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CODES AND VERTEX OPERATOR ALGEBRAS

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1. Introduction

Vertex operator algebra (VOA) is essentially the chiral algebra in conformal field theory (cf. [22]). It provides a very powerful tool for studying the general structure of conformal field theory. Nevertheless, the notion of vertex operator algebra is also well-motivated by the study of the Monster simple group — the largest sporadic group. In fact, vertex operator algebra was introduced in an attempt of explaining certain mysterious relations between the Monster and some modular functions (cf. [1], [2], [3], and [13]). In particular, vertex operator algebra is a very useful tool in studying certain properties of the Monster and other sporadic groups.

Unfortunately, vertex operator algebras are, in general, very difficult to construct. In fact, almost all known examples are constructed from some auxiliary structures such as lattices and Lie algebras or their variations (e.g. orbifold construction). In this article, we shall discuss a method of constructing vertex operator algebras from certain type of codes and modules of Virasoro VOAs. Some important examples will also be discussed. Most of the results we mentioned here have been appeared in the literature (cf. [16], [17], [18], [19], [24], [25], and [26]). Please refer to the corresponding references for more details. As we have mentioned, this type of VOAs will be very useful for studying finite groups because one can easily define certain automorphisms of the vertex operator algebra by using the defining codes (cf. [23] and [26]).

2. Definitions and Terminologies

In this section, we shall review some necessary definitions and terminologies. First, let us recall the definition of vertex operator algebra.

Definition 2.1. A vertex operator algebra (VOA) is a $\mathbb{Z}$-graded vector space $V=\amalg_{n\in\mathbb{Z}}V_n$ equipped with a linear map

$$\begin{aligned} Y(\ ,z):V &\longrightarrow \text{End}\, V\left[[z,z^{-1}]\right] \\ v &\longrightarrow Y(v,z)=\sum_{i\in\mathbb{Z}} v_i z^{-i-1} \end{aligned}$$

such that the following conditions hold:

1. (Vacuum condition) there is a vector $\mathbf{1}$ such that
$$Y(\mathbf{1},z)=id|_V;$$
2. (Creation property) $Y(v,z)\cdot\mathbf{1}\in V[[z]]$ and $\lim_{z\to 0}Y(v,z)\cdot\mathbf{1}=v$ for any $v\in V$ (i.e., $Y(v,z)\cdot\mathbf{1}$ involves only non-negative integral powers of z and the constant term is v);
3. $\dim V_n<\infty$ and $V_n=0$ for sufficiently small n;

4. for any $u, v \in V$,
$$u_n v = 0 \text{ for } n \text{ sufficiently large;}$$
5. (Virasoro condition) there is a vector ω such that the operators $L_i = \omega_{i+1}$ satisfy the Virasoro relation:
$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c$$
where c is a scalar and is called the rank of V;
6. $L_0 v = nv = (wt\ v)\, v$ for $v \in V_n$;
7. (L_{-1}-derivative property)
$$Y(L_{-1}v, z) = \frac{d}{dz} Y(v, z);$$
8. (Jacobi Identity) for any $u, v \in V$,
$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1) Y(v, z_2) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) Y(v, z_2) Y(u, z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2)$$

Remark 2.2. The Jacobi identity (8) can be equivalently replaced by the following commutativity (see Dong and Lepowsky [7] and Li [20]):

$$(z_1 - z_2)^n \left(Y(u, z_1) Y(v, z_2) - Y(v, z_2) Y(u, z_1)\right) = 0 \tag{2.1}$$

for a sufficiently large positive integer n. Here, n depends on both u and v.

Remark 2.3. By the Jacobi Identity, one can also obtain the following commutator formula (cf. [13]): for any $u, v \in V$ and $m, n \in \mathbb{Z}$,
$$[u_m, v_n] = \sum_{i \geq 0} \binom{m}{i} (u_i v)_{m+n-i}\,.$$

We can define the notion of modules and intertwining operators in a similar way.

