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Hopf-Lax formulas for Hamilton-Jacobi equations with semicontinuous initial data

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§0 Introduction

This is an outline of a recent joint work [1] with O. Alvarez and E. N. Barron.

We are concerned with finding an explicit solution of the Hamilton-Jacobi equation

$$\text{(IVP)} \qquad \begin{cases} u_t + H(u, D_x u) = 0 \quad \text{in } \mathbf{R}^N \times (0,\infty), & (0.1)\\ u(0,x) = g(x) \quad (x \in \mathbf{R}^N). & (0.2) \end{cases}$$

Under the assumptions that $H = H(p)$ is independent of u and convex, and g is at least continuous, the Lax formula gives the explicit solution

$$u(x,t) = \inf_{y\in\mathbf{R}^N} \left(g(y) + tH^* \left(\frac{x-y}{t} \right) \right), \tag{0.3}$$

where $*$ means the Legendre-Fenchel conjugate.

One way to find this formula is to consider an associated optimal control problem. A more difficult problem emerges when one wants to move the convexity off of H and onto the initial data g since then the associated control problem is a differential game. Nevertheless, assuming that H is at least continuous and g is convex and finite, the Hopf formula

$$u(x,t) = [g^* + tH]^*(x) \tag{0.4}$$

gives the solution of (IVP).

We call these formulas the classical Hopf and Lax formulas. These results are originally due to Hopf, Lax, and Oleinik but are proved in the context of viscosity solutions under assumptions leading to a continuous solution u, refer to Bardi-Evans [3] and Lions-Rochet [12].

We shall recall sufficient conditions for H and g in the next section under which either formulas (0.3) or (0.4) give a solution of (IVP).

It was recent that Barron, Jensen, and Liu [7, 8, 9, 10] extended these formulas (0.3) and (0.4) to equation (0.1) with u dependence and allowed more general initial data, namely the class of quasiconvex functions, which are defined by the property that they have convex level sets.

Under the assumption that g is at least continuous and $H(u,p)$ is continuous, nondecreasing in u, convex and positively homogeneous degree one in p, they obtained the Lax type formula

$$u(x,t) = \min_{y \in \mathbf{R}^N} g(y) \vee H^{\#}\left(\frac{x-y}{t}\right),$$

which uses the (second) quasiconvex conjugate # of $H(u,p)$, given by

$$H^{\#}(z) = \inf\{r \mid \sup_{p \in \mathbf{R}^N} p \cdot z - H(r,p) \leq 0\}.$$

Moving the convexity off of H and onto g in the weakened form of quasiconvexity, they obtained the Hopf type formula

$$u(x,t) = [g^{\#} + tH]^{\#}(x),$$

where the first quasiconvex conjugate of the function g is

$$g^{\#}(r,p) = \sup\{p \cdot x \mid x \in E(r,g)\}, \qquad E(r,g) = \{x \in \mathbf{R}^N \mid g(x) \leq r\}.$$

The assumptions to derive this formula involved $H(u,p)$ continuous, nondecreasing in u, positively homogeneous degree one in p, and g at least continuous and quasiconvex. One of the main points of these new formulas is that with u dependence of the Hamiltonian, one must have homogeneity of H in p.

We will refer to the formulas for $u_t + H(u, D_x u) = 0$ where quasiconvexity is involved as the quasiconvex formulas.

The purpose of the joint work [1] is to extend all of these considerations to initial data which is only semicontinuous and possibly infinite. The assumptions on the Hamiltonian remain unchanged. This extension results in very weak and natural assumptions on g.

In the Lax formula case, when we assume a convex Hamiltonian in p for either $H(p)$ or $H(u,p)$, we may use the theory of lower semicontinuous (lsc for short) viscosity solutions introduced by Barron-Jensen [5, 6] which extended the classical Crandall-Lions definition to lsc functions possibly taking the value, ∞.

Precisely, a lsc function u with values in $(-\infty, \infty]$ is a *lsc solution* of an equation $u_t + H(x,t,u,D_x u) = 0$, if

$$q + H(x,t,u,p) = 0 \quad \forall (p,q) \in D^- u(x,t) \quad \text{if } u(x,t) < +\infty.$$

Using this notion of solution we can prove that the Lax formulas give a lsc solution of the problem with lsc initial data.

In the more difficult case of the Hopf formula, when no convexity of the Hamiltonian is assumed, we do not have a good characterization of what it means to be a lsc solution in the sense that uniqueness