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Another proof of Hiramine's theorem on three-dimensional Schur rings

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1 Introduction

Let G be a finite group. For a subset S of G, let $S^{-1} = \{x^{-1} | x \in S\}$, $\bar{S} = \sum_{x\in S} x(\in C[G])$. Let $G = S_0 \cup S_1 \cup S_2$ be a partition of G of order n^2 such that $S_0 = \{1\}, S_1 = S_1^{-1}, S_2 = S_2^{-1}$ and $\bar{S}_i\bar{S}_j = \sum_{k=0}^{2} p_{i\,j}^{k}\bar{S}_k$, where $p_{i\,j}^{k}$,s are nonnegative integers $(0 \leq i,\ j \leq 2)$. The subring $\Re =< \bar{S}_0, \bar{S}_1, \bar{S}_2 >$ of $Z[G]$ is called a three-dimensional (3D) *Schur ring* over G. It is well known that the concept of a (3D) Schur ring is equivalent to that of a strongly regular Cayley graph(cf.[1]). We say that $\Re$ is *rational* if the eigenvalues of the corresponding strongly regular Cayley graph are rational. $\Re$ is called primitive if S_i generates G for all $i \neq 0$. $\Re$ is said to be of (n, r)-type if $|S_1| = r(n-1)$ for some r $(1 \leq r \leq n)$. We here note that by definition $\Re$ is a Schur ring of (n, r) - type if and only if it is of $(n, n-r+1)$-type.

We now give an example.

Example 1 *Let G be an group of order n^2 . Let $\{H_1, H_2, \ldots, H_r\}$ $(1 \leq r \leq n)$ be a partial spread of G with degree r. We set $S_0 = \{1\}, S_1 = H_1 \cup H_2 \cup \ldots H_r - \{1\}, S_2 = G - S_0 \cup S_1$. Then $< \bar{S}_0, \bar{S}_1, \bar{S}_2 >$ is a Schur ring of (n, r)-type over G.*

We note that the Schur ring of the example above satisfies an equation

$$\bar{S}_1^2 = r(n-1)\bar{S}_0 + (n + r^2 - 3r)\bar{S}_1 + r(r-1)\bar{S}_2. \qquad [A]$$

A Schur ring of (n,r)-type is said to be of Latin square type [2] if it satisfies [A].

We state a conjecture due to [2].

Conjecture 1 *Let $\Re =< \bar{S}_0, \bar{S}_1, \bar{S}_2 >$ be a Schur ring of (n,r)-type over an abelian group G of order n^2. Then $\Re$ is of Latin square type .*

Hiramine [2] verified the conjecture for the case $n > f'(r)$, where $f'(r) = 4r^5 - 8r^4 - 2r^3 - 10r^2 - 3r - 1$.

In this note we shall verify the conjecture for the case $n > f(r)$, where $f(r) = r^5 - 2r^4 + r^3 + 3r^2 - r$.

Notation. We follow the notation and terminology of [2].

2 Preliminary results

Assume that $\Re =< \bar{S}_0, \bar{S}_1, \bar{S}_2 >$ is a Schur ring of (n,r)-type over a group G of order n^2. By [3] we have

Lemma 1 *The following hold.*

(i) *$\Re$ is primitive unless $r \in \{1, n\}$.*

(ii) *$\Re$ is rational.*

In the rest of paper let us assume that $\Re =< \bar{S}_0, \bar{S}_1, \bar{S}_2 >$ is a Schur ring of (n,r)-type over an abelian group G of order n^2. We have the following, which is due to [2].

Lemma 2 *Set $\bar{S}_1^2 = a\bar{S}_0 + b\bar{S}_1 + c\bar{S}_2$, where a, b and c are some nonnegative integers. Then,*

(i) *$a = r(n-1)$ and $(c - r^2)n + r^2 + (b - c + 1)r + c = 0$.*

(ii) *If $n > 2r - 1$, then c is even.*

(iii) *Set $m = \sqrt{(b-c)^2 + 4(rn - r - c)}$. Then m is an integer and $m | n^2$.*

Lemma 3 $c \neq 0$.

