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On Balanced Sets and Related Structures

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1 Motivation

I have been studying distance-regular graphs since 1990. Distance-regular graphs are also known as P-polynomial association schemes as there is a natural correspondence between them. Distance-regular graphs are defined to satisfy ideal regularity conditions, and there are lots of excellent researches using the combinatorial regularity. (See for example [10, 11, 17, 21].) Moreover, these developments in the study of combinatorial aspect of distance-regular graphs have been successfully applied to develop the theory of association schemes which are not necessarily P-polynomial ([12, 24, 25, 26]). On the other hand, association schemes have both combinatorial and algebraic structures, it is natural to use algebraic properties of the association scheme associated to each distance-regular graph. However, besides the use of the integrality condition of multiplicities of eigenvalues of the adjacency matrix, the only analysis successfully applied using the algebraic structures is made under extra algebraic condition such as Q-polynomial property. I strongly feel the need of the development of the study on the algebraic properties of association schemes and their representation theory. As the combinatorial theory is developing with the combinatorial analysis of P-polynomial association scheme, which has the ideal combinatorial structure among association schemes, I feel that it should be very important to study algebraic properties and their representation theory of association schemes which have ideal algebraic condition. Q-polynomial association schemes and balanced conditions which define Q-polynomial property of distance-regular graphs seem to be the structure we should study first ([18, 19]).

In 1970's, Q-polynomial association schemes were defined as schemes 'dual' in a sense to P-polynomial association schemes by P. Delsarte and they were studied in connection with the design theory and the tight condition ([4]). In 1980's, P. Terwilliger introduced balanced conditions in order to describe the Q-polynomial properties of distance-regular graphs ([22, 23]). He defined balanced condition and strongly balanced condition. He showed that a distance-regular graph satisfies the balanced condition if and only if the association scheme is Q-polynomial. In 1990's, P. Terwilliger with the aid of his student G. Dickie classified all distance-regular graphs with strongly balanced condition ([5, 6]).

These are all excellent results, but the general theory of Q-polynomial association schemes and that of balanced conditions have not been much studied. For example, we do not have many examples of Q-polynomial association schemes, which are not P-polynomial. (Most of the classical P-polynomial association schemes are Q-polynomial as well.) We know neither a primitive Q-polynomial association scheme, which is not P-polynomial, nor a class of Q-polynomial association schemes with unbounded diameter, which is not P-polynomial. Moreover, though Q-polynomial association schemes satisfy balanced condition but we have very few examples of association schemes with balanced condition which are not Q-polynomial ([13]).

Balanced conditions can be defined for any finite set on the real Euclidean sphere. But all examples we have are associated with association schemes. Is this always the case?

Finally, balanced conditions are originally defined on the real Euclidean sphere or they are defined using a real primitive idempotent of a commutative association scheme. Is it possible to define such a condition for Hermitian space or using a non-real primitive idempotent. This is also connected to the problem to define the dual of distance-regular digraphs. Distance-regular digraphs were studied in 1980's and it was shown that besides the coclique extension of a directed cycle, there is an upper bound of the number of classes of the corresponding association schemes ([3, 14]). We may be able to define a little wider class containing distance-regular digraphs, if we can successfully define nonsymmetric balanced conditions.

I hope that this note serves to give a startpoint to study one of the problems listed above.

2 Balanced Conditions

We start with a definition of representation diagrams.

Definition 1 Let $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a commutative association scheme, and let $E_0, E_1, \ldots, E_d$ be its primitive idempotents. Let $q_{i,j}^h$ be the Krein parameters defined by:

$$E_i \circ E_j = \frac{1}{|X|}\sum_{h=0}^{d} q_{i,j}^h E_h.$$

1. *The representation diagram* $\mathcal{D}^* = \mathcal{D}^*(\mathcal{X}, E)$ of $\mathcal{X}$ with respect to $E = E_h$, is a diagram with $\{0, 1, \ldots, d\}$ as the vertex set such that the adjacency is defined as follows.
$$i \sim j \Leftrightarrow E({}^tE_i \circ E_j) \neq O \Leftrightarrow q_{h,i}^j \neq 0.$$
The adjacent pair (i, j) is said to be an arc if $i \neq j$ and a loop if $i = j$.