Definition 2.4. A module for a VOA is a $\mathbb{Q}$-graded vector space $M = \amalg_{n \in \mathbb{Q}} M_n$ equipped with a linear map
$$\begin{aligned} Y_M(\ ,z) : V &\longrightarrow \text{End}\, M\left[\left[z, z^{-1}\right]\right] \\ v &\longrightarrow Y_M(v, z) = \sum_{i \in \mathbb{Z}} v_i z^{-i-1} \end{aligned}$$
such that all the conditions mentioned in Definition 2.1 also hold, provided that they make sense.

Definition 2.5. Let (V, Y) be a VOA and let $(W^1, Y^1), (W^2, Y^2)$ and (W^3, Y^3) be V-modules. An intertwining operator of type $\begin{pmatrix} & W^1 & \\ W^2 & & W^3 \end{pmatrix}$ is a linear map
$$\begin{aligned} I(\ ,z) : W^2 &\longrightarrow \text{Hom}\left(W^3, W^1\right)\{z\} \\ u &\longrightarrow I(\,u, z) = \sum_{n \in \mathbb{Q}} u_n z^{-n-1} \end{aligned}$$

satisfying:

1. for any $u \in W^2$ and $v \in W^3$,

$$u_n v = 0 \text{ for } n \text{ sufficiently large;}$$

2. $I(L_{-1}v, z) = \frac{d}{dz} I(v, z)$;
3. (Jacobi Identity) for any $u \in V, v \in W^2$

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1) I(v, z_2) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) I(v, z_2) Y(u, z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) I(Y(u, z_0)v, z_2).$$

The set of all intertwining operators of type $\begin{pmatrix} & W^1 & \\ W^2 & & W^3 \end{pmatrix}$ is denoted by $I_V\begin{pmatrix} & W^1 & \\ W^2 & & W^3 \end{pmatrix}$. We shall often omit the V in $I_V\begin{pmatrix} & W^1 & \\ W^2 & & W^3 \end{pmatrix}$ in order to simplify the notation.

Remark 2.6. Let $N^{W^1}_{W^2,W^3} = \dim I_V\begin{pmatrix} & W^1 & \\ W^2 & & W^3 \end{pmatrix}$. These integers $N^{W^1}_{W^2,W^3}$ are usually called the "fusion rule". For convenience, we shall often consider the fusion product

$$W^2 \times W^3 = \sum_W N^W_{W^2,W^3} W$$

where W runs over the set of equivalence classes of irreducible V-modules.

Definition 2.7. Let $(V^1, Y^1, \omega^1, 1^1)$ and $(V^2, Y^2, \omega^2, 1^2)$ be vertex operator algebras. The tensor product of V^1 and V^2 is the quadruple

$$\left(V^1 \otimes V^2, Y^1 \otimes Y^2, \omega^1 \otimes 1^2 + 1^1 \otimes \omega^2, 1^1 \otimes 1^2\right)$$

with the vertex operator $Y^1 \otimes Y^2$ defined as follows:

$$Y^1 \otimes Y^2 (u \otimes v, z) = Y^1(u, z) \otimes Y^2(v, z) \in \operatorname{End}\left(V^1 \otimes V^2\right)[[z, z^{-1}]]$$

for any $u \otimes v \in V^1 \otimes V^2$.

Remark 2.8. (cf. [7] and [12]) The tensor product of vertex operator algebras is also a vertex operator algebra.

Proposition 2.9. *(cf. [12]) Let $V = V^1 \otimes \cdots \otimes V^n$ be the tensor product of the vertex operator algebras $V^1, \cdots, V^n$ and W an irreducible module of V. Then,*

$$W \cong W^1 \otimes \cdots \otimes W^n$$

where $W^1, \cdots, W^n$ are irreducible modules of $V^1, \cdots, V^n$ respectively.

Before we go on, let us first discuss certain important examples of vertex operator algebras.

2.1. **Vertex operator algebra associated with Virasoro algebras.**
Let

$$Vir = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \bigoplus \mathbb{C}\mathbf{c}$$

be a Virasoro algebra, i.e.,

$$\begin{aligned} [L_m, L_n] &= (m-n) L_{m+n} + \frac{1}{12}\left(m^3 - m\right)\delta_{m+n,0}\mathbf{c}; \\ [L_m, \mathbf{c}] &= 0. \end{aligned}$$

Then, $\mathfrak{b} = \left(\bigoplus_{n\geq 1} \mathbb{C}L_n\right) \bigoplus \mathbb{C}L_0 \bigoplus \mathbb{C}\mathbf{c}$ is a subalgebra of Vir.

