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Metrizability of GO-spaces, and k-spaces

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In this paper, first we shall make a survey of metrizability theorems by means of spaces with certain k-networks, GO-spaces, or topological groups. Then, we will give some metrizability theorems on GO-spaces or topological groups in terms of weak topology.

Definition 1: As is well-known, a *linearly ordered topological space* (abbreviated LOTS) is a triple $(X, \mathcal{T}, \leq)$, where $(X, \leq)$ is a linearly ordered (= totally ordered) set, and $\mathcal{T}$ is the order topology by the order $\leq$; that is, $\{(\alpha, +\infty), (-\infty, \alpha) : \alpha \in X\}$ is a subbase for $\mathcal{T}$, here $(\alpha, +\infty) = \{x \in X : x > \alpha\}, (-\infty, \alpha) = \{x \in X : x < \alpha\}$.

A space X is a *generalized ordered space* (abbreviated GO-space) if X is a subspace of a LOTS Y, where the order of X is the one induced by the order of Y. For many important properties of GO-spaces, see [8] or [10], for example.

We recall that a space $(X,\mathcal{T})$ is *orderable* if there exists a linear order $\leq$ on X such that the order topology on X given by $\leq$ coincides with the topology $\mathcal{T}$. Obviously, a space X is orderable iff it is homeomorphic to a LOTS. A space X is called *suborderable* if it is homeomorhic to a GO-space.

Examples: (1) The Sorgenfrey line, or the Michael line, etc. is a GO-space, but it is not a LOTS with the usual ordering, not even orderable. Also, any Stone-Čech compactification $\beta(X)$ of a completely regular, non-countably compact space X is not orderable ([17]).

(2) Let $S = \{0\} \cup (1, 2)$ be a subspace of the real line R, and let D be an infinite countable discrete space. Then, S is the topological sum $\{0\} \cup (1, 2)$ of LOTS' in R, and also, S is an open and closed subset of the product space $S \times D$ which is orderable. However, S is not orderable.

(3) A subspace $\{0\} \cup (1, 2]$ of R with the usual ordering is not a LOTS, but it is orderable. Also, a space $X = [0, \omega_1]$ obtained by isolating every countable limit ordinal is not a LOTS with the usual ordering, but X is orderable (cf. [8]).

(4) Let X be the quotient space of the topological sum of three unit intervals $[0, 1]$ by identifying the point 0 to a point. Then, X is a union of two closed LOTS', but X is not suborderable.

In this paper, however, let us say that a space X is a LOTS (resp. GO-space) if X is orderable (resp. suborderable), for it will cause no confusion.

Definition 2: A space X *is determined by* a cover $\mathcal{C}$ if $F \subset X$ is closed in X iff $F \cap C$ is closed in C for every $C \in \mathcal{C}$. We use "X is determined by $\mathcal{C}$" instead of the usual "X has the weak topology with respect to $\mathcal{C}$".

A space is a *k-space* (resp. *sequential space*) if it is determined by a cover of compact subsets (resp. compact metric subsets). A space is a *quasi-k-space* (Nagata [9]) if it is determined by a cover of countably compact subsets.

As is well-known, every k-space (resp. sequential space) is precisely a quotient image of a locally compact space (resp. metric space). Also, every quasi-k-space (resp. k-space) is characterized as a quotient image of an M-space (resp. paracompact M-space); see [9].

A space X *has countable tightness* (or, $t(X) \leq \omega$) if, whenever $x \in clA$, then $x \in clB$ for some countable subset B with $B \subset A$. It is well-known that $t(X) \leq \omega$ iff X is determined by a cover of countable subsets.

Sequential spaces are k-spaces of countable tightness, and k-spaces are quasi-k-spaces.

Definition 3: Let $\mathcal{P}$ be a cover of a space X. Then, $\mathcal{P}$ is a *k-network* for X, if whenever $K \subset U$ with K compact and U open in X, $K \subset \cup\mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. Also, $\mathcal{P}$ is a *wcs*-network* if whenever L is a sequence converging to a point $x \in X$ and U is a nbd of x, some $P \in \mathcal{P}$ is contained in U, but contains the sequence L frequently. Every k-network is a *wcs**-network.

CW-complexes, Lašnev spaces, or quotient s-images of metric spaces are sequential spaces having a point-countable k-network.

Definition 4: A space X is a *$w\Delta$-space* if there exists a sequence $\{\mathcal{U}_n; n \in N\}$ of open covers of X such that if $x \in X$ and $x_n \in \mathrm{St}(x,\mathcal{U}_n)$, then the sequence $\{x_n\}$ has an accumulation point in X. Every developable space, or every M-space is a $w\Delta$-space. Recall that a space X is a Σ-space, if there exist a σ-locally-finite closed cover $\mathcal{F}$ in X, and a cover $\mathcal{C}$ of countably compact closed subsets in X such that, for $C \subset U$ with $C \in \mathcal{C}$ and U open in X, $C \subset F \subset U$ for some $F \in \mathcal{F}$. Every M-space, σ-space, or locally compact GO-space [8] is a Σ-space.

A collection $\mathcal{C}$ in X is *compact-finite* if any compact subset of X meets only finitely many elements of $\mathcal{C}$.

We assume that spaces are regular T_1, and maps are continuous and onto.

