

## Double group construction for $W^*$ -quantum groups

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### Introduction.

In [Dr], Drinfeld devised an ingenious method, called the double group construction, which generates a quasitriangular Hopf algebra out of any finite-dimensional Hopf algebra. This method was used to find solutions to the quantum Yang-Baxter equation in statistical mechanics. It was Podleś and Woronowicz [PW] who employed this method from the viewpoint of operator algebras in order to define a quantum deformation of Lorentz group. Later, Baaj and Skandalis [BS] introduced a notion of a Kac system, using (regular and irreducible) multiplicative unitaries. They showed that one can equally define the quantum double of a Kac system, and that the framework of Kac systems is stable under the construction of the quantum double. Afterwards, Nakagami [N] discussed the double group construction for Woronowicz algebras. The category of Woronowicz algebras can be naturally regarded as a “subcategory” of Kac systems. In [N], Nakagami was able to define the quantum double of a *compact* Woronowicz algebra, and to show that the double group is again a (noncompact, unimodular) Woronowicz algebra. It is, however, not so transparent how Nakagami’s double construction is related to Baaj-Skandalis’.

The purpose of this note is to define (construct) the quantum double for a *general* (quasi) Woronowicz algebra, and to prove that the category of (quasi) Woronowicz algebras is stable under this construction. We also give a brief description of the dual of the quantum double, which was left untouched in [N].

### 1. Definition of a quasi Woronowicz algebra.

In this section, we give a quick review on quasi Woronowicz algebras, introduced in [Y1]. Quasi Woronowicz algebras are almost like Woronowicz algebras introduced in [MN]. It is not too much to say that what is true for Woronowicz algebras is equally true for quasi Woronowicz algebras. Thus, for the general theory of quasi Woronowicz algebras, we may refer readers to [MN] and [N] (also see [Y1]).

A *coinvolutive Hopf-von Neumann algebra* is a triple  $(\mathcal{M}, \delta, R)$  in which:

- (1)  $\mathcal{M}$  is a von Neumann algebra;

- (2)  $\delta$  is an injective normal  $*$ -homomorphism, called a *coproduct* (or a *comultiplication*), from  $\mathcal{M}$  into  $\mathcal{M} \otimes \mathcal{M}$  with the coassociativity condition:  $(\delta \otimes id_{\mathcal{M}}) \circ \delta = (id_{\mathcal{M}} \otimes \delta) \circ \delta$ ;
- (3)  $R$  is a  $*$ -antiautomorphism of  $\mathcal{M}$ , called a *coinvolution* or a *unitary antipode*, such that  $R^2 = id_{\mathcal{M}}$  and  $\sigma \circ (R \otimes R) \circ \delta = \delta \circ R$ , where  $\sigma$  is the usual flip.

A *quasi Woronowicz algebra* is a family  $\mathbb{W} = (\mathcal{M}, \delta, R, \tau, h)$  in which:

- (1)  $(\mathcal{M}, \delta, R)$  is a coinvolutive Hopf-von Neumann algebra;
- (2)  $\tau$  is a continuous one-parameter automorphism group of  $\mathcal{M}$ , called the *deformation automorphism*, which commutes with the coproduct  $\delta$  and the antipode  $R$ ;
- (3)  $h$  is a  $\tau$ -invariant faithful normal semifinite weight on  $\mathcal{M}$ , called the *Haar measure* of  $\mathbb{W}$ , satisfying the following conditions:
  - (a) *Quasi left invariance*: For any  $\phi$  in  $\mathcal{M}_*^+$ , we have  $(\phi \otimes h) \circ \delta(x) = h(x)\phi(1)$  for all  $x \in \mathfrak{m}_h^+$ ;
  - (b) *Strong left invariance*: For any  $x, y \in \mathfrak{n}_h$  and  $\phi \in \mathcal{M}_*$  which is analytic with respect to the adjoint action of the deformation automorphism  $\tau$  on  $\mathcal{M}_*$ , the following equality holds:

$$(\phi \otimes h)((1 \otimes y^*)\delta(x)) = (\phi \circ \tau_{-i/2} \circ R \otimes h)(\delta(y^*)(1 \otimes x)).$$

- (c) *Commutativity*:  $h \circ \sigma_t^{h \circ R} = h$  for all  $t \in \mathbf{R}$  (or, equivalently,  $h \circ R \circ \sigma_t^h = h \circ R$ ).