For any h and c in $\mathbb{C}$, define a 1-dimensional $\mathfrak{b}$-module $\mathbb{C}\cdot 1$ by

$$\begin{aligned} L_n \cdot 1 &= 0 \text{ for } n \geq 1 \\ L_0 \cdot 1 &= h \cdot 1 \text{ and } \mathbf{c} \cdot 1 = c1. \end{aligned}$$

The Verma module of weight h and central charge c is the Vir-module given by

$$M(c,h) = U(Vir) \otimes_{U(\mathfrak{b})} \mathbb{C} \cdot 1.$$

If $h = 0$, then the Vir-module generated by $L_{-1} \cdot 1$ is a proper submodule of $M(c,h)$. We shall denote

$$V(c,0) = M(c,0) / < L_{-1} \cdot 1 > .$$

It was shown by Frenkel and Zhu [14] that $V(c,0)$ is a VOA. Moreover, $V(c,0)$ has a unique maximal ideal J (i.e., maximal $V(c,0)$-submodule of $V(c,0)$). Denote

$$L(c,0) = V(c,0)/J.$$

Then, $L(c,0)$ is a simple VOA, i.e., $L(c,0)$ is irreducible as a $L(c,0)$-module. This class of VOAs is often referred to as simple Virasoro VOA. It is well known (cf. [11], [15] and [27]) that $L(c,0)$ has only finitely many irreducibles and all of its modules are completely reducible if the central charge

$$c = 1 - \frac{6}{(m+2)(m+3)}, \text{ for } m = 1, 2, 3, \cdots .$$

Such kind of VOA is called rational.

2.2. **Fock spaces associated with lattices.** Let L be a lattice with a nondegenerate $\mathbb{Q}$-valued bilinear form denoted by $< \, , \, >$ (although it is not necessary, we shall assume that $< \, , \, >$ is positive definite). Let L_0 be an even sublattice of L such that $< \alpha, L > \subset \mathbb{Z}$ for $\alpha \in L_0$ and rank $L_0 =$ rank L. Let $\hat{L}$ be a central extension of L

$$1 \to < \omega_p > \to \hat{L} \to L \to 1,$$

with the commutator map $c(\alpha,\beta)$ for $\alpha, \beta \in L$ such that $c(\alpha,\beta) = (-1)^{<\alpha,\beta>}$ if $\alpha,\beta \in L_0$, where ω_p is a primitive p-th root of unity and p is an even positive integer.

We shall view $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ as an abelian Lie algebra and consider its affine Lie algebra

$$\tilde{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}\left[z, z^{-1}\right] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

$\tilde{\mathfrak{h}}$ has a Heisenberg subalgebra

$$\hat{\mathfrak{h}}_{\mathbb{Z}} = \oplus_{n\neq 0} \left(\mathfrak{h} \otimes t^n\right) \oplus \mathbb{C}c.$$

Let $\hat{\mathfrak{h}}_{\mathbb{Z}}^{\pm} = \oplus_{\mathbb{Z}^{\pm}} (\mathfrak{h} \otimes t^n)$. Then, we have a triangular decomposition

$$\hat{\mathfrak{h}}_{\mathbb{Z}} = \hat{\mathfrak{h}}_{\mathbb{Z}}^{+} \oplus \mathbb{C}c \oplus \hat{\mathfrak{h}}_{\mathbb{Z}}^{-}.$$

Let $M(1) = M_L(1)$ be the $\hat{\mathfrak{h}}_{\mathbb{Z}}$-module induced from the 1-dimensional $\hat{\mathfrak{h}}_{\mathbb{Z}}^{+} \oplus \mathbb{C}c$-module $\mathbb{C}\cdot 1$ such that $\alpha \otimes t^n \cdot 1 = 0$, for $n > 0$ and $c\cdot 1 = 1$.

