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Classification of primitive association schemes with small vertices

Akihide Hanaki (花木 章秀、 信州大)
Izumi Miyamoto (宮本 泉、山梨大)

After we intended to give a talk on this title, we have computed a little more association schemes, not only primitive association schemes but also all the association schemes with small vertices. The present result is as follows.

1 Association schemes with n vertices, $n \leq 23$

The number of the isomorphism classes of the association schemes.

number of vertices	number of isomorphism classes			
5	3			
6	8			
7	4			
8	21			
9	12		up to 10	
10	13		by Nomiyama	
11	4		by Hirasaka	
12	59		by Hirasaka	
13	6		by Hirasaka and Suga	
14	16		by Sakita	
15	25		by Hirasaka and Suga	
16	222			
17	5			
18	95			
19	7		(Pentium II 400MHz)	(SS5 75MHz)
20	95	new	(one day)	(two weeks*)
21	32	new	(a few days)	(two weeks)
22	16	new	(ten days)	
23	21	new		

Let $(X, \{R_i\}_{0\leq i\leq d})$ be an association scheme. Let $A_0,\ A_1, \cdots, A_d$ be its adjacency matrix. Let $X = \{x_1,\ x_2, \cdots, x_n\}$ be the set of vertices. So $A_k(i,j) = 1$, if $(x_i, x_j) \in R_k$, 0, otherwise.

The property of adjacency matrices

1. A_i has $\{0,\ 1\}$−entris and has constant collumn and row sum. The sum v_i is called its *valency* . A_0 =identity matrix.
2. $A_0 + A_1 + \cdots + A_d = J$(all 1 matrix)
3. For all i there exists i' such that ${}^tA_i = A_{i'}$.
4. $A_iA_j = \sum_{0\le k\le d} p_{ijk}A_k$ ($A_0,\ A_1, \cdots, A_d$ are a basis of the algebra generated by themselves.)

By an association scheme A we mean the associatin scheme of which relation matrix is A. Let $A_i^m = \sum_{0\le k\le d} a_{i,k}^{[m]}A_k$. Let $supp(\langle A_i\rangle) = \{k | a_{i,k}^{[m]} \neq 0$ for some $m \in \{0,1,2,\cdots d\}\}$. An association scheme A is said to be connected with respect to A_i, if $supp(\langle A_i\rangle) = \{0,1,2,\cdots d\}$. An association scheme A is *primitive* if and only if A is connected with respect to A_i for all $i \in \{1,2,\cdots d\}$. Easy primitive association schemes are
association schemes of class 1 (association schemes with $d = 1$),
association schemes with prime vertices, especially, cyclotomic schemes with prime vertices,
association schemes defined from primitive permutation groups.

2 Primitive schemes with up to 24 vertices

Association schemes of class 1 and cyclotomic schemes with prime vertices always exist, so we omit them. The names of primitive groups are in the library of `GAP`.

number of vertices	valencies	primitive permutation group
9	1, 4, 4	`3^2:4`, `3^2:D8`
10	1, 3, 6	`A(5)`, `S(5)`
15	1, 6, 8	`A(6)`, `S(6)`
	1, 7, 7	non (prim15)
16	1, 5, 10	`(2^4:5).4`, `2^4:A_5`, `2^4:S_5`
	1, 6, 9	`(A_4xA_4):2`, `2^4.3^2:4`, `2^4.S_3xS_3`, `(S_4xS_4):2`
	1, 6, 9	non (prim16_1)
	1, 5, 5, 5	`2^4:5`, `2^4:D_10`
	1, 5, 5, 5	non (prim16_2)
19	1, 9, 9	non (prim19)
21	1, 10, 10	`A(7)`, `S(7)`
	1, 4, 8, 8	`PGL(2,7)`
23	1, 11, 11	non (from prim23_1 to prim23_18)

3 Computation, up to 19 vertices

We use group algorithm programming system GAP [7] to compute isomorphism classes of association schemes. Let $[v_1, v_2, \cdots, v_d]$ is the list of valencies arranged in incleasing order $v_1 \leq v_2 \leq \cdots \leq v_d$. The list $[1', 2', \cdots, d']$ indicates the numbers such that $A_{i'} = {}^tA_i$. Here we note that always $v_0 = 1$ and $0' = 0$, so they are omitted in the lists. Let n be the number of the vertices $|X|$ of an association scheme A. Then $[v_1, v_2, \cdots, v_d]$ is a partition of $n - 1$. We input these two lists $[v_1, v_2, \cdots, v_d]$ and $[1', 2', \cdots, d']$ to our computer program in order to make relation matrices. The following facts show some impossible inputs.

