbb{T}^2$, $\mathbb{P}^2$, $\mathbb{K}^2$, $\mathbb{D}^2$ and $\mathbb{M}$ denote the 2-sphere, torus, projective plane, Klein bottle, 2-disk and Möbius band respectively.

Theorem 3.3. (M. E. Hamstrom, G. P. Scott et.al. [11, 12, 21, 25])
(1) $\mathcal{H}(\mathbb{S}^2)_0 \simeq SO(3)$,
(2) $\mathcal{H}(\mathbb{T}^2)_0 \simeq \mathbb{T}^2$,
(3) $\pi_i\mathcal{H}(\mathbb{P}^2)_0$: $\pi_1 = \mathbb{Z}_2, \pi_2 = 0, \pi_i = \pi_i\mathbb{P}^2$ $(i \geq 3)$,
(4) $\mathcal{H}_X(M)_0 \simeq \mathbb{S}^1$ *if* $(M,X) \cong (\mathbb{D}^2,\emptyset)$, $(\mathbb{D}^2,0)$, $(\mathbb{S}^1 \times [0,1],\emptyset)$, $(\mathbb{M},\emptyset)$, $(\mathbb{S}^2, 1pt)$, $(\mathbb{S}^2, 2pts)$, $(\mathbb{P}^2, 1pt)$, $(\mathbb{K}^2,\emptyset)$.
(5) $\mathcal{H}_X(M)_0 \simeq *$ *if (M,X) is not the cases* (1)–(4) (*i. e.* $(M,X) \not\cong (\mathbb{D}^2,\emptyset)$, $(\mathbb{D}^2,0)$, $(\mathbb{S}^1 \times [0,1],\emptyset)$, $(\mathbb{M},\emptyset)$, $(\mathbb{S}^2,\emptyset)$, $(\mathbb{S}^2, 1pt)$, $(\mathbb{S}^2, 2pts)$, $(\mathbb{T}^2,\emptyset)$, $(\mathbb{K}^2,\emptyset)$, $(\mathbb{P}^2,\emptyset)$, $(\mathbb{P}^2, 1pt)$).

Theorem 3.4. (R. Geoghegan - W. E. Haver, K. Sakai - R. Y. Wong, T. Yagasaki, et.al. [10, 24, 28])
(i) *$\mathcal{H}_X^{\mathrm{PL}}(M)$ has the h. n. complement in $\mathcal{H}_X(M)$. Hence, $\mathcal{H}_X(M) \simeq \mathcal{H}_X^{\mathrm{LIP}}(M) \simeq \mathcal{H}_X^{\mathrm{PL}}(M)$.*
(ii) *$(\mathcal{H}_X(M), \mathcal{H}_X^{\mathrm{LIP}}(M), \mathcal{H}_X^{\mathrm{PL}}(M))$ is an (s,Σ,σ)-manifold.*

When M is a compact connected Riemann surface, we can consider the subgroup $\mathcal{H}^{\mathrm{QC}}(M)$ of QC-homeomorphisms of M. Since the quasiconformality is a kind of boundedness condition, the corresponding model space is Σ:

Theorem 3.5. *$(\mathcal{H}(M)_+, \mathcal{H}^{\mathrm{QC}}(M))$ is an (s,Σ)-manifold.*

From Theorem 3.3 and Homotopy invariance of (s,Σ,σ)-manifolds (Proposition 2.1), we can detect the topological types of these homeomorphism groups and subgroups.

3.2. **The noncompact case.** (T. Yagasaki [29, 32, 33])

Suppose M is a noncompact connected PL 2-manifold and X is a compact subpolyhedron of M.

Theorem 3.6. *$\mathcal{H}_X(M)_0$ is an ℓ_2-manifold.*

Theorem 3.7.
(1) $\mathcal{H}_X(M)_0 \simeq \mathbb{S}^1$ *if* $(M,X) \cong (\mathbb{R}^2,\emptyset)$, $(\mathbb{R}^2, 1pt)$, $(\mathbb{S}^1\times\mathbb{R}^1,\emptyset)$, $(\mathbb{S}^1\times[0,1),\emptyset)$ *or* $(\mathbb{P}^2\setminus 1pt,\emptyset)$,
(2) $\mathcal{H}_X(M)_0 \simeq *$ *if* (M,X) *is not the case* (1).