2. A representation diagram is said to be a *representation graph* if we do not consider the directions of arcs nor loops.

Let $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a finite set X in a complex Hermitian space $\boldsymbol{C}^m$ with the inner product $< x, y >= {}^t\bar{x}y$ together with some relations on it:

$$\emptyset \neq R_i \subset X \times X,\ i = 0, 1, \ldots, d.$$

We assume the following conditions.

1. $\mathrm{Span}(X) = \boldsymbol{C}^m$.
2. $R_0 = \{(x,x) \mid x \in X\}$.
3. $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$, (disjoint union).
4. ${}^tR_i = R_{i'}$ for some $i' \in \{0,1,\ldots,d\}$, where ${}^tR_i = \{(x,y) \mid (y,x) \in R_i\}$.
5. For $x, y \in X$, the inner product $< x, y >= \gamma(i)$ depends only on i such that $(x,y) \in R_i$.

Let $A_i \in Mat_X(\boldsymbol{C})$ defined by

$$(A_i)_{xy} = \begin{cases} 1 & (x,y) \in R_i \\ 0 & \text{otherwise.} \end{cases}$$

A_i's are called adjacency matrices of the configuration $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$.

For $x, y \in X$, $0 \leq i, j \leq d$, let

$$\begin{aligned} P_{i,j}(x,y) &= \{z \in X \mid (x,z) \in R_i, (z,y) \in R_j\}, \\ \widehat{P_{i,j}(x,y)} &= \sum_{z \in P_{i,j}(x,y)} z = \sum_{z \in X, (x,z) \in R_i, (z,y) \in R_j} z \in \boldsymbol{C}^m. \end{aligned}$$

The configuration $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ (with $X \subset \boldsymbol{C}^m$) is said to be *real* if $X \subset \boldsymbol{R}^m$.

Since $< x, x >= \gamma(0)$, X can be embedded as a subset of the unit sphere S^{m-1}, in this case. Conversely if a finite subset S is given in $S^{m-1} \subset \boldsymbol{R}^m$ and

$$\Delta = \{< x, y > \mid x, y \in X\} = \{1 = \gamma(0), \gamma(1), \ldots, \gamma(d)\},$$

then X together with $R_i = \{(x,y) \mid < x, y >= \gamma(i)\}$ with $i = 0, 1, \ldots, d$ define a configuration $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ satisfying the conditions $1 \sim 5$ above. In this case, ${}^tR_i = R_i$ for every i.

Definition 2 Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be as above with $X \subset \boldsymbol{R}^m$, i.e, *real.*

1. $\mathcal{X}$ is *balanced,* if the following are satisfied.
   (a) For every $0 \leq i \leq d$ and $x \in X$, there exists a constant $\alpha_i(x) \in \boldsymbol{R}$ such that
   $$\widehat{P_{i,i'}(x,x)} = \alpha_i(x)x.$$
   (b) For all $0 \leq h, i, j \leq d$ and $(x,y) \in R_h$, there exists a constant $\alpha_{i,j}^h(x,y) \in \boldsymbol{R}$ such that
   $$\widehat{P_{i,j}(x,y)} - \widehat{P_{j,i}(x,y)} = \alpha_{i,j}^h(x,y)(x-y).$$

2. $\mathcal{X}$ is *strongly balanced*, if for all $0 \le h, i, j \le d$ and $(x, y) \in R_h$, there exist constants $\beta^h_{i,j}(x,y)$, $\gamma^h_{i,j}(x,y) \in \boldsymbol{C}$ such that

$$P_{i,j}\widehat{(x,y)} = \beta^h_{i,j}(x,y)x + \gamma^h_{i,j}(x,y)y.$$

3. A balanced [resp. strongly balanced] set is said to be *homogeneous* if $\alpha_i(x)$ and $\alpha^h_{i,j}(x,y)$ [resp. $\beta^h_{i,j}(x,y)$, and $\gamma^h_{i,j}(x,y)$] can be chosen so that they depend only on i, j and h.

Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be a commutative association scheme, $V = \boldsymbol{C}X$ and $E = E_h$ be a primitive idempotent. Then E defines a projection on V. If

$$E = E_h = \frac{1}{|X|}\sum_{i=0}^{d} q_h(i)A_i,$$

then the map $x \mapsto Ex$ is an injection if and only if $q_h(i) \ne q_h(0)$ for every $i \ne 0$. Moreover,

$$< Ex, Ey >= {}^t\overline{(Ex)}Ey = E_{x,y} = \frac{1}{|X|}q_h(i),$$

whenever $(x, y) \in R_i$. Hence if we set $\hat{R}_i = \{(\hat{x}, \hat{y}) \mid (x, y) \in R_i\}$ for $i = 0, 1, \ldots, d$, then the configuration $\widetilde{\mathcal{X}} = (\tilde{X}, \{\tilde{R}_i\}_{0 \le i \le d})$ satisfies the conditions above, where $\tilde{X} = \{Ex \mid x \in X\}$.

Moreover, if $E = {}^tE = \bar{E}$, then $\tilde{X}$ can be regarded as a subset of $E\boldsymbol{R}X \subset \boldsymbol{R}X$. Hence this configuration becomes real in this case. In particular, $\tilde{X}$ is a subset of a sphere of radius $\sqrt{q_h(0)/|X|} = \sqrt{m/|X|}$, where m is the dimension of the space $E\boldsymbol{R}X$.

In this way the configuration we defined above is naturally associated with the eigenspaces of commutative association schemes.

Let $\mathrm{Mat}_X(\boldsymbol{C})$ denote the set of square matrices of size $|X|$, whose rows and columns are indexed by the elements of X. For a matrix $M \in \mathrm{Mat}_X(\boldsymbol{C})$ and $x \in X$, x_M denotes a diagonal matrix defined by $(x_M)_{y,y} = M_{x,y}$.

P. Terwilliger obtained a beautiful theorem for the case when balanced set is defined on the eigenspace of a commutative association scheme.

Theorem 1 (P. Terwilliger [22, 23]) *Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be a commutative association scheme and let*

$$E = E_h = \frac{1}{|X|}\sum_{i=0}^{d} q_h(i)A_i$$

be a primitive idempotent of the Bose-Mesner algebra. Suppose $E = {}^tE = \bar{E}$, i.e., real. Let $\hat{x} = Ex$ and $\hat{X} = \{\hat{x} \mid x \in X\}$. Let $\hat{R}_i = \{(\hat{x}, \hat{y}) \mid (x, y) \in R_i\}$ for $i = 0, 1, \ldots, d$. If $q_h(i) \ne q_h(0)$ for all $i > 0$, then the following hold.

(1) *The following are equivalent.*

(*i*) $\hat{\mathcal{X}} = (\hat{X}, \{\hat{R}_i\}_{0\le i\le d})$ *is balanced.*

(*ii*) $\hat{\mathcal{X}} = (\hat{X}, \{\hat{R}_i\}_{0\le i\le d})$ *is balanced and homogeneous.*

(*iii*) *For some* $x \in X$, *there exist constants* $\alpha_{i,j}^l(x) \in \boldsymbol{R}$ *satisfying:*

$$A_i x_E A_j - A_j x_E A_i = \sum_{l=0}^{d} \alpha_{i,j}^l(x)(x_E A_l - A_l x_E).$$

