

Four-weight Spin Model With Exactly Two Values on W_4 and Regular Hadamard Designs

九州大学大学院数理学研究科
澤野 光弘 (Mitsuhiro Sawano)

Abstract.

In this paper, we will show that every four-weight spin model with exactly two values on W_2 is associated with a regular Hadamard design with parameters $(16r^2, 8r^2 - 2r, 4r^2 - 2r)$. We also show the necessary and sufficient condition for a regular Hadamard design to admit a four-weight spin model with exactly two values on W_2 .

1. Introduction. Spin model introduced by V.F.R. Jones is the concept to construct invariants of knots and links[8]. It was generalized by Kawagoe, Munemasa and Watatani[9]. Finally, Bannai and Bannai introduced the much more general four-weight spin model[2]. Guo constructed examples of some four-weight spin models with exactly two values on W_4 (see [4,5]). M.Yamada constructed two-weight spin models of symmetric Hadamard type from Hadamard matrices[11]. Yamada's two-weight spin models have exactly two values on W_+ , too. All four-weight spin models with exactly two values on W_2 , including Guo's example and Yamada's spin models, have parameters $(16r^2, 8r^2 - 2r, 4r^2 - 2r)$. We will see the following theorem.

Theorem 1 The followings are equivalent.

- i) There exists a four-weight spin model with exactly two values on W_2
- ii) There exists a regular Hadamard design (X, \mathcal{B}) with the parameters $(16r^2, 8r^2 - 2r, 4r^2 - 2r)$ satisfying the following conditions.
 - a) The residual design of (X, \mathcal{B}) is a quasi-symmetric design with exactly two intersection numbers $2r^2 - 2r$ and $2r^2 - r$.
 - b) For any four blocks B_x, B_y, B_z and B_w in \mathcal{B} , the following equation holds.

$$f(B_y \cap B_z \cap B_w) = \frac{f(B_x \cap B_y \cap B_z)}{f(B_x \cap B_y \cap B_w)f(B_x \cap B_z \cap B_w)} \quad (1)$$

,where f is a function defined by the following

$$f(B_x \cap B_y \cap B_z) = \begin{cases} -1 & (|B_x \cap B_y \cap B_z| = 2r^2 - 2r) \\ 1 & (|B_x \cap B_y \cap B_z| = 2r^2 - r) \end{cases} \quad (2)$$

We can express W_2 of a four-weight spin model with exactly two values on W_2 as $\alpha(2E - J)$, where E is an incidence matrix of the regular Hadamard design of ii) in Theorem 1 and α is non-zero complex number. W_1, W_3 and W_4 are determined by the definition of four-weight spin models. In section 3, we give the construction of W_1 . In section 2, we will see definitions and known facts about four-weight spin models and regular Hadamard designs.

2. Preliminaries.

2.1 Four-weight spin models

The concept of four-weight spin models was defined by Bannai and Bannai[2]. It was represented by four non-symmetric matrices and finite non-empty set. Let X be a finite set with $|X| = n$ and $M_C(X)$ be the set of all the matrices over complex number field C with rows and columns indexed by X . In this paper, we denote the identity matrix with I and the matrix whose entries are all equal to 1 with J .

Definition 1(Bannai and Bannai). Four-weight spin model on a finite non-empty set X is a 6-tuple $(X, W_1, W_2, W_3, W_4; D)$, where $D^2 = |X|$ and W_1, W_2, W_3, W_4 are in $M_C(X)$ satisfy the following equations for all $a, b, c \in X$,

- 1) $\sum_{x \in X} W_1(a, x)W_3(x, b) = \sum_{x \in X} W_2(a, x)W_4(x, b) = |X|\delta_{a,b}$,
- 2) $W_1(a, b)W_3(b, a) = W_2(a, b)W_4(b, a) = 1$,
- 3)-a $\sum_{x \in X} W_2(a, x)W_2(b, x)W_4(x, c) = DW_1(b, a)W_3(a, c)W_3(c, b)$,
- 3)-b $\sum_{x \in X} W_2(x, a)W_2(x, b)W_4(c, x) = DW_1(a, b)W_3(b, c)W_3(c, a)$.

The equation 2) in the Definition 1 shows that four-weight spin models are determined by two matrices W_1 and W_2 .

The equations 3)-a and 3)-b imply that there exists a non zero complex

number μ , called the modulus of the spin model, such that for all $a \in X$ the following conditions hold.