We shall denote the action of $\alpha \otimes t^n$ on $M(1)$ by $\alpha(n)$.

Let

$$\mathbb{C}\{L\} = \mathbb{C}[\hat{L}] \otimes_{\mathbb{C}[\omega_p]} \mathbb{C} \cong \mathbb{C}[L] \text{ (linearly)}$$

where $\mathbb{C}[L] = \operatorname{span}\{e^{\alpha} | \ \alpha \in L\}$ is the group algebra of L and $\mathbb{C}$ is an 1-dimensional module of the group algebra $\mathbb{C}[\omega_p]$ such that ω_p acts as multiplication by ω_p.

The Fock space of L is the space

$$V_L = M(1) \otimes_{\mathbb{C}} \mathbb{C}\{L\}.$$

For a subset $M \subset L$, we shall also denote

$$V_M = M(1) \otimes_{\mathbb{C}} \mathbb{C}\{M\}$$

where $\mathbb{C}\{M\}$ is the subspace spanned by $e^{\alpha}, \alpha \in M$.

Remark 2.10. If L is doubly even and $L = L_0$, then $< \alpha, \beta >= 0$ for all $\alpha, \beta \in L$ and thus $c(\alpha, \beta) = 1$. In this case, $\mathbb{C}\{L\} \cong \mathbb{C}[L]$ as algebras.

An vertex operator $Y(\cdot, z)$ can be defined on V_L as follows:
For $\alpha \in L$,

$$Y(e^{\alpha}, z) = E^{-}(-\alpha, z) E^{+}(-\alpha, z) e^{\alpha} z^{\alpha}$$

where $E^{\pm}(\alpha, z) = \exp\left(\sum_{n \in \mathbb{Z}^{\pm}} \frac{\alpha(n)}{n} z^{-n}\right)$ and $z^{\alpha} \cdot e^{\beta} = z^{<\alpha,\beta>} e^{\beta}$;

For a general element $v = \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes e^{\alpha}$, we define

$$\begin{aligned} Y(v, z) =: \quad & \left(\frac{1}{(n_1 - 1)!} \left(\frac{d}{dz}\right)^{n_1 - 1} \alpha_1(z)\right) \cdots \\ & \left(\frac{1}{(n_k - 1)!} \left(\frac{d}{dz}\right)^{n_k - 1} \alpha_k(z)\right) Y(e^{\alpha}, z) : \end{aligned}$$

where $: \cdots :$ is the normal ordered product.

Proposition 2.11. *(cf.* [4] *and* [13]*) If L is an even positive definite lattice, then (V_L, Y) is a vertex operator algebra.*

Next, we shall recall the notion of commutant (cf. [14]).

Definition 2.12. Let $(V, Y, \omega, 1)$ be a vertex operator algebra and $(W, Y, \omega', 1)$ be a vertex operator subalgebra of V. Note that the Virasoro elements of V and W are different. The commutant of W in V is defined to be the subspace

$$W^c = \{v \in V | \ w_n v = 0, \text{ for all } w \in W \text{ and } n \geq 0\}$$

Proposition 2.13. $(W^c, Y, \omega'', 1)$ *is a vertex operator algebra where* $\omega'' = \omega - \omega'$.

Proof. First, we shall show that W is closed under the actions of the vertex operators.

Let $x, y \in W^c$ and $m \in \mathbb{Z}$. Then, for any $w \in W$ and $n \geq 0$,

$$w_n\left(x_m y\right) = x_m\left(w_n y\right) + \sum_{i=0}^{\infty} \binom{n}{i} \left(w_i x\right)_{n+m-i} y = 0. \qquad \text{(cf. Remark 2.3)}$$

Therefore, $x_m y \in W^c$ and W is closed under the actions of the vertex operators.