We say that a quasi Woronowicz algebra  $\mathbb{W} = (\mathcal{M}, \delta, R, \tau, h)$  is *unimodular* (resp. *compact*) if  $h = h \circ R$  (resp.  $h$  is bounded).

Readers should note that only difference between a Woronowicz algebra and a quasi Woronowicz algebra is whether one requires that the weight  $h$  should be left invariant or that it should be quasi left invariant. In other words, in the definition of a Woronowicz algebra, one requires that  $h$  should satisfy  $(\phi \otimes h) \circ \delta(x) = h(x)\phi(1)$  for all  $\phi \in \mathcal{M}_*^+$  and *all*  $x \in \mathcal{M}_+$ . Let us briefly tell the reason why we work with quasi Woronowicz algebras rather than with Woronowicz algebras in this note. In the paper [MN], there is a crucial gap at the end of the proof of Proposition 3.8. Because of this gap, we do *not* yet know that the dual Woronowicz algebra in the sense of [MN] is really a Woronowicz algebra. One can, however, easily see that the dual *is* a quasi Woronowicz algebra. Moreover, the argument in [MN] proving the “duality” goes through perfectly without any change even if we start with a quasi Woronowicz algebra, not with a Woronowicz algebra. This is why we stick to working with quasi Woronowicz algebras. Besides, as shown in [Y1], every matched pair of (locally compact) groups gives rise to a quasi Woronowicz algebra. Hence there are plenty of examples of quasi Woronowicz algebras.

**Remark.** After the conference “Hilbert  $C^*$ -modules and groupoid  $C^*$ -algebras” at R.I.M.S., I was informed by Prof. Nakagami that quasi left invariance is actually equivalent to left invariance. Thus a quasi Woronowicz algebra is the same as a Woronowicz algebra. However, the proof of this equivalence has not yet been available to us, as of June, 1999. Hence we will distinguish these two objects(notions) at least in this note.

Throughout the remainder of this note, we fix a quasi Woronowicz algebra  $\mathbb{W} = (\mathcal{M}, \delta, R, \tau, h)$ . We always think of  $\mathcal{M}$  as represented on the Hilbert space  $\mathfrak{H}$ , constructed

from the weight  $h$  by the GNS construction. We denote by  $\Delta$  and  $J$  the modular operator and the modular conjugation of  $h$ , respectively. By the commutativity of  $h$ , there exists a non-singular positive self-adjoint operator  $Q$  on  $\mathfrak{H}$  affiliated with the centralizer  $\mathcal{M}_h = \{x \in \mathcal{M} : \sigma_t^h(x) = x \ (t \in \mathbf{R})\}$  of  $h$  such that the Connes' Radon-Nikodym derivative  $(D(h \circ R) : Dh)_t$  satisfies  $(D(h \circ R) : Dh)_t = Q^{it}$  for  $t \in \mathbf{R}$ . In the notation in [MN], we have  $Q = \rho^{-1}$ . We write  $W$  for the *Kac-Takesaki operator* of  $\mathbb{W}$ . As usual,  $\widehat{\mathbb{W}} = (\widehat{\mathcal{M}}, \widehat{\delta}, \widehat{R}, \widehat{\tau}, \widehat{h})$  stands for the quasi Woronowicz algebra dual to  $\mathbb{W}$ . The Kac-Takesaki operator of  $\widehat{\mathbb{W}}$  is denoted by  $\widehat{W}$ . Let  $\widehat{\Delta}$  and  $\widehat{J}$  designate the modular objects associated with the Haar measure  $\widehat{h}$ , which are regarded as acting on the Hilbert space  $\mathfrak{H}$ .

## 2. Hopf-von Neumann algebra structure on $\mathcal{M} \bar{\otimes} \widehat{\mathcal{M}}$ .

In this section, we shall equip the tensor product  $\mathcal{N} := \mathcal{M} \bar{\otimes} \widehat{\mathcal{M}}$  with a Hopf-von Neumann algebraic structure. The method for this is exactly the same as the one set out in Section 2 of [N]. But, here, we will reconsider it more carefully along the line of argument given in [BS, Section 8].