Proof. If $c = 0$, then $\Re$ is non-primitive. This fact contradicts Lemma 1 (ii). ∎

Lemma 4 *If $r = 1$, then the conjecture is true.*

Proof. If $r = 1$, then $(n-1)^2 = (n-1) + b(n-1) + c(n^2 - (n-1))$. From this we see that $c = 0$ and $b = n - 2$, which show that $\Re$ is of Latin square type. ■

3 Sketch of Proof

If $c = r^2 - r$, then $b = n + r^2 - 3r$ and so the conjecture is true. Our proof is by contradiction. Therefore, we assume that $2 \leq r \leq n-1$, and $c \neq r^2 - r$.

Lemma 5 $c \neq r^2$.

Proof. See [2]. ■

Lemma 6 $2 \leq c \leq r^2 - 1$.

Proof. By Lemma 2 (i),

$$\begin{aligned} c &= r^2 + \frac{r^3 - 2r^2 - (b+1)r}{n - r + 1} \\ &< r^2 + \frac{r^3 - 2r^2 - r}{f(r) - r + 1} \\ &< r^2 + 1. \end{aligned}$$

Hence $c \leq r^2 - 1$ by Lemma 5. Lemmas 3 and 2 show that $2 \leq c$. ■

Assume $g = r^2 - c$, where $1 \leq g \leq r^2 - 2$. Set $d = g(n+1)/r$. Then d is a positive integer. After some calculations we have the following lemma, which is due to Hiramine [2].

Lemma 7

$$(gd + 2r^2 - 2rg - g + gm) | 2(r-g)^2(r^2 - g).$$

Proof. See [2]. ■

We now distinguish two cases.

(i) The case when $2 \leq c < r^2 - r$. The following is a key to our proof of the conjecture.

Lemma 8 *If $n > f(r)$, then*

$$\begin{aligned} m^2 - n^2 &= ((r - c/r)^2 - 1)n^2 + (2c^2/r^2 + 2c/r + 2r - 2r^2)n \\ &+ 1 - 2c + c^2/r^2 + 2c/r - 2r + r^2 \\ &> 0. \end{aligned}$$

Proof. Set $h(n) = r^2(m^2 - n^2)$. Recall that $g = r^2 - c$. So $r + 1 \le g < r^2 - 1$. Hence

$$r^2(1 - 2c + c^2/r^2 + 2c/r - 2r + r^2) > 0. \tag{B}$$

Observe that in case (i)

$$(r^2 - c)^2 - r^2 > 0. \tag{C}$$

From (B) and (C) it follows that

$$\begin{aligned} h(n) &> h'(n) = ((r^2 - c)^2 - r^2)n^2 + (2c^2 + 2cr + 2r^3 - 2r^4)n \\ &= n[((r^2 - c)^2 - r^2)n + 2c^2 + 2cr + 2r^3 - 2r^4] \\ &> 0, \quad \text{when } n \ge -1(2c^2 + 2cr + 2r^3 - 2r^4)/((r^2 - c)^2 - r^2). \end{aligned}$$

On the other hand, since $r + 1 \le g < r^2 - 1$, it follows that $2r^3 - 3r - 1 > -1(2c^2 + 2cr + 2r^3 - 2r^4)/((r^2 - c)^2 - r^2)$. Hence if $n(> f(r)) > 2r^3 - 3r - 1$, then $h(n) > 0$. This completes the proof of this lemma. ∎

So if $n > f(r)$, then $m > n$. From this inequality and Lemma 7 we have

$$gd + 2r^2 - 2rg - g + gn < 2(r - g)^2(r^2 - g). \tag{D}$$

Since $gd > gn$, substitution of gn in gd of the inequality (D) yields

$$2gn < 2(r - g)^2(r^2 - g) - 2r^2 - 2rg + g.$$

So

$$n < [(r - g)^2(r^2 - g) - r^2 - rg + g/2]/g. \tag{E}$$

Since $r + 1 \le g \le r^2 - 2$, the right hand side of (E) is less than $r^4 + r^3 - 5r^2 - 7r - 1/2$, which contradicts our assumption. So we complete the proof of our conjecture in this case.

(ii) The case when $r^2 - r < c \le r^2 - 1$. Elaborate arguments show that if $n > f(r)$, then $gn/r \le m$. From this inequality and Lemma 7 we have a contradiction, so we complete the proof of our conjecture. ∎

References

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[3] J. J. Seidel: *Strongly regular graphs with (-1,1,0) adjacency matrix having eigenvalue 3*, Linear Algebra Appl.**1**(1968). 281–298.