(*iv*) *For every* $x \in X$, *there exist constants* $\alpha_{i,j}^l(x) \in \boldsymbol{R}$ *satisfying:*

$$A_i x_E A_j - A_j x_E A_i = \sum_{l=0}^{d} \alpha_{i,j}^l(x)(x_E A_l - A_l x_E).$$

(*v*) *The representation graph* $\mathcal{D}^*(\mathcal{X}, E)$ *is a tree.*

(2) *The following are equivalent.*

(*i*) $\hat{\mathcal{X}} = (\hat{X}, \{\hat{R}_i\}_{0\le i\le d})$ *is strongly balanced.*

(*ii*) $\hat{\mathcal{X}} = (\hat{X}, \{\hat{R}_i\}_{0\le i\le d})$ *is homogeneously strongly balanced.*

(*iii*) *For some* $x \in X$, *there exist constants* $\beta_{i,j}^l(x), \gamma_{i,j}^l(x) \in \boldsymbol{R}$ *satisfying:*

$$A_i x_E A_j = \sum_{l=0}^{d} (\beta_{i,j}^l(x) x_E A_l + \gamma_{i,j}^l(x) A_l x_E).$$

(*iv*) *For every* $x \in X$, *there exist constants* $\beta_{i,j}^l(x), \gamma_{i,j}^l(x) \in \boldsymbol{R}$ *satisfying:*

$$A_i x_E A_j = \sum_{l=0}^{d} (\beta_{i,j}^l(x) x_E A_l + \gamma_{i,j}^l(x) A_l x_E).$$

(*v*) *The representation graph* $\mathcal{D}^* = \mathcal{D}^*(\mathcal{X}, E)$ *is a tree and the diagram has at most one loop.*

(3) *If* $\mathcal{X}$ *is strongly balanced, then it is balanced.*

Remarks.

1. Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a P-polynomial association scheme with balanced condition with respect to a primitive idempotent $E = E_h$. P. Terwilliger showed that in this case the representation graph $\mathcal{D}^*(\mathcal{X}, E)$ becomes a path. Hence $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ is a Q-polynomial association scheme in this case ([22]).

2. Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a P-polynomial association scheme with strongly balanced condition with respect to a primitive idempotent $E = E_h$. G. Dickie and P. Terwilliger classified such association schemes with $d \ge 7$ ([5, 6]).

3 Examples of Balanced Sets

In this section, we give lists of (real) balanced sets.

Example 1 Suppose $X \subset S^1 \subset \boldsymbol{R}^2$, i.e., $m = 2$. Then it is easy to see that the following are equivalent.

(*i*) X is the set of vertices of a regular n-gon with $n \geq 3$.

(*ii*) X is balanced.

(*iii*) X is strongly balanced.

(*iv*) For every $x \in X$, $P_{i,i'}\widehat{(x,x)} \in \mathrm{Span}(x)$.

As the example above shows, the first non-trivial case is when $m = 3$. Suppose $X \subset S^2 \subset \boldsymbol{R}^3$. Since

$$P_{i,j}(x,y) = \{z \in X \mid (x,z) \in R_i, (z,y) \in R_j\} \subset \{z \in S^2 \mid < x,z >= \gamma(i), < z,y >= \gamma(j)\},$$

the set $P_{i,j}(x,y)$ consists of at most two elements if x and y are linearly independent. Hence a balanced [resp. strongly balanced] set X is invariant by the action of certain reflections. More precisely, let

$$S = \{x - y \mid x, y \in X, \dim \mathrm{Span}(x,y) = 2\}, \; T = \{x \times y \mid x, y \in X, \dim \mathrm{Span}(x,y) = 2\},$$

where $x \times y$ denotes the outer product of three dimensional vectors. For a nonzero vector a let σ_a denote the reflection with respect to the hyperplane orthogonal to the vector a. Then we have the following.