Let  $\mathbf{X} = (\widehat{W}')^*$ , where  $\widehat{W}'$  stands for the Kac-Takesaki operator associated with the commutant of the dual of the given quasi Woronowicz algebra  $\mathbb{W}$ . Then set

$$\mathbf{Y}_0 := \Sigma \mathbf{X}^* \Sigma, \quad Z_0 := \Sigma \mathbf{X}(u \otimes u) \mathbf{X}^*(u \otimes u) \Sigma.$$

Here  $u$  is the self-adjoint unitary given by  $u = J\widehat{J} = \widehat{J}J$ . Then, by [BS, Théorème 8.17], the family  $\{(\mathfrak{H}, \mathbf{X}, u), (\mathfrak{H}, \mathbf{Y}_0, u), Z_0\}$  forms a matched pair of Kac systems. (Precisely speaking, each of these systems may not be a Kac system, since the Kac-Takesaki operator  $W$  is not in general *regular* in the sense of [BS]. But this will not be any harm for our purpose. We remark that the Kac-Takesaki operator is always a *manageable* multiplicative unitary in the sense of [W].) Hence, by [BS, Proposition 8.14], if we set  $V_0 := (Z_0)_{12}^* \mathbf{X}_{13} (Z_0)_{12} (\mathbf{Y}_0)_{24}$ , then the map  $\delta_\tau$  given by

$$\delta_\tau(X) := V_0(X \otimes 1)V_0^* \quad (X \in S_{\mathbf{X}}'' \bar{\otimes} S_{\mathbf{Y}_0}'')$$

defines a coproduct on the von Neumann algebra  $S_{\mathbf{X}}'' \bar{\otimes} S_{\mathbf{Y}_0}''$ . In our notation, we have

$$S_{\mathbf{X}}'' = \mathbb{W}, \quad S_{\mathbf{Y}_0}'' = \widehat{W}' \sigma.$$

Since we want to work with  $\mathcal{N} := \mathcal{M} \bar{\otimes} \widehat{\mathcal{M}}$  rather than  $\mathcal{M} \bar{\otimes} \widehat{\mathcal{M}}'$ , we modify the above construction in the following way. First we note that the map  $\text{Ad } u$  gives a quasi Woronowicz algebra isomorphism from  $\widehat{\mathbb{W}}$  onto  $\widehat{W}' \sigma$  (cf. [N, Section 4]). So, through the isomorphism  $\text{id}_{\mathcal{M}} \otimes \text{Ad } u$ , everything that is true for the above construction can be translated in terms of our setting  $\mathcal{N} = \mathcal{M} \bar{\otimes} \widehat{\mathcal{M}}$ . Thus we put

$$\mathbf{Y} := (u \otimes u) \mathbf{Y}_0 (u \otimes u), \quad Z := (1 \otimes u) Z_0 (1 \otimes u).$$

Then the family  $\{(\mathfrak{H}, \mathbf{X}, u), (\mathfrak{H}, \mathbf{Y}, u), Z\}$  forms a matched pair. Hence the map  $\gamma$  given by  $\gamma := \sigma \circ \text{Ad } Z$  defines an “inversion” on  $\mathcal{M}$  and  $\widehat{\mathcal{M}}$  (in the sense of [BS]). Namely,  $\gamma$  is an isomorphism from  $\mathcal{M} \otimes \widehat{\mathcal{M}}$  onto  $\widehat{\mathcal{M}} \otimes \mathcal{M}$  satisfying

$$(2.1) \quad \begin{cases} (\gamma \otimes id_{\mathcal{M}}) \circ (id_{\mathcal{M}} \otimes \gamma) \circ (\delta \otimes id_{\widehat{\mathcal{M}}}) = (id_{\widehat{\mathcal{M}}} \otimes \delta) \circ \gamma, \\ (id_{\widehat{\mathcal{M}}} \otimes \gamma) \circ (\gamma \otimes id_{\widehat{\mathcal{M}}}) \circ (id_{\mathcal{M}} \otimes \hat{\delta}) = (\hat{\delta} \otimes id_{\mathcal{M}}) \circ \gamma. \end{cases}$$