Theorem 3.8.
(i) $\mathcal{H}_X^{\mathrm{PL,c}}(M)_0 \subset \mathcal{H}_X(M)_0$ *has the h. n. complement. Hence* $\mathcal{H}_X(M)_0 \simeq \mathcal{H}_X^{\mathrm{PL}}(M)_0 \simeq \mathcal{H}_X^{\mathrm{PL,c}}(M)_0$.
(ii) $(\mathcal{H}_X(M)_0, \mathcal{H}_X^{\mathrm{PL}}(M)_0, \mathcal{H}_X^{\mathrm{PL,c}}(M)_0)$ *is an* $(s^\infty,\sigma^\infty,\sigma_f^\infty)$*-manifold.*

Corollary 3.1.
(1) $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{PL}}(M)_0, \mathcal{H}^{\mathrm{PL,c}}(M)_0) \cong \mathbb{S}^1\times(s^\infty,\sigma^\infty,\sigma_f^\infty)$ *if* $(M,X) \cong (\mathbb{R}^2,\emptyset)$, $(\mathbb{R}^2,1pt)$, $(\mathbb{S}^1\times\mathbb{R}^1,\emptyset)$, $(\mathbb{S}^1\times[0,1),\emptyset)$ *or* $(\mathbb{P}^2\setminus 1pt,\emptyset)$,
(2) $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{PL}}(M)_0, \mathcal{H}^{\mathrm{PL,c}}(M)_0) \cong (s^\infty,\sigma^\infty,\sigma_f^\infty)$ *in the case of not* (1).

When M is a noncompact connected Euclidean PL 2-manifold, we have the Lipschitz - version:

Proposition 3.1. (1) $(\mathcal{H}_X(M)_0, \mathcal{H}_X^{\mathrm{LIP}}(M)_0)$ *is an* (s,Σ)*-manifold.*
(2) $(\mathcal{H}_X(M)_0, \mathcal{H}_X^{\mathrm{LLIP}}(M)_0, \mathcal{H}_X^{\mathrm{LIP,c}}(M)_0)$ *is an* $(s^\infty,\Sigma^\infty,\Sigma_f^\infty)$*-manifold.*

When M is a noncompact connected Riemann surface, we can consider the subgroup $\mathcal{H}^{\mathrm{LQC}}(M)$ of locally QC-homeomorphisms [30].

Theorem 3.9. (1) $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{QC}}(M)_0)$ *is an* (s,Σ)*-manifold.*
(2) $(\mathcal{H}_X(M)_0, \mathcal{H}_X^{\mathrm{LQC}}(M)_0, \mathcal{H}_X^{\mathrm{QC,c}}(M)_0)$ *is an* $(s^\infty,\Sigma^\infty,\Sigma_f^\infty)$*-manifold.*

Finally we determine the condition on the end of M under which the whole group $\mathcal{H}(M)$ is an ℓ^2-manifold [29]. Consider the following condition on M:

$(*)$ $M = N\setminus(F\cup A)$, where N is a compact connected 2-manifold, F is a finite subset of $\mathrm{Int}\,N$ and A is a 0-dimensional compact subset of ∂N.

When M has the form of $(*)$, we can consider $\mathcal{H}_+(A)$, the group of order preserving homeomorphisms of A: Let $C_1,\cdots,C_m$ be the circle components of ∂N which meet A, and set $A_i = A\cap C_i$. We choose a component U_i of $C_i\setminus A_i$ and set $I_i = C_i\setminus U_i$. Then I_i is an arc (or a single point) and any orientation on I_i induces a linear order on A_i. Let $\mathcal{H}_+(A) = \{f\in\mathcal{H}(A) : f|_{A_i}\in\mathcal{H}_+(A_i)\}$, where the subscript "+" means "order preserving".

Theorem 3.10. (1)(i) $\mathcal{H}_X(M)$ *is locally connected iff* M *takes the form of* $(*)$ *and* $\mathcal{H}_+(A)$ *is discrete,*
(ii) *in this case* $(\mathcal{H}_X(M), \mathcal{H}_X^{\mathrm{PL}}(M)) \cong (\mathcal{H}_X(M), \mathcal{H}_X^{\mathrm{LIP}}(M))$ *and they are* $(s^\infty, \sigma^\infty)$*-manifold.*
(2)(i) $\mathcal{H}_{\partial\cup X}(M)$ *is locally connected iff* M *takes the form of* $(*)$,
(ii) *in this case* $(\mathcal{H}_{\partial\cup X}(M), \mathcal{H}_{\partial\cup X}^{\mathrm{PL}}(M)) \cong (\mathcal{H}_{\partial\cup X}(M), \mathcal{H}_{\partial\cup X}^{\mathrm{LIP}}(M))$ *and they are* $(s^\infty, \sigma^\infty)$*-manifolds.*

4. Spaces of embeddings into 2-manifolds

Suppose M is a Euclidean PL 2-manifold and $K \subset X$ are compact subpolyhedra of M.