- If X is balanced, then the group $\Sigma(S) =< \{\sigma_s \mid s \in S\} >$ acts transitively on X.
- If X is strongly balanced, then the group $\Sigma(T) =< \{\sigma_t \mid t \in T\} >$ acts transitively on X.

Since finite groups generated by reflections are well-known, by inspection, we have the following.

Proposition 2 *Let $X \subset S^2 \subset \boldsymbol{R}^3$ be a finite set. Then the following are equivalent.*

(*i*) *X is the vertex set of the tetrahedron, the cube, the octahedron or the icosahedron.*

(*ii*) *X is balanced.*

(*iii*) *X is strongly balanced.*

Remarks. The connection with the finite reflection groups in the case of $m = 3$ was first observed by T. Ito. K. Goto, his student, studied balanced sets on which finite reflection groups act transitively, and obtained some partial results.

Let W be a finite reflection group with the root system $\Phi \subset V = \boldsymbol{R}^m$, the simple system Δ. It is well-known that the association scheme defined on the orbit of W, is isomorphic to the left cosets representation of W on $W/W(\Pi)$ for some subset $\Pi \subset \Delta$, where $W(\Pi)$ denotes the corresponding reflection group. Moreover, the association scheme is isomorphic to the Hecke algebra. The list of commutative Hecke algebras corresponding to the maximal parabolic subgroups, i.e., $|\Pi| = |\Delta| - 1$, is given in [2, Theorem 10.4.11]. The corresponding list for Chevalley groups is also given in [2, Proposition 10.9.2]. As for the case of exceptional type, the character tables of the Hecke algebras, which are nothing but the first eigenmatrices of the corresponding association schemes, have been computed by Y. Gomi [7, 8], we can check the balanced condition using Theorem 1.

Commutative Hecke Algebras of Exceptional Type

Thin Case with $q = 1$						Thick Case	
Name	BCN	Graph	d	P-str.	Q-str.	P-str.	Q-str.
E_6/D_5	$E_{6,1}$	Schäfli	2	P	Q	P	Q
E_6/A_5	$E_{6,2}$	E_6 roots	4	no	Q	no	no
E_7/D_6	$E_{7,1}$	E_7 roots	4	no	Q	no	no
E_7/A_6	$E_{7,2}$		9	no	no	no	no
E_7/E_6	$E_{7,7}$	Gosset	3	P	Q	P	Q
E_8/D_7	$E_{8,1}$		9	no	no	no	no
E_8/E_7	$E_{8,8}$	E_8 roots	4	no	Q	no	no
F_4/C_3	$F_{4,1}$		4	no	Q	no	no
	$F_{4,4}$					no	no
$I(m)$		m-gon	d	P	Q		
$G_{2,1}, {}^3D_{4,1}, {}^3D_{4,2}, {}^2F_{4,1}, {}^2F_{4,2}$						P	Q
$H_3/I(5)$	$H_{3,1}$	Icosahedron	3	P	Q		
H_3/A_2	$H_{3,3}$	Dodecahedron	5	P	no		
H_4/H_3	$H_{4,1}$	600cell	6	no	?		

Remarks.

1. The second column of the table above shows the name in the book [2].

2. Using the intersection numbers, we can define the distribution diagrams $\mathcal{D}(\mathcal{X}, A_h)$ similar to the representation diagrams. The vertex set is $\{0, 1, \ldots, d\}$ and $i \sim j \Leftrightarrow p^i_{h,j} \neq 0$. We can consider the condition when the diagram becomes a tree. When

it becomes a path, it is P-polynomial. The fifth and the seventh columns give whether the scheme satisfies this condition. The table above shows that all cases which satisfy tree condition are P-polynomial.

3. 'no' above shows that it does not satisfy the respective tree condition.