Then the map

$$\delta^{\mathcal{N}} := (id_{\mathcal{M}} \otimes \gamma \otimes id_{\widehat{\mathcal{M}}}) \circ (\delta \otimes \hat{\delta})$$

defines a coproduct on  $\mathcal{N}$ . Moreover, with  $V = Z_{12}^* \mathbf{X}_{13} Z_{12} \mathbf{Y}_{24}$ , we have

$$\delta^{\mathcal{N}}(x) = V(x \otimes 1_{\mathfrak{H} \otimes \mathfrak{H}})V^* \quad (x \in \mathcal{N}).$$

It can be verified that  $Z = \hat{\mathbf{X}}\tilde{\mathbf{X}}^*$  with the notation in [BS, Section 6]. Since  $\tilde{\mathbf{X}} = (u \otimes u)W^*(u \otimes u)$  belongs to  $\mathcal{M}' \otimes \widehat{\mathcal{M}'}$ , the map  $\gamma$  actually equals  $\sigma \circ \text{Ad } \tilde{\mathbf{X}}$ . Since  $\hat{\mathbf{X}} = W^*$ , the inversion  $\gamma$  coincides with  $\sigma_W$  introduced in [N, Section 2]. Therefore, our coproduct  $\delta^{\mathcal{N}}$  is the same as Nakagami’s.

**Theorem 2.2.** *Retain the notation established above. Let*

$$\begin{aligned} R^{\mathcal{N}} &:= (R \otimes \hat{R}) \circ \text{Ad } W^*, \\ \tau_t^{\mathcal{N}} &:= \tau_t \otimes \hat{\tau}_t. \end{aligned}$$

Then  $(\mathcal{N}, \delta^{\mathcal{N}}, R^{\mathcal{N}})$  is a coinvolutive Hopf-von Neumann algebra. Each  $\tau_t^{\mathcal{N}}$  is a coinvolutive Hopf-von Neumann algebra automorphism, i.e., it satisfies

$$\begin{aligned} (\tau_t^{\mathcal{N}} \otimes \tau_t^{\mathcal{N}}) \circ \delta^{\mathcal{N}} &= \delta^{\mathcal{N}} \circ \tau_t^{\mathcal{N}}, \\ R^{\mathcal{N}} \circ \tau_t^{\mathcal{N}} &= \tau_t^{\mathcal{N}} \circ R^{\mathcal{N}} \end{aligned}$$

for any  $t \in \mathbf{R}$ .

*Proof.* This follows from a combination of Lemma 5 and Lemma 6 of [N].  $\square$

### 3. Haar measure for $(\mathcal{N}, \delta^{\mathcal{N}}, R^{\mathcal{N}}, \tau^{\mathcal{N}})$ .

The purpose of this section is to give a Haar measure  $h^{\mathcal{N}}$  for the coinvolutive Hopf-von Neumann algebra  $(\mathcal{N}, \delta^{\mathcal{N}}, R^{\mathcal{N}})$  with the deformation automorphism  $\tau^{\mathcal{N}}$  defined in the previous section. We can learn from [N] what  $h^{\mathcal{N}}$  should be. Thus, following [N], we let  $h^{\mathcal{N}}$  be the faithful normal semifinite weight on  $\mathcal{N} = \mathcal{M} \otimes \widehat{\mathcal{M}}$  defined by

$$h^{\mathcal{N}} := h \otimes \hat{h} \circ \hat{R}.$$

The reason why  $h^{\mathcal{N}}$  worked in Nakagami’s case is that  $h^{\mathcal{N}}$  equals the weight  $\Psi := (h \otimes \hat{h}) \circ \text{Ad } W^*$  when  $\mathbb{W}$  is a compact Woronowicz algebra. This fact enabled him to prove that  $h^{\mathcal{N}}$  in fact satisfies left invariance, strong left invariance and commutativity. However, this equality can be proven to be false in general by looking at examples obtained from matched pairs of groups. Thus the question is “*How much are  $h^{\mathcal{N}}$  and  $\Psi$  different in the general setting?*” Our philosophy is that the difference can be measured in terms of the Radon-Nikodym derivative. This derivative can be explicitly computed as follows:

**Proposition 3.1.** *The weight  $h^{\mathcal{N}}$  is  $\sigma^{\Psi}$ -invariant. The Radon Nikodym derivative  $(Dh^{\mathcal{N}} : D\Psi)$  of  $h^{\mathcal{N}}$  with respect to  $\Psi$  is given by*

$$(Dh^{\mathcal{N}} : D\Psi)_t = W(Q^{it} \otimes 1)W^*.$$

In particular, we have  $h^{\mathcal{N}} = \Psi(P \cdot)$ , where  $P$  is the nonsingular positive self-adjoint operator defined by  $P := W(Q \otimes 1)W^*$ .