- 4) $W_3(a, a) = \mu^{-1}$, $\sum_{x \in X} W_2(a, x) = \sum_{x \in X} W_2(x, a) = D\mu^{-1}$.
 5) $W_1(a, a) = \mu$, $\sum_{x \in X} W_4(a, x) = \sum_{x \in X} W_4(x, a) = D\mu$.

If we assume $W_1 \in \{W_2, {}^tW_2\}$, (consequently $W_3 \in \{W_4, {}^tW_4\}$) then the conditions 1), 2) and 3) give the conditions of two-weight spin model (of Jones type) $(X, W_+, W_-; D)$ [3]. conversely if we have a two-weight spin model $(X, W_+, W_-; D)$, then $(X, W_+, W_+, W_-, W_-; D)$ is a four-weight spin model.

If we assume, $W_1 \in \{W_4, {}^tW_4\}$ and $W_3 \in \{W_2, {}^tW_2\}$, then the conditions 1), 2) and 3) give the conditions of the two-weight spin models of psuedo Jones type.

If we assume $W_1 \in \{W_3, {}^tW_3\}$ and $W_2 \in \{W_4, {}^tW_4\}$, then the conditions 1), 2) and 3) give the conditions of the two-weight spin models of Hadamard type.

Although not all four-weight spin models have a symmetric W_1 , it is known that any four-weight spin model is gauge equivalent to a four-weight spin model with symmetric W_1 and W_3 (see[7]). Therefore a spin model of Hadamard type is always gauge equivalent to a four-weight spin model satisfying $W_1 = W_3$, $W_2 = {}^tW_4$ and ${}^tW_1 = W_1$. These spin models have exactly two values on W_2 .

The main theorem of this paper, Theorem 1, shows that every four-weight spin model with exactly two values on W_2 is gauge equivalent to a four-weight spin model which is naturally obtained from a two-weight spin model of Hadamard type.

2.2 The Regular Hadamard Design and Quasi-symmetric Design

In this chapter, we see some definitions and notations of designs. For more details, see[1,10].

Let $X = \{x_1, x_2, \dots, x_v\}$ be a finite set of elements called points and $\mathcal{B} = \{B_1, B_2, \dots, B_b\}$ be a finite set of distinct k -subsets of X called blocks. Then the pair $D = (X, \mathcal{B})$ is called a t - (v, k, λ) design if every t distinct points of X incident with precisely λ blocks.

Suppose (X, \mathcal{B}) is a t - (v, k, λ) design. The cardinality $|B_i \cap B_j|$, $B_i, B_j \in \mathcal{B}$ ($i \neq j$), is called an intersection number of (X, \mathcal{B}) . If a 2-design has exactly one intersection number, then it must necessarily be symmetric design. A 2-design with exactly two intersection numbers is called a quasi-symmetric design.[10]

Let $D = (X, \mathcal{B})$ be a design, $x \in X$ and $B \in \mathcal{B}$. Define sets as follows; $X^B = X - \{x | x \in X, x \in B\}$; $\mathcal{B}_B = \mathcal{B} - \{B\}$. Then $D^B = (X^B, \mathcal{B}_B)$ are called the residual design of D at the block B .

Let $D = (X, \mathcal{B})$ be a 2- (v, k, λ) design. Then residual design D^B is a 2- $(v - k, k - \lambda, \lambda)$ design, where $B \in \mathcal{B}$.

A matrix H is an Hadamard matrix provided $H(i, j) = \pm 1$ for all i and j and $H^t H = m I_m$, where I_m is the $m \times m$ identify matrix, H being also $m \times m$. We call m the size of the Hadamard matrix. It is known that if $m > 2$, then m is divisible by 4.

Theorem 2

Suppose H is an Hadamard matrix of size $m = 4n$ with the property that $\sum_{x \in X} H(i, x) = \sum_{x \in X} H(x, j) = \text{const.}$ for all $i, j \in X$. Then n is a square and the constant is either $2\sqrt{n}$ or $-2\sqrt{n}$. Setting $n = N^2$, then $\log_{-1}(H)$, whose (i, j) -entry is defined by $\log_{-1}(H)(i, j) = \log_{-1}(H(i, j))$ for all $i, j \in X$, is an incidence matrix of either a $(4N^2, 2N^2 - N, N^2 - N)$ -design or a $(4N^2, 2N^2 + N, N^2 + N)$ -design, depending on whether the constant is positive or negative.

We call the design of Theorem 2 the regular Hadamard design.