4. There are four schemes which are not P-polynomial but Q-polynomial, and there are no examples of the similar schemes in the thick case. In the following we list the information of the each case. Here

$$b_i^* = q_{1,i+1}^i,\ a_i^* = q_{1,i}^i,\ c_i^* = q_{1,i-1}^i, \text{ and } v = |X|.$$

   (a) The root system of E_6 with $v = 72$.

$$\left\{\begin{array}{c} c_i^* \\ a_i^* \\ b_i^* \end{array}\right\} = \left\{\begin{array}{ccccc} * & 1 & 3/2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 5 & 9/2 & 3 & * \end{array}\right\}.$$

   (b) The root system of E_7 with $v = 126$.

$$\left\{\begin{array}{c} c_i^* \\ a_i^* \\ b_i^* \end{array}\right\} = \left\{\begin{array}{ccccc} * & 1 & 14/9 & 21/8 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 7 & 6 & 49/9 & 35/8 & * \end{array}\right\}.$$

   (c) The root system of E_8 with $v = 240$.

$$\left\{\begin{array}{c} c_i^* \\ a_i^* \\ b_i^* \end{array}\right\} = \left\{\begin{array}{ccccc} * & 1 & 8/5 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 8 & 7 & 32/5 & 6 & * \end{array}\right\}.$$

   (d) Short or long roots of the system of F_4 with $v = 24$.

$$\left\{\begin{array}{c} c_i^* \\ a_i^* \\ b_i^* \end{array}\right\} = \left\{\begin{array}{ccccc} * & 1 & 4/3 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 8/3 & 1 & * \end{array}\right\}.$$

5. The all examples above are dual bipartite, which is antipodal, i.e., $X = -X$.

J. McKay observed that the representation graph of the group association scheme of a group is a tree if it has a faithful irreducible self-dual representation of dimension 2. Moreover the representation graphs appear there are the Dynkin diagrams of the affine root systems ([9, 16]).

The following result gives a classification of Q-polynomial group association schemes. For more detail, see the original paper.

Theorem 3 **([13])** *Let G be a finite group. If the symmetrization of its group association scheme is Q-polynomial, then one of the following holds.*

(*i*) $G \simeq \boldsymbol{Z}_n$, *the cyclic group of order* n.

(*ii*) $G \simeq S_3$, *the symmetric group of degree* 3.

(*iii*) $G \simeq A_4$, *the alternating group of degree* 4.

(*iv*) $G \simeq SL(2,3)$, *the two dimensional special linear group over the field with three elements.*

(*v*) $G \simeq F_{21}$, *the Frobenius group of order* 21.

4 Nonsymmetric Balanced Conditions

In this section, we consider nonsymmetric case. We first give a definition for the nonsymmetric case.

Definition 3 Let $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a configuration in the first section satisfying the conditions 1–5.

(1) $\mathcal{X}$ is *balanced*, if the following are satisfied.

(*a*) For every $0 \leq i \leq d$ and $x \in X$, there exists a constant $\alpha_i(x) \in \boldsymbol{C}$ such that

$$P_{i,i'}\widehat{(x,x)} = \alpha_i(x)x.$$

(*b*) For all $0 \leq h, i, j \leq d$ and $(x,y) \in R_h$, there exists a constant $\alpha^h_{i,j}(x,y) \in \boldsymbol{C}$ such that

$$P_{i,j}\widehat{(x,y)} - P_{j,i}\widehat{(x,y)} = \alpha^h_{i,j}(x,y)x - \overline{\alpha^h_{i,j}(x,y)}y.$$

(2) $\mathcal{X}$ is *strongly balanced*, if for all $0 \leq h, i, j \leq d$ and $(x,y) \in R_h$, there exist constants $\beta^h_{i,j}(x,y)$, $\gamma^h_{i,j}(x,y) \in \boldsymbol{C}$ such that

$$P_{i,j}\widehat{(x,y)} = \beta^h_{i,j}(x,y)x + \gamma^h_{i,j}(x,y)y.$$

(3) A balanced [resp. strongly balanced] set is said to be *homogeneous* if $\alpha_i(x)$ and $\alpha^h_{i,j}(x,y)$ [resp. $\beta^h_{i,j}(x,y)$, and $\gamma^h_{i,j}(x,y)$] can be chosen so that they do not depend on x, y.