Now, denote

$$L''(n) = \left(\omega''\right)_{n+1} = \left(\omega - \omega'\right)_{n+1} = L(n) - L'(n)$$

where $L(n)$ and $L'(n)$ are the Virasoro operators of the VOAs V and W respectively. Then, $L(n) = L'(n) + L''(n)$ and we have

$$\left[L(m), L(n)\right] = \left[L'(m) + L''(m), L'(n) + L''(n)\right].$$

Since

$$\left[L'(m), L''(n)\right] = \left[\left(\omega'\right)_{m+1}, \omega''_{n+1}\right] = \sum_{i=0}^{\infty} \binom{m+1}{i} \left(\left(\omega'\right)_i \omega''\right)_{n+m+2-i}$$

and $\left(\omega'\right)_i \omega'' = 0$ for all $i \geq 0$, we have

$$\left[L'(m), L''(n)\right] = 0.$$

Thus,

$$\left[L(m), L(n)\right] = \left[L'(m), L'(n)\right] + \left[L''(m), L''(n)\right].$$

Therefore,

$$\begin{aligned}
\left[L''(m), L''(n)\right] &= \left[L(m), L(n)\right] - \left[L'(m), L'(n)\right] \\
&= (m-n) L(m+n) + \frac{1}{12}\left(m^3 - m\right) \delta_{m+n,0} c \\
&\quad - (m-n) L'(m+n) + \frac{1}{12}\left(m^3 - m\right) \delta_{m+n,0} c' \\
&= (m-n)\left[L(m+n) - L'(m+n)\right] + \frac{1}{12}\left(m^3 - m\right) \delta_{m+n,0}\left[c - c'\right] \\
&= (m-n) L''(m+n) + \frac{1}{12}\left(m^3 - m\right) \delta_{m+n,0}\left[c - c'\right].
\end{aligned}$$

Moreover, for any $v \in W^c$,

$$\left[L''(0), v\right] = \left[L(0) - L'(0), v\right] = \left[L(0), v\right]$$

and

$$Y\left(L''(-1) v, z\right) = Y\left(\left(L(-1) - L'(-1)\right) v, z\right) = Y\left(L(-1) v, z\right) = \frac{d}{dz} Y(v, z).$$

Therefore, ω'' is a Virasoro element of W^c with the central charge $c - c'$. The other axioms of VOA clearly hold in this case as $W \subset V$. □

3. VOA ASSOCIATED WITH CODES

In this section, we shall discuss the methods of constructing VOAs from codes.

Let L_0 be an even lattice (i.e., $< x, x >\in 2\mathbb{Z}$ for all $x \in L_0$) and L a sublattice of

$$L_0^{\perp} = \{x \in \mathbb{Q}\otimes_{\mathbb{Z}} L_0 | < x, L_0 >\in \mathbb{Z}\}.$$

Denote $G = L/L_0$. Then, G is a finite abelian group. We shall denote $G^n = G \oplus \cdots \oplus G$ be the direct sum of n-copies of G.

Let $\{L^g | g \in G = L/L_0\}$ be the set of all cosets of L/L_0. Note that $V^0 = V_{L_0}$ is a VOA and $V^g = V_{L^g}, g \in G$ are irreducible modules of V^0 (cf. [4]).

Let α be a representative of L^g. We shall define the norm of g by

$$|g| =< \alpha, \alpha > \mod 2\mathbb{Z}$$

and for any $\delta = (\delta^1, \cdots, \delta^n) \in G^n$, the norm of δ is defined by

$$|\delta| = \sum_{i=1}^{n} |\delta^i| \mod 2\mathbb{Z}.$$

An element δ is called even if $|\delta| \equiv 0 \mod 2\mathbb{Z}$.

Remark 3.1. The definition of $|g|$ is independent of the choice of the representative α.

Proof. Let $\gamma \in L^0$. Then,

$$\begin{aligned} < \alpha+\gamma, \alpha+\gamma > &= < \alpha, \alpha > +2 < \alpha, \gamma > + < \gamma, \gamma > \\ &\equiv < \alpha, \alpha > \qquad \mod 2\mathbb{Z}. \end{aligned}$$

□

Now, let $L^n = L \oplus \cdots \oplus L$ be the orthogonal sum of n-copies of G.
For any $\delta = (\delta^1, \cdots, \delta^n) \in G^n$, define

$$L^{\delta} = \oplus_{i=1}^{n} L^{\delta^i} \subset L^n.$$

Moreover, for any subgroup $D \subset G^n$, we define

$$L_D = \bigcup_{\delta \in D} L^{\delta}.$$

Clearly, L_D is sublattice of L^n. In fact, we have the following:

Theorem 3.2. *Let $D \subset G^n$ be even subgroup (i.e., all elements of D have even norms). Then, L_D is an even lattice.*

Proof. Let $x \in L^{\delta}$. Then, $x = (x^1, \cdots, x^n)$ where $x^i \in L^{\delta^i}$.
Therefore,

$$\begin{aligned} < x, x > &= \sum_{i=1}^{n} < x^i, x^i > \\ &= \sum_{i=1}^{n} |\delta^i| \mod 2\mathbb{Z} \\ &= 0 \mod 2\mathbb{Z}. \end{aligned}$$

Hence, $L_D = \bigcup_{\delta \in D} L^{\delta}$ is an even lattice. □

Therefore, if D is even, then V_{L_D} is a VOA. Moreover, we have

$$V_{L_D} = \bigoplus_{\delta \in D} V^{\delta}$$

where $V^{\delta} = V^{\delta^1} \otimes \cdots \otimes V^{\delta^n}$.

Now, let $(W, Y, \omega', 1)$ be any subalgebra of V^0. Define

$$M^g = \{v \in V^g |\ w_n v = 0 \text{ for all } w \in W, n \geq 0\}, \ g \in G.$$

For any $D \subset G^n$, we shall denote

$$M_D = \bigoplus_{\delta \in D} M^{\delta}$$

where $M^{\delta} = \otimes_{i=1}^{n} M^{\delta^i} \subset V^{\delta}$ for $\delta = (\delta^1, \cdots, \delta^n) \in D$.

Let $W^n = \otimes_{i=1}^{n} W$ be the tensor product of n-copies of the vertex operator algebra W. Then,

$$M_D = (W^n)^c \text{ in } V_{L_D}.$$

Therefore, M_D is also a VOA if D is even.

3.1. **VOA associated with binary codes.** Let $L = \mathbb{Z}x$ with $< x, x >= 1$ and $L_0 = 2\mathbb{Z}x$. Then, L_0 is a doubly even lattice and $L/L_0 \cong \mathbb{Z}_2$. Let

$$\omega^1 = \frac{1}{4} x(-1)^2 + \frac{1}{4}\left(e^{2x} + e^{-2x}\right)$$

and

$$\omega^2 = \frac{1}{4} x(-1)^2 - \frac{1}{4}\left(e^{2x} + e^{-2x}\right).$$

Then, ω^1 and ω^2 are two mutually orthogonal conformal vectors of central charge $\frac{1}{2}$, i.e., the VOA generated by ω^1 and ω^2 is isomorphic to $L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{1}{2}, 0\right)$. Moreover, $\omega^1 + \omega^2$ is the Virasoro element of V_{L_0} (cf. Miyamoto [24]). In fact,

$$V_{L_0} \cong L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{1}{2}, 0\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{1}{2}, \frac{1}{2}\right)$$

and

$$V_{x+L_0} \cong L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{1}{2}, \frac{1}{2}\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{1}{2}, 0\right)$$

as $L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{1}{2}, 0\right)$-modules.

Define

$$M^0 = \left\{v \in V_{L_0} |\ \left(\omega^1\right)_1 v = 0\right\}$$

and

$$M^1 = \left\{v \in V_{x+L_0} |\ \left(\omega^1\right)_1 v = 0\right\}.$$

Then,

$$M^0 \cong L\left(\frac{1}{2}, 0\right) \text{ and } M^1 \cong L\left(\frac{1}{2}, \frac{1}{2}\right).$$

Note that $M^0 =< \omega^1 >^c$ in V^0 where $< \omega^1 >$ denotes the VOA generated by ω^1.