Moreover, we can prove

**Theorem 3.2.** *The weight  $h^{\mathcal{N}}$  is  $R^{\mathcal{N}}$ -invariant, i.e.,*

$$h^{\mathcal{N}} = h^{\mathcal{N}} \circ R^{\mathcal{N}}.$$

In [N, Lemma 8], Nakagami proved that  $h^{\mathcal{N}}$  is nothing but  $\Psi$ , by using the structure of the dual algebra  $\widehat{\mathcal{M}}$  when  $\mathbb{W}$  is compact. But Proposition 3.1 fully explains why  $h^{\mathcal{N}}$  equals  $\Psi$  in the case of a compact Woronowicz algebra, and that one does not need to use the structure of  $\widehat{\mathcal{M}}$  to obtain the equality. It also says that  $h^{\mathcal{N}}$  coincides with  $\Psi$  so long as  $\mathbb{W}$  is unimodular. Powerfulness of Proposition 3.1 is that we can go “back and forth” between  $h^{\mathcal{N}}$  and  $\Psi$ , since we have  $h^{\mathcal{N}} = \Psi(P \cdot)$ . Thanks to this, we can prove that  $h^{\mathcal{N}}$  is both quasi left invariant and strong left invariant, by applying Nakagami’s argument in [N] with a suitable modification. Therefore we get

**Theorem 3.3.** *The system  $(\mathcal{N}, \delta^{\mathcal{N}}, R^{\mathcal{N}}, \tau^{\mathcal{N}}, h^{\mathcal{N}})$  is a unimodular quasi Woronowicz algebra.*

**Definition 3.4.** We call the unimodular quasi Woronowicz algebra constructed above the *quantum double (group)* of the given quasi Woronowicz algebra  $\mathbb{W}$ , and denote it by  $D(\mathbb{W})$ . The construction is referred to as the *double group construction*.

**Corollary 3.5.** *If a quasi Woronowicz algebra  $\mathbb{W}$  is a Kac algebra, then so is the quantum double  $D(\mathbb{W})$ .*

*Proof.* We retain the notation introduced so far. Note first that a quasi Woronowicz algebra  $\mathbb{W}$  is a Kac algebra if and only if  $\sigma^h = \sigma^{h \circ R}$  and the deformation automorphism is trivial. By Theorem 3.2, we certainly have  $\sigma^{h^{\mathcal{N}}} = \sigma^{h^{\mathcal{N}} \circ R^{\mathcal{N}}}$ . If  $\mathbb{W}$  is a Kac algebra, then  $\tau^{\mathcal{N}}$  is trivial. Hence the quantum double  $D(\mathbb{W})$  is a Kac algebra.  $\square$

#### 4. The dual of $D(\mathbb{W})$ .

This section is concerned with the dual of the quantum double  $D(\mathbb{W})$ . For this, we first clarify how  $W$  and  $\widehat{W}$  etc. are related to the Kac-Takesaki operator  $W^{\mathcal{N}}$  of the double group  $D(\mathbb{W})$ . This is not at all a trivial task, since  $W^{\mathcal{N}}$  “lives” in a different Hilbert space from the one where  $W$  and  $\widehat{W}$  live. But, by definition, these two Hilbert spaces are *canonically* isomorphic. Thus, for our purpose, one first has to identify this canonical isomorphism. In any case, through this isomorphism, the Kac-Takesaki operator  $W^{\mathcal{N}}$  is described as follows:

**Theorem 4.1.** *The Kac-Takesaki operator  $W^{\mathcal{N}}$  of the quantum double  $D(\mathbb{W})$  equals  $Z_{34}^* \widehat{W}_{24} Z_{34} W_{13}$ .*

**Remark.** Theorem 4.1 fully answers the problem raised in Section 2 of [N, Page 532]. In other words, Theorem 4.11 gives an explicit relation between the Kac-Takesaki operators for a *general* quasi Woronowicz algebra  $\mathbb{W}$  and its quantum double  $D(\mathbb{W})$ .