For a fixed primitive idempotent E of the Bose-Mesner algebra of a commutative association scheme $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ and $x \in X$, we define the following subspaces of $\mathrm{Mat}_X(\boldsymbol{C})$.

$$\begin{aligned}\mathcal{L}(x) &= \mathrm{Span}(A_i x_E A_j \mid 0 \leq i < j \leq d)\\ \mathcal{N}(x) &= \mathrm{Span}(x_E A_i, A_i x_E \mid 0 \leq i \leq d)\end{aligned}$$

Theorem 4 *Let $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a commutative association scheme and let*

$$E = E_h = \frac{1}{|X|}\sum_{i=0}^{d} q_h(i)A_i$$

be a primitive idempotent of the BM-algebra. Let $\hat{x} = Ex$ and $\hat{X} = \{\hat{x} \mid x \in X\}$. Let $\hat{R}_i = \{(\hat{x},\hat{y}) \mid (x,y) \in R_i\}$ for $i = 0,1,\dots,d$. If $q_h(i) \neq q_h(0)$ for all $i > 0$, then the following hold.

(1) *The following are equivalent.*

(*i*) *$\hat{\mathcal{X}} = (\hat{X}, \{\hat{R}_i\}_{0\leq i\leq d})$ is balanced.*

(*ii*) *$\hat{\mathcal{X}} = (\hat{X}, \{\hat{R}_i\}_{0\leq i\leq d})$ is balanced and homogeneous.*

(*iii*) *For every $x \in X$, there exist constants $\alpha^l_{i,j}(x) \in C$ satisfying the following.*

$$A_i x_E A_j - A_j x_E A_i = \sum_{l=0}^{d} \alpha^l_{i,j}(x) x_E A_l - \overline{\alpha^l_{i,j}(x)} A_l x_E.$$

(2) *The following are equivalent.*

(*i*) *$\hat{\mathcal{X}} = (\hat{X}, \{\hat{R}_i\}_{0\leq i\leq d})$ is strongly balanced.*

(*ii*) *$\hat{\mathcal{X}} = (\hat{X}, \{\hat{R}_i\}_{0\leq i\leq d})$ is homogeneously strongly balanced.*

(*iii*) *$\mathcal{L}(x) = \mathcal{N}(x)$ for every $x \in X$.*

(*iv*) *$\mathcal{L}(x) = \mathcal{N}(x)$ for some $x \in X$.*

(*v*) *The representation graph $\mathcal{D}^* = \mathcal{D}^*(\mathcal{X}, E)$ satisfies the following:*

$$2d + 1 - \text{defect} = \text{arc}(\mathcal{D}^*) + \text{loop}(\mathcal{D}^*)$$

(3) *If $\mathcal{X}$ is strongly balanced, then it is balanced.*

The *defect* of $\mathcal{X}$ denoted by defect($\mathcal{X}$) is defined by the number one less than the number of relations with valency 1, i.e., defect $:= |\{i \mid k_i = 1,\ i = 1,2,\dots,d\}|$.

Remarks.

1. The interpretation of the balanced condition using the representation diagram is missing, thus the condition (v) is missing in Theorem 4 (1).
2. Is this the right generalization? There should be many ways to consider similar condition in the nonsymmetric case.
3. The directed cycle is strongly balanced. Moreover, if $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ is strongly balanced, then the clique extension of it is also strongly balanced. See [15].
4. Classify the case $m = 2$.
5. Study the dual. Does it include the distance regular di-graphs?
6. Construct non-homogeneous (strongly) balanced sets.
7. Classify group schemes which has a (strongly) balanced embedding.

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