Moreover, we have

Theorem 3.3. *Let D be an even binary code. Then, $M_D = \bigoplus_{\delta\in D} M^\delta$ is a VOA where $M^\delta = \otimes_{i=1}^n M^{\delta^i}$.*

Proof. First, we shall note that $<x,x>=1$. Therefore,

$$\begin{aligned} <\alpha,\alpha> &\equiv 0 \quad \bmod 2\mathbb{Z} \text{ if } \alpha\in L_0 = 2\mathbb{Z}x; \\ <\alpha,\alpha> &\equiv 1 \quad \bmod 2\mathbb{Z} \text{ if } \alpha\in L_1 = x + L_0. \end{aligned}$$

It is now clear that L_D is even if D is an even binary code. Thus, M_D is a VOA. □

Remark 3.4. The above VOA was first constructed by Miyamoto [24] using a slightly different method.

3.2. **VOA associated with ternary codes.** Next, we shall consider the case for ternary codes. This kind of VOA was first constructed by Kitazume, Miyamoto and Yamada [16].

Let $L_0 = \sqrt{2}A_2$ where A_2 denote the root lattice of type A_2. Then, the dual of L_0

$$L_0^\perp = \{\alpha\in \mathbb{Q}\otimes_{\mathbb{Z}} L_0 |\ <\alpha, L_0> \subset \mathbb{Z}\}$$

has exactly 12 cosets modulo L_0. We shall consider the sublattice

$$L = 2L_0^\perp = L^0\cup L^1\cup L^2$$

where $L^0 = L_0, L^1 = \frac{-x+y}{3} + L_0$ and $L^2 = \frac{x-y}{3} + L_0$ and $x=\sqrt{2}\alpha_1, y = \sqrt{2}\alpha_2, \{\alpha_1,\alpha_2\}$ are simple roots of A_2. Note that $L/L_0\cong \mathbb{Z}_3$.

It was shown by Dong, Li, Mason and Norton [8] that the Virasoro element of $V^0 = V_{L_0} = V_{\sqrt{2}A_2}$ can be written as a sum of three mutually orthogonal conformal vectors,

$$\begin{aligned} \omega^1 &= \frac{1}{8}\alpha_1(-1)^2 - \frac{1}{4}x_{\alpha_1}, \\ \omega^2 &= \frac{1}{40}\left(-\alpha_1(-1)^2 + 4\alpha_2(-1)^2 + 4\alpha_3(-1)^2\right) \\ &\quad -\frac{1}{20}\left(-x_{\alpha_1} + 4x_{\alpha_2} + 4x_{\alpha_3}\right), \\ \omega^3 &= \frac{1}{15}\left(\alpha_1(-1)^2 + \alpha_2(-1)^2 + \alpha_3(-1)^2\right) + \frac{1}{5}\left(x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3}\right) \end{aligned}$$

where $x_{\alpha_i} = e^{\sqrt{2}\alpha_i} + e^{-\sqrt{2}\alpha_i}$, $\alpha_3 = \alpha_1+\alpha_2$ and the central charge of ω^1, ω^2 and ω^3 are $\frac{1}{2}$, $\frac{7}{10}$ and $\frac{4}{5}$ respectively. In other words, V_{L_0} contains a subalgebra T (with the same Virasoro element) such that

$$T\cong L\left(\frac{1}{2},0\right)\otimes L\left(\frac{7}{10},0\right)\otimes L\left(\frac{4}{5},0\right).$$

Now, define

$$M^i = \left\{v\in V_{L^i}|\ \left(\omega^1\right)_1 v = \left(\omega^2\right)_1 v = 0\right\},\ i=0,1,2.$$

Note that

$$M^0 \cong L\left(\frac{4}{5},0\right) \oplus L\left(\frac{4}{5},3\right);$$
$$M^1 \cong L\left(\frac{4}{5},\frac{2}{3}\right);$$
$$M^2 \cong L\left(\frac{4}{5},\frac{2}{3}\right)$$

as $L\left(\frac{4}{5},3\right)$-modules (see [16] for more details).

Theorem 3.5. *(cf.* [16]*) Let D be a self-orthogonal code over $\mathbb{Z}_3$ of length n. Then,*

$$M_D = \bigoplus_{\delta\in D} M^\delta \quad \text{with } M^\delta = \otimes_{i=1}^n M^{\delta^i}$$

is a VOA.

