数理解析研究所講究録
1109 巻 1999 年 136-140

A Generalization of Prime Graphs of Finite Groups

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1999

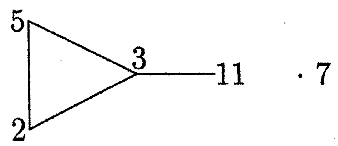
1 Introduction

There are a lot of ways to characterize a finite group by orders of its elements. Considering a prime graph is one of such ways. In a prime graph $\Gamma(G)$ of a finite group G, edges p and q are defined to be joined when there exists an element x of G whose order is pq. This condition can be interpreted that G includes a cyclic subgroup of order pq. So it seems natural to consider some other graphs in which the condition "being cyclic" is replaced to other ones. We will discuss solvable graph which will be defined afterward in this paper and will show some applications of the graphs. Every group appearing in this paper is a finite group. Following the notation in Iiyori-Yamaki[4] and Williams[9], π_i stands for the ith connected components of prime graphs of G in tables of [4],[9] and we let com(G) stand for the number of connected components of prime graph of G.

2 Definitions and Remarks

Definition 1 *Let Λ be a set of positive rational integers. We denote Λ-graph by Γ_Λ and the set of vertices of Γ_Λ by V_Λ which is the set of primes which divide an element of Λ. For vertices p and q of Γ_Λ, p is joined to q if and only if there exists an element a in Λ such that $pq|a$.*

For example, let $\Lambda = \{6, 7, 30, 33\}$. Then Γ_Λ is the following;

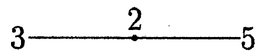


Definition 2 *Let Ξ be a group theoretical property. For a group G, $\mathcal{S}_\Xi(G)$ is the set of Ξ-subgroups of G. $\mathcal{S}_\Xi^*(G)$ is the set of Ξ-subgroups of G which do not coincide G. Let ρ be a mapping of $\mathcal{S}_\Xi(G)$ to the set of natural numbers.*

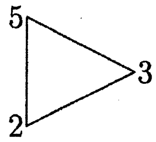
$\Gamma_{\rho(\mathcal{S}_\Xi(G))}$ stands for the (ρ, Ξ)-graph of G and $\Gamma_{\rho(\mathcal{S}_\Xi^*(G))}$ stands for the $(\rho, \Xi)^*$-graph of G.

We can consider several types of the mappings as follows: for $H \in \mathcal{S}_\Xi(G)$, "ord" : $H \mapsto |H|$, "ind" : $H \mapsto |G : H|$, "conj" : $H \mapsto$ the number of conjugacy classes of H to construct H and so on.

Let G be the alternating group A_5 of degree 5 and Ξ be "solvable". Then an element of $\mathcal{S}_\Xi(G)$ is isomorphic to one of the following groups: the alternating A_4 of degree 4, the dihedral group D_{10} of order 10, the symmetric group S_3 of degree 3. Hence the (ord, Ξ)-graph of A_5 is



Let Ξ' be "abelian ". Then the (ind, Ξ')-graph of A_5 is as follows.

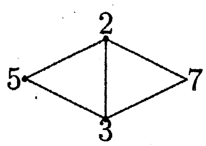


This time we focus on a mapping "ord" and disregard the rest. We denote the image of ord by $\mathrm{Ord}_\Xi(G)$ for convenience.

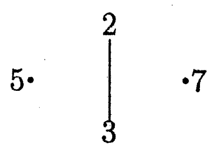
$$\mathrm{Ord}_\Xi(G) = \mathrm{ord}(\mathcal{S}_\Xi(G)) \subseteq \mathbf{N}$$

We simply call the (ord, Ξ)-graph of G *the Ξ-graph of G*. According to this rule, a prime graph $\Gamma(G)$ can be called a *cyclic graph*, which is denoted by $\Gamma_{cyc}(G)$. If Ξ stands for "solvable", then we call the $\mathrm{Ord}_\Xi(G)$-graph *a solvable graph of a group G*, which is denoted by $\Gamma_{sol}(G)$. $\Gamma_{nil}(G), \Gamma_{abel}(G)$ and so on can be defined in a same way where *nil* and *abel* stand for "nilpotent" and "abelian" respectively. It is easy to see that $\Gamma_{nil}(G), \Gamma_{abel}(G)$ and $\Gamma_{cyc}(G)$ are same things. Note that $\Gamma_{sol}(G)$ is different from $\Gamma_{cyc}(G)$ in general, although $V_{sol}(G) = V_{cyc}(G)$.

Example. The solvable graph and cyclic graph of $S_6(2)$ are drawn as below:



$\Gamma_{sol}(S_6(2))$



$\Gamma_{cyc}(S_6(2))$

The following two remarks are very important for this studies though they can be shown very easily.

Remark 1 Let G be a group.

(1) If G is not solvable, then $\Gamma_{sol}(G) = \Gamma^*_{sol}(G)$.

(2) If G is solvable, then $\Gamma_{sol}(G)$ is complete.
(3) If G is solvable and $|\pi(G)| \geq 3$, $\Gamma^*_{sol}(G)$ is complete.