Survey of Metrizability theorems

Basic Metrizability theorem: (1) *Every M-space X is metrizable if X has a σ-hereditarily closure-preserving network (more generally, X has a G_δ-diagonal), or X has a point-countable base.* This is well-known; see [10], for example.

(2) (Lutzer [8]) *Every GO-space X is metrizable if X has a σ-hereditarily closure-preserving network, more generally, X is a semi-stratifiable space.*

(3) (Birkhoff-Kakutani (1936)) *Every first countable topological group is metrizable.*

(4) (Gruenhage-Michael-Tanaka [3]) *Every paracompact M-space with a point-countable k-network is metrizable.*

(5) (Filippov (1969)) *Every quotient s-image X of a locally separable metric space is metrizable if X is first countable.*

Metrizability theorem A: (1) Every M-space X with a point-countable k-network is metrizable if X is a k-space.

(2) Every M-space X with a σ-locally countable k-network is metrizable.

(3) Every $w\Delta$-space X with a σ-closure-preserving k-network is metrizable (Tanaka and Murota [16]).

(4) Every GO-space X with a G_δ-diagonal is metrizable if X is a $w\Delta$-space or a Σ-space.

(5) Every k-space X with a σ-compact-finite k-network is metrizable if X contains no closed copy of the *sequential fan* S_ω, and no the *Arens' space* S_2. In particular, every first countable space with a σ-hereditarily closure-preserving k-network is metrizable.

Metrizability theorem B: Let G be a topological group, and a k-space with a point-countable k-network. Then, G is metrizable if one of the following (a), (b), and (c) holds.

(a) G contains no closed copy of S_ω.

(b) G contains no closed copy of S_2.

(c) G has the sequentially order $\sigma(G) < \omega_1$ (Shibakov [14]).

Metrizability theorem C: (1) Let $f : X \to Y$ be a quotient compact map such that X is metric. If Y is a GO-space or topological group, then Y is metrizable.

(2) Let $f : X \to Y$ be a quotient s-map such that X is metric.

(i) If Y is a GO-space, then Y has a point-countable base. If X or Y is locally separable, then Y is metrizable.

(ii) If Y is a topological group satisfying one of (a), (b), and (c) in *Metrizability theorem B*, then Y is metrizable.

Remark: (1) Every topological group G satisfying the followind (a) and (b) need not be metrizable (by a topological group $G = \varinjlim\{R^n : n \in N\}$, the inductive limit of n-dimensional Euclidean spaces R^n).

(a) G is a sequential, $\aleph_0$-space with $\sigma(G) = \omega_1$.

(b) G is a quotient, countable-to-one image of a locally compact, separable metric space.

(2) Every GO-space M with a G_δ-diagonal, which is an open s-image of a metric space need not be metrizable (by the Michael line M).

Main result and Related matters

Metrizability theorem: Let X be a GO-space. If X has a point-countable wcs^*-network, then the following (1) and (2) hold.

(1) Suppose that one of the following properties (a), (b), and (c) holds. Then, X is a paracompact space with a point-countable base.

In particular, if X has a σ-compact-finite wcs^*-network, then X is metrizable.

(a) Each point of X is a G_δ-set.

(b) X is a quasi-k-space.

(c) $t(X) \leq \omega$.

(2) Suppose that one of the following properties (d), (e), (f) holds. Then, X is metrizable.

(d) X is locally separable.

(e) X is a (locally) $w\Delta$-space.

(f) X is a (locally) Σ-space.

Remark 1: Related to the previous theorem, the following metrizability theorem holds ([6]).

Metrizability theorem: Let G be a topological group. If G is a GO-space, then G is metrizable if one of the above properties (a) $\sim$ (f) holds.

Remark 2: (1) Not every countably compact space with a point-countable k-network is metrizable ([3]).

(2) Not every countably compact, first countable, LOTS X with a locally countable network is metrizable (by the order space $X = [0, \omega_1)$).

(3) Not every LOTS M^* with a σ-point-finite base is metrizable (by the usual LOTS M^* containing the Michael line M. Here, for a GO-space X, the usual LOTS X^* contaning X, see [8] for example).

Corollary 1: Every GO-space with a σ-locally countable wcs^*-network is metrizable.

Corollary 2: Let $X = \varinjlim\{X_n : n \in N\}$ such that X_n are metric spaces (resp. metric spaces of covering dimension zero), here X_n are not necessarily closed in X. Then, X is a GO-space $\Rightarrow$ X is a metrizable space (resp. X is a GO-space $\Leftrightarrow$ X is a metrizable space of covering dimension zero).

In particular, for locally separable metric, zero-dimensional spaces X_n, X is a GO-space $\Leftrightarrow$ X is the topological sum of subspaces of the Cantor set 2^ω.

Remark 3: Let $X = \varinjlim\{X_n : n \in N\}$ such that X_n are locally compact, topological groups. Here, X_n are not necessarily closed in X. Then, X is a GO-space $\Leftrightarrow$ (a), (b), or (c) below holds.

(a) X is a discrete space.

(b) X is the topological sum of the real lines R.

(c) X is the topological sum of the Cantor sets.

More details and other properties of GO-spaces and topological groups are investigated in the author's joint papers [6] *and* [7] *with C. Liu and M. Sakai.*

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