Proof. We shall show that L_D is an even lattice if D is a self-orthogonal ternary code. First, let us note that

$$<\alpha,\alpha>\equiv 0 \mod 4\mathbb{Z} \text{ if } \alpha \in L^0$$

and

$$<\alpha,\alpha>\equiv \frac{4}{3} \mod 4\mathbb{Z} \text{ if } \alpha \in L^1 \text{ or } L^2.$$

Therefore, if D is a self-orthogonal ternary code,

$$<\gamma,\gamma>\equiv 0 \mod 4\mathbb{Z} \text{ for any } \gamma \in L^\delta, \delta \in D.$$

Hence, L_D is, in fact, a doubly even lattice. (cf. [16]) □

3.3. **VOA associated with codes over $\mathbb{Z}_2 \times \mathbb{Z}_2$.** As in the last section, $L_0 = \sqrt{2}A_2$ and $L_0^\perp = \{\alpha \in \mathbb{Q}\otimes_{\mathbb{Z}} L_0 |\ <\alpha, L_0>\subset \mathbb{Z}\}$ is the dual of L_0.

Consider the sublattice

$$L = 3L_0^\perp = L^0 \cup L^a \cup L^b \cup L^c$$

where $L^0 = L_0, L^a = \frac{y}{2}+L_0, L^b = \frac{x+y}{2}+L_0$ and $L^c = \frac{x}{2}+L_0$ and $K = \{0,a,b,c\} \cong \mathbb{Z}_2\times\mathbb{Z}_2$. Define

$$M^i = \left\{v \in V_{L^i} |\ \left(\omega^3\right)_1 v = 0\right\},\ i = 0,a,b,c.$$

Then,

$$M^0 \cong L\left(\frac{1}{2},0\right) \otimes L\left(\frac{7}{10},0\right) \oplus L\left(\frac{1}{2},\frac{1}{2}\right) \otimes L\left(\frac{7}{10},\frac{3}{2}\right);$$
$$M^a \cong L\left(\frac{1}{2},\frac{1}{16}\right) \otimes L\left(\frac{7}{10},\frac{7}{16}\right);$$
$$M^b \cong L\left(\frac{1}{2},\frac{1}{16}\right) \otimes L\left(\frac{7}{10},\frac{7}{16}\right);$$
$$M^c \cong L\left(\frac{1}{2},\frac{1}{2}\right) \otimes L\left(\frac{7}{10},0\right) \oplus L\left(\frac{1}{2},0\right) \otimes L\left(\frac{7}{10},\frac{3}{2}\right)$$

as $L\left(\frac{1}{2},0\right) \otimes L\left(\frac{7}{10},0\right)$-modules (cf. [19]).

Definition 3.6. A code C over $\mathbb{Z}_2 \times \mathbb{Z}_2$ of length n is simply a subgroup of K^n. An element $\delta \in C$ is called even if the number of non-zero entities in δ is even. A code is called even if all of its elements are even.

Theorem 3.7. *(cf. [19]) Let D be an even code over $\mathbb{Z}_2 \times \mathbb{Z}_2$. Then,*

$$M_D = \bigoplus_{\delta \in D} M^\delta \text{ with } M^\delta = \otimes_{i=1}^n M^{\delta^i}$$

is a VOA.

Proof. The proof is similar to Theorem 3.3. We shall note that

$$\begin{aligned} <\alpha,\alpha> &\equiv 0 \mod 2\mathbb{Z} \text{ if } \alpha \in L^0; \\ <\alpha,\alpha> &\equiv 1 \mod 2\mathbb{Z} \text{ if } \alpha \in L^a, L^b \text{ or } L^c. \end{aligned}$$

Therefore, L_D is an even lattice if D is an even code over $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

Final remark: By the above method, one can, in principle, construct vertex operator algebras associated with codes over $\mathbb{Z}_p$ and $\mathbb{Z}_2^{p-1}$ by using the lattices $\sqrt{2}A_{p-1}$ for any prime p. Nevertheless, the decomposition of the lattice VOA $V_{\sqrt{2}A_{p-1}}$ into Virasoro modules is quite complicated when $p > 3$ (cf. [9] and [28]). Therefore, the actual structures of these VOAs and their representations are still not very well understood.

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