Remark 2 Let G be a group, H a subgroup of G and N a normal subgroup of G.
(1) Let $p, q \in \pi(H)$. If p and q are not joined in $\Gamma_{sol}(G)$, then p and q are not joined in $\Gamma_{sol}(H)$.
(2) Let N a normal subgroup of G. For $p \in \pi(N)$ and $q \in \pi(G) - \pi(N)$, p and q are joined in $\Gamma_{sol}(G)$.
(3) Let $p, q \in \pi(G/N)$. If p and q are not joined in $\Gamma_{sol}(G)$, then p and q are not joined in $\Gamma_{sol}(G/N)$.

Especially (2) in Remark 2 makes the proof of the connectivity of solvable graph of G attribute that of a simple group which is included in G.

3 Some Results on Solvable Graphs

One of the most striking features of a solvable graph of a non abelian simple group is "always connected though always incomplete" as shown below. We show theorems which gives such properties using the classification of finite simple groups, that is;
(1) G is isomorphic to the alternating group of degree $n (n \geq 5)$,
(2) G is a simple group of Lie type,
(3) G is a sporadic finite simple group.

Theorem 1 *Let G be a non abelian simple group. Then $\Gamma_{sol}(G)$ is connected.*

In order to prove Theorem 1, some properties of prime graphs play crucial roles.

Lemma 1 (Williams[9]) *Let G be a non abelian simple group such that* $\mathrm{com}(G) \geq 2$. *Then the following hold.*
(1) *G has a Hall π_i-subgroup H_i for a connected component $\pi_i (i \geq 2)$ of prime graph of G*
(2) *H_i is an isolated abelian subgroup.*

Lemma 2 *Let G be a non abelian simple group such that* $\mathrm{com}(G) \geq 2$ *and H_i is an isolated π_i-subgroup. Then H_i is a proper subgroup of $N_G(H_i)$.*

(2) in Remark 2 also says that any prime which divides $|N_G(H_i) : H_i|$ is connected with any prime in π_i. Therefore, if $|N_G(H_i) : H_i|$ and $|N_G(H_j) : H_j| (i \neq j)$ share a common prime divisor q, then any two primes $p_i \in \pi_i$ and $p_j \in \pi_j$ are connected via q. In order to get such common primes, we calculated all $|N_G(H_i) : H_i|$'s for any simple group of Lie type. If $\mathrm{com}(G) \leq 2$, any prime divisor of $|N_G(H_i) : H_i|$ has to belong to π_1. This means that any primes of $\pi(G)$ are connected via p. These situation are described in the following. Following theorems describe such situations.

Corollary 1 *Let G be a non abelian simple group such that* $\mathrm{com}(G) \leq 2$. *Then* $\Gamma_{sol}(G)$ *is connected.*

As mentioned before, previous propositions for simple group are extended to that for any group by (2) in Remark 2.

Corollary 2 *Let G be a finite group. Then $\Gamma_{sol}(G)$ is connected.*

Theorem 2 *Let G be a non-abelian simple group. Then $\Gamma_{sol}(G)$ is not a complete graph.*

The following theorem describes a sufficient and necessary condition for two primes to be joined in a solvable graph.

Theorem 3 *Let G be a finite group and $p, q \in \pi(G)$. p and q are not joined in $\Gamma_{sol}(G)$ if and only if there exists a normal series*

$$G \trianglerighteq N \trianglerighteq M \trianglerighteq 1,$$

of G such that G/N and M are $\{p,q\}'$−group and N/M is a non abelian simple group such that p and q are not joined in $\Gamma_{sol}(N/M)$.

4 Applications

If $X = \{i \in \mathbf{N} | 1 \leq i \leq n\}$, then we say that X is *consecutive up to n*. The following theorem is shown as an application of prime graphs.

Theorem 4 (Brandl − Shi)[1] *Let G be a finite group. If* $\mathrm{Ord}_{cyc}(G)$ *is consecutive up to n, Then $n \leq 8$ and G can be classified.*

Using same arguments in Brandl-Shi[1], a similar result for $\mathrm{Ord}_{abel}(G)$ was shown by N. Chigira. This is one of applications of abelian graphs, which should be regarded as that of prime graphs, since an abelian graph of a group G is nothing but a prime graph of G a group as mentioned before.

Using some properties of solvable graphs, we got following theorems.

Theorem 5 *Let G be a finite group. If* $\mathrm{Ord}_{sol}(G)$ *is consecutive up to n, Then* $G \simeq \mathbf{Z}_2$ *or* 1.

Theorem 6 *If* $\mathrm{Ord}^*_{sol}(G)$ *is consecutive up to n, Then $n \leq 4$ and*

$$\begin{array}{rcll} G & \simeq & A_4 & (n=4), \\ & & S_3,\ \mathbf{Z}_6 & (n=3), \\ & & \mathbf{Z}_2 \times \mathbf{Z}_2,\ \mathbf{Z}_4 & (n=2), \\ & & 1,\ \mathbf{Z}_p & (n=1), \end{array}$$

for any prime p.

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