Proposition 4.1. *If* $\dim(X \setminus K) \geq 1$, *then* $(\mathcal{E}_K(X,M),\ \mathcal{E}_K^{\mathrm{LIP}}(X,M),\ \mathcal{E}_K^{\mathrm{PL}}(X,M))$ *is an* (s, Σ, σ)*-manifold* [31].

Let $\mathcal{E}_K(X,M)_0$ denote the connected component of the inclusion $i : X \subset M$ in $\mathcal{E}_K(X,M)$. We can determine the homotopy type of $\mathcal{E}(X,M)_0$ for $X =$ an arc I, a disk D or a circle C [33]:

Theorem 4.1.
(1) $\mathcal{E}(I,M) \simeq S(TM)$ (the unit circle bundle of the tangent bundle of M).
(2) $\mathcal{E}(D,M) \simeq S(T\tilde{M})$, where $\tilde{M}$ is the orientation double cover of M.
(3-1) If $M \not\cong \mathbb{S}^2$ then $\{f \in \mathcal{E}(C,M) : f \text{ is inessential}\} \simeq S(T\tilde{M})$ and if $M \cong \mathbb{S}^2$ then $\mathcal{E}(C,M) \simeq S(TM)$.
(3-2) Suppose C is an essential simple closed curve in M.
(a) If $M \neq \mathbb{P}^2, \mathbb{T}^2, \mathbb{K}^2$, then $\mathcal{E}(C,M)_0 \simeq \mathbb{S}^1$.
(b) If $M \cong \mathbb{T}^2$, then $\mathcal{E}(C,M)_0 \simeq \mathbb{T}^2$.
(c) Suppose $M \cong \mathbb{K}^2$ and $M \setminus C$ is connected.
(c-i) If C preserves the orientation then $\mathcal{E}(C,M)_0 \simeq \mathbb{T}^2$,
(c-ii) If C reverses the orientation then $\mathcal{E}(C,M)_0 \simeq \mathbb{S}^1$.
(c-iii) If $M \setminus C$ is not connected (i.e., C is a common boundary of two Möbius bands) then $\mathcal{E}(C,M)_0 \simeq \mathbb{S}^1$.
(d) If $M \cong \mathbb{P}^2$, then $\pi_1\mathcal{E}(C,M)_0 \cong \mathbb{Z}_4$, $\pi_2\mathcal{E}(C,M)_0 = 0$ and $\pi_k\mathcal{E}(C,M)_0 = \pi_k(\mathbb{P}^2)$ $(k \geq 3)$.

When X is an arbitrary compact subpolyhedron of M, we can take a regular neighborhood N of X in M and consider a core K of N. If X is neither an arc nor a circle which preserves orientation, the restriction map $\pi : \mathcal{E}(N,M)_0 \to \mathcal{E}(X,M)_0$, $\pi(f) = f|_X$, is a homotopy equivalence. Since we can choose the core K to be a disk, a circle or a one-point union of circles, the classification of homotopy types of $\mathcal{E}(X,M)_0$ will be completed when we finish writing down the homotopy types of $\mathcal{E}(X,M)_0$ for $X =$ a one-point union of circles. This is our remaining task.

5. Principal bundle $\mathcal{H}_K(M)_0 \to \mathcal{E}_K(X, M)_0^*$

In this final section we show that the restriction maps from homeomorphism groups to embedding spaces are principal bundles in the 2-dimensional case, and then we seek some conditions under which the fibers of these bundles are connected [31, 32]. These principal bundles enable us to derive the results on homeomorphism groups of noncompact 2-manifolds and embedding spaces into 2-manifolds in Sections 3.2 and 4 from the corresponding results on homeomorphism groups of compact 2-manifolds [31, 32, 33]. To exhibit principal bundles, we need to show existance of sections. In our case, this is equivalent to obtain some extension theorem for embeddings of a compact 2-polyhedron X into a 2-manifold M to ambient homeomorphisms of M. Since every graph can be decomposed into ads (i.e., cones over finite points) and arcs connecting them, it suffices to study the embeddings of trees into a disk. The key ingredients are the conformal mapping theorems, extension to boundary and continuity (cf. [20, Ch.1,2]). The proper embedding case is a consequence of a direct application of the mapping theorem on simply connected domains (and seems to be well known ([12, 17])). Thus our interest is in the case of embeddings into the interior of a disk, where we need to apply the mapping theorem on a doubly connected domain one boundary circle of which is collapsed to a tree. The conclusion is summarized as follows: Suppose M is a PL 2-manifold and X is a compact subpolyhedron of M. We say that an embedding $f : X \to M$ is proper if $f(X \cap \partial M) \subset \partial M$ and $f(X \cap \operatorname{Int} M) \subset \operatorname{Int} M$. Let $\mathcal{E}_K(X, M)^*$ denote the subspace of proper embeddings of X into M, and let $\mathcal{E}_K(X, M)_0^*$ denote the connected component of the inclusion $i : X \subset M$ in $\mathcal{E}_K(X, M)^*$.