2.3 Four-weight spin models with exactly two values on W_2 .

It is known that the elements of each row or each column of DW_4 give the set of eigenvalues including multiplicities[4,5]. Guo showed the following in his paper[4,5].

If there exist a four-weight spin models with exactly two values on W_2 , then we may express W_2 as $\alpha E + \beta(J - E)$, where α, β are complex numbers and E is an $(0,1)$ -matrix satisfying $EJ = kJ$. k is a positive integer.

By the definition of four-weight spin models, we obtain a positive integer λ such that

$$E^t E = (k - \lambda)I + \lambda J$$

It is easy to see that there exist a symmetric design (X, \mathcal{B}) which has the incidence matrix E . The numbers of points and blocks of (X, \mathcal{B}) are both n . Every blocks contains precisely k points. Every two distinct points are together incident with precisely λ blocks. And we denote the intersection number of any three blocks in \mathcal{B} with s , then we obtain the following equation.

$$s = n^{-1}(k\lambda + \lambda - k \pm (k - \lambda)\sqrt{k - \lambda})$$

Because s is a positive integer, $k - \lambda$ is a square of a positive integer. In addition, by using parameters (n, k, λ, s) in the conditions 3) of definition 1, we obtain the condition b) of Theorem 1.

2.4 Two-weight spin models of Hadamard type

A.A. Ivanov and I.V. Chuvavaeva showed that symmetric amorphous association schemes of class 4 obtained from Hadamard matrices. An infinite family of Hadamard matrices can be constructed by fusing the relations of these amorphous association schemes. M.Yamada show the necessary and sufficient condition that these Hadamard matrices give two-weight spin models of symmetric Hadamard type.[11]

A.A. Ivanov and I.V. Chuvavaeva proved the following theorem[6].

Theorem 3

Let $H = (h_{i,j})_{i,j \in \Omega}$ be an Hadamard matrix of size $4n$ and $\Omega = \{0, 1, 2, \dots, 4n - 1\}$. Put $X = \Omega \times \Omega$. The subsets $R_i, (0 \leq i \leq 4)$ of $X \times X$ are

defined by

$$R_0 = \{(x, x) | x \in X\}$$

$$R_1 = \{(x_1, x_2), (y_1, y_2) | x_1 = y_1, x_2 \neq y_2\}$$

$$R_2 = \{(x_1, x_2), (y_1, y_2) | x_1 \neq y_1, x_2 = y_2\}$$

$$R_3 = \{(x_1, x_2), (y_1, y_2) | x_1 \neq y_1, x_2 \neq y_2, h_{x_1, x_2} h_{y_1, y_2} h_{x_1, y_2} h_{y_1, x_2} = 1\}$$

$$R_4 = \{(x_1, x_2), (y_1, y_2) | x_1 \neq y_1, x_2 \neq y_2, h_{x_1, x_2} h_{y_1, y_2} h_{x_1, y_2} h_{y_1, x_2} = -1\}$$

Then $(X, R_0, R_1, R_2, R_3, R_4)$ is an amorphous association scheme of class 4.

Yamada showed the following theorem by using the amorphous association schemes given above.

Theorem 4

Let $A_i (0 \leq i \leq 4)$ be adjacency matrices of an amorphous association scheme obtained from an Hadamard matrix of size $4n$ by Theorem 3.

Then

$$M_1 = A_0 + A_1 + A_2 + A_3 - A_4,$$

$$M_2 = A_0 + A_1 - A_2 - A_3 + A_4,$$

$$M_3 = A_0 - A_1 + A_2 - A_3 + A_4,$$

are regular symmetric Hadamard matrices of size $(4n)^2$.

Yamada gave a necessary and sufficient condition that each Hadamard matrices $M_i (1 \leq i \leq 3)$ in Theorem 4 give a two-weight spin model of symmetric Hadamard type.

Theorem 5

Let H be a normalized Hadamard matrix of size $4n$ and $A_i (0 \leq i \leq 4)$ be adjacency matrices obtained from H .

(1) $W_+ = W_- = M_1 = A_0 + A_1 + A_2 + A_3 - A_4$ gives a two-weight spin model of symmetric Hadamard type if and only if the following condition (\star) is satisfied for any $\beta_1, \beta_2, \gamma_1$ and $\gamma_2 \in \Omega^* = \{0, 1, \dots, 4n - 1\}$:

$$(\star) \sum_{l=-n}^n \theta_l l = (h_{\beta_1, \beta_2} h_{\gamma_1, \gamma_2} h_{\beta_1, \gamma_2} h_{\gamma_1, \beta_2} + 1)n/2$$

, where $\theta_l = \#\{x_1 | h_{x_1, \beta_2} h_{x_1, \gamma_2} = 1, \sum_{j=0}^{4n-1} h_{0, j} h_{\beta_1, j} h_{\gamma_1, j} h_{x_1, j} = 4l\}$.