Once we have Theorem 4.1, we can describe the dual  $\widehat{\mathcal{N}}$  and its commutant  $\widehat{\mathcal{N}}'$ .

**Corollary 4.2.** *The quasi Woronowicz algebra  $\widehat{\mathcal{N}}$  dual to  $\mathcal{N}$  is generated by  $\widehat{\mathcal{M}} \otimes \mathbf{C}$  and  $Z^*(\mathbf{C} \otimes \mathcal{M})Z$ .*

**Proposition 4.3.** *The modular conjugation  $\widehat{J}_{\mathcal{N}}$  associated with  $\widehat{D(\mathbb{W})}$  is  $(\widehat{J} \otimes J)Z = Z^*(\widehat{J} \otimes J)$ .*

**Corollary 4.4.** *The commutant  $\widehat{\mathcal{N}}'$  of the dual  $\widehat{\mathcal{N}}$  is generated by  $Z^*(\widehat{\mathcal{M}}' \otimes \mathbf{C})Z$  and  $\mathbf{C} \otimes \mathcal{M}'$ .*

*Proof.* The assertion easily follows from a combination of Corollary 4.2 and Proposition 4.3.  $\square$

In what follows, we set  $\Sigma_{34}^{12} := \Sigma_{13}\Sigma_{24}$ . In other words,  $\Sigma_{34}^{12}$  is the unitary on  $\mathfrak{H} \otimes \mathfrak{H} \otimes \mathfrak{H} \otimes \mathfrak{H}$  given by  $\Sigma_{34}^{12}(\xi \otimes \eta) = \eta \otimes \xi$  for  $\xi, \eta \in \mathfrak{H} \otimes \mathfrak{H}$ .

**Theorem 4.5.** *With the unitary  $V$  defined in Section 2, we have*

$$(\widehat{J}_{\mathcal{N}} \otimes \widehat{J}_{\mathcal{N}})\Sigma_{34}^{12} V \Sigma_{34}^{12} (\widehat{J}_{\mathcal{N}} \otimes \widehat{J}_{\mathcal{N}}) = W^{\mathcal{N}}.$$

Therefore,  $V$  is the adjoint of the Kac-Takesaki operator  $W(\widehat{D(\mathbb{W})}' )$  of the commutant of the dual quasi Woronowicz algebra.

As a direct consequence of Theorem 4.5, we immediately obtain the proposition that follows. It is merely a rephrase of a part of Proposition 8.14 and Proposition 8.19 in [BS].

**Proposition 4.6.** (1) *For any  $z \in \widehat{\mathcal{M}}'$  and  $b \in \mathcal{M}'$ , put  $\pi(z) := Z^*(z \otimes 1)Z$ ,  $\pi'(b) := 1 \otimes b$ . Then  $\pi: \widehat{\mathcal{M}}' \rightarrow \widehat{\mathcal{N}}'$  and  $\pi': \mathcal{M}' \rightarrow \widehat{\mathcal{N}}'$  are Hopf-von Neumann algebra morphisms, i.e., we have*

$$\begin{aligned} (\pi \otimes \pi) \circ \widehat{\delta}'(z) &= (\widehat{\delta}^{\mathcal{N}})' \circ \pi(z), \\ (\pi' \otimes \pi') \circ \delta'(b) &= (\widehat{\delta}^{\mathcal{N}})' \circ \pi'(b). \end{aligned}$$

(2) *Set  $\mathcal{R} := Z_{12}^* \mathbf{X}_{14} Z_{12}$ . Then  $\mathcal{R}$  is a unitary in  $\widehat{\mathcal{N}}'$ . One also has*

$$(id \otimes (\widehat{\delta}^{\mathcal{N}})' )(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12}, \quad ((\widehat{\delta}^{\mathcal{N}})' \otimes id)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{23}.$$

For any  $x \in \widehat{\mathcal{N}}'$ , we have  $\sigma \circ (\widehat{\delta}^{\mathcal{N}})'(x) = \mathcal{R}(\widehat{\delta}^{\mathcal{N}})'(x)\mathcal{R}^*$ . Moreover, it satisfies the quantum Yang-Baxter equation:  $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$ .

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