Theorem 5.1. *For every $f \in \mathcal{E}_K(X, M)^*$ and every neighborhood U of $f(X)$ in M, there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{E}_K(X, M)^*$ and a map $\varphi : \mathcal{U} \to \mathcal{H}_{K\cup(M\setminus U)}(M)_0$ such that $\varphi(g)f = g$ for each $g \in \mathcal{U}$ and $\varphi(f) = id_M$.*

Suppose U is an open neighborhood of X in M and $\pi : \mathcal{H}_{K\cup(M\setminus U)}(M)_0 \to \mathcal{E}_K(X, U)_0^*$, $\pi(h) = h|_X$, denote the restriction map. The group $\mathcal{G} = \mathcal{H}_{K\cup(M\setminus U)}(M)_0 \cap \mathcal{H}_X(M)$ acts on $\mathcal{H}_{K\cup(M\setminus U)}(M)_0$ by right composition.

Corollary 5.1. *The map $\pi : \mathcal{H}_{K\cup(M\setminus U)}(M)_0 \to \mathcal{E}_K(X, U)_0^*$ is a principal bundle with fiber $\mathcal{G}$.*

Next we investigate some condition which implies that $\mathcal{G} = \mathcal{H}_X(M)_0$. Suppose M is a 2-manifold and N is a 2-submanifold of M. In [6] it is shown that (i) two homotopic essential simple closed curves in $\operatorname{Int} M$ and two proper arcs homotopic rel ends in M are ambient isotopic rel ∂M, (ii) every homeomorphism $h : M \to M$ homotopic to id_M is ambient isotopic to id_M.

Using these results or arguments we can show that if, in addition, $h|_N = id_N$ then h is isotopic to id_M rel N under some restrictions on disks, annuli and Möbius bands components (i.e. the pieces which admit global rotations). The symbol $\#X$ denotes the number of elements (or cardinal) of a set X.

Theorem 5.2. *Suppose M is a connected 2-manifold, N is a compact 2-submanifold of M and X is a subset of N such that*
(i) $M \neq \mathbb{T}^2, \mathbb{P}^2, \mathbb{K}^2$ *or* $X \neq \emptyset$.
(ii)(a) *if H is a disk component of N, then $\#(H \cap X) \geq 2$,*
(b) *if H is an annulus or Möbius band component of N, then $H \cap X \neq \emptyset$,*
(iii) (a) *if L is a disk component of $cl(M \setminus N)$, then* FrL *is a disjoint union of arcs or $\#(L \cap X) \geq 2$,*
(b) *if L is a Möbius band component of $cl(M \setminus N)$, then* FrL *is a disjoint union of arcs or $L \cap X \neq \emptyset$.*
If $h_t : M \to M$ is an isotopy rel X such that $h_0|_N = h_1|_N$, then there exists an isotopy $h'_t : M \to M$ rel N such that $h'_0 = h_0$, $h'_1 = h_1$ and $h'_t = h_t$ $(0 \leq t \leq 1)$ on $M \setminus K$ for some compact subset K of M.

Corollary 5.2. *Under the same condition as in Theorem* 5.2, *we have $\mathcal{H}_N(M) \cap \mathcal{H}_X(M)_0 = \mathcal{H}_N(M)_0$.*

We conclude this section with some problems. Suppose M is a compact PL n-manifold, $n \geq 3$, and X is a compact subpolyhedron of M.

Problem. (1) Is $\mathcal{H}_X(M)$ always an ℓ^2-manifold ?
(2) Is the triple $(\mathcal{E}(X,M), \mathcal{E}^{\mathrm{LIP}}(X,M), \mathcal{E}^{\mathrm{PL}}(X,M))$ always an (s, Σ, σ)-manifold ?
(3) When I is an arc and D^n is an n-disk, calculate the homotopy group of $\mathcal{E}(I, D^n)$ for $n \geq 3$
(4) Extend the theory of topological embeddings from the viewpoint of spaces of embeddings.

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