Proposition 3.1 *There exist no regular graphs of odd valencies with odd vertices, which means that if $|X|$ is odd and $v_i =$ odd ,then $i \neq i'$.*

Proposition 3.2 ([3]) *Let A be an association scheme on a vertex set X. Let $n = |X|$ and let m be the number of valencies such that $v_i = 1$. Then the adjacency matrices of valency 1 make a semiregular group on X and m divides n.*

Proposition 3.3 ([3]) *Let m be the number of valencies equal to 1. If s is the number of valencies equal to l, then m divides sl.*

Association schemes of all of its valencies equal to 1 are given by regular representation of groups (cf.[6]). So we omitted to compute this case and relied on the classification of groups. Then we computed all the possible inputs by our GAP-programs to get isomorphism classes of association schemes.

4 Computation of primitive schemes

The method of our computation was same as in up to 19 vertices. But we used the following facts and we were able to compute all the possible inputs for the primitive association schemes.

Proposition 4.1 (cf. [10, Theorem 1.4.2]) *Let $\mathcal{X}$ be a primitive association scheme. Let k be the minimal valency greater than 1. Then any prime factor of any valency is less than or equal to k.*

Proposition 4.2 *Let E and F be disjoint sets such that the union $E \cup F$ is the set of the valencies of an association scheme $\mathcal{X}$. Let v_i be maximal in E and let v_j be minimal in E. Suppose that any valency v_k in F satisfies one of the following.*
(i) v_k is coprime with any valency in E and $v_j < v_k$.
(ii) v_k is greater than $v_i v_j$.
If F is not empty, then the association scheme $\mathcal{X}$ is not primitive.

We note that many conditions for imprimitivity are given in [1], which are also sufficient for our computation.

We have 271 inputs with 23 vertices. Among them Proposition 4.1 gives 36 inputs $[[v_1, v_2, \cdots, v_d], [1', 2', \cdots, d']]$ for primitive schemes and Proposition 4.2 shows 16 of them imprimitive.

For some cases we applyed the following argument to prove the imprimitivity. For instance we have cases $[[v_1, v_2, \cdots, v_d] = [3, 3, 4, 6, 6]$ and $[3, 3, 3, 3, 4, 6]$ imprimitive.

For any adjacency matrix A_i, define $A_i^{[0]} = A_0$ and

$$A_i^{[n]} = \begin{cases} A_i^{[n-1]} A_i & \text{if } n \text{ is odd} \\ A_i^{[n-1]} A_{i'} & \text{if } n \text{ is even} \end{cases}$$

Let $A_i^{[n]} = \Sigma a_{[n],k} A_k$ and let $\delta_i(j) = minimum\{n | a_{[n],j} \neq 0\}$ or $\delta_i(j) = \infty$ if $a_{[n],j} = 0$ for all n (see [1] Section 5).

Proposition 4.3 (cf. [1] Section 5) *Suppose that v_i and v_j, the valencies of A_i and A_j, are coprime mutually and that $\delta_i(j) = n$. Then $A_j A_{\hat{i}} = p_{j,\hat{i},k} A_k$ for some k with $\delta_i(k) = n - 1$, where $A_{\hat{i}} = A_i$ or $A_{i'}$ if n is even or odd respectively, $v_k \geq v_i$, $v_k \geq v_i v_j / (v_i - 1)$ and if the last equality holds, then $A_k A_{\hat{i}'} = p_{k,\hat{i}',m} A_m + v_i A_j$.*

Proving the above facts, the fundamental property $p_{ijk} v_k = p_{kj'i} v_i$ plays a most important role. We had not recognized this property well in our computer program.

5 Computation, from 20 vertices

There is another fundamental property $p_{ijk} = p_{j'i'k'}$. These properties imply that in course of constructing a relation matrix, if one p_{ijk} is determined, then at the same time $p_{j'i'k'}$, $p_{k'ij'}$, $p_{i'kj}$, $p_{jk'i'}$ and $p_{kj'i}$ are determined. We also have conditions that $p_{ijk} v_k / v_i$ and $p_{ijk} v_k / v_j$ are integers. As we have checked p_{ijk} as soon as possible, this improved our program.

6 Survey of related topics

Let G be a transitive permutation group on X. Then the orbits of G on $X \times X$ define an association scheme. For any association scheme A, each adjacency matrix A_i can be seen an adjacency matrix of a regular graph, not necessarily connected..

A. Hulpke (1996) computed transitive groups up to degree 30. By Pentium PC 133Mhz, degree 8, 9, 10 take 2 or 3 minites, degree 12 half an hour, degree 18 or 20 one or 2 days. Up to degree 22 the list is in the library of GAP. (http://www-gap.dcs.st-and.ac.uk/~ahulpke/publ.html)

M. Meringer (1996) computed regular graphs by DEC-Alpha UNIX-Workstation and DEC-Station 5000/200. (http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html)

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