(2) $W_+ = W_- = M_2 = A_0 + A_1 - A_2 - A_3 + A_4$ gives a two-weight spin model of symmetric Hadamard type if and only if the above condition (\star)

is satisfied for any $\beta_1, \beta_2, \gamma_1$ and $\gamma_2 \in \Omega^*$.

(3) $W_+ = W_- = M_3 = A_0 - A_1 + A_2 - A_3 + A_4$ gives a two-weight spin model of symmetric Hadamard type if and only if the transpose matrix tH satisfies the above condition (\star) for any $\beta_1, \beta_2, \gamma_1$ and $\gamma_2 \in \Omega^*$.

2.5 Guo's examples

Guo constructed examples of four-weight spin models by using the symmetric design with the symmetric difference property [4,5]. In this section, we study his construction. At first, we give the definition of the symmetric difference.

Definition 2(Symmetric difference)

Let S and T be subsets of the point set X of a block design (X, \mathcal{B}) . Symmetric difference $S\Delta T$ is defined by $S\Delta T = (S \cup T) - (S \cap T)$

A symmetric design is said to be have the symmetric difference property when $B\Delta C\Delta D$ is either a block or the complement of a block for any three blocks B, C and D of the design.

It is known that if (X, \mathcal{B}) is a symmetric design with the symmetric difference property then its parameters (v, k, λ) are of the form $(4\mu^2, 2\mu^2 - \mu, \mu^2 - \mu)$, where $\mu = 2^{m-1}$ and m is any positive integer.

In the symmetric design (X, \mathcal{B}) with the symmetric difference property, the number of points containing in an intersection of any three blocks of \mathcal{B} has exactly two values. Let those be x and y . Let f be a function defined by the following equation for any three blocks A, B and C in \mathcal{B} .

$$f(A \cap B \cap C) = \begin{cases} -1 & (|A \cap B \cap C| = x) \\ 1 & (|A \cap B \cap C| = y) \end{cases} \quad (3)$$

Guo's example

Guo showed that if a symmetric design (X, \mathcal{B}) with symmetric difference

property satisfying the certain condition, then we can construct a four-weight spin model on X . We can express the condition he gave in [4] as follows.

Condition

For any four blocks A, B, C and D in \mathcal{B}

$$f(A \cap B \cap C)f(A \cap B \cap D)f(A \cap C \cap D)f(B \cap C \cap D) = 1$$

3. The construction of W_1

Let $(X, W_1, W_2, W_3, W_4; D)$ be a four-weight spin model with exactly two values on W_2 . Then we can express W_2 as $\alpha(2E - J)$, where E is an incidence matrix of the regular Hadamard design of theorem 1 with the parameter $(16r^2, 8r^2 - 2r, 4r^2 - 2r)$. Since $\frac{1}{D} \sum_{x \in X} W_4(a, x) = \alpha^{-1}$ by the equation 4) in section 2.1, we define $W_1(x, x) = \alpha^{-1}$ for any $x \in X$.

Fix $c \in X$ and Let $a = b \neq c$ in the conditions 3) of the definition 1, then

$$\sum_{x \in X} \frac{W_2(a, x)^2}{W_2(c, x)} = \sum_{x \in X} \frac{W_2(c, x)^2}{W_2(a, x)} = D\alpha^2 \quad (4)$$

So we define $W_1(c, x) = W(x, c) = \epsilon_x \alpha^{-1}$.

By the equations 3) of the definition 1 and the condition (a) of the theorem 1, we can define the other entries of W_1 as the following.

$$\begin{aligned} W_1(a, b) &= \frac{W_1(a, c)W_1(c, b)}{D} \sum_{x \in X} \frac{W_2(a, x)W_2(b, x)}{W_2(c, x)} \\ &= f(B_a \cap B_b \cap B_c)\alpha \end{aligned}$$

,where f is a function defined in theorem 1 and B_x is a block of the regular Hadamard design of theorem 1. By the condition (b) of Theorem 1, we can prove that these W_1 and W_2 give a four-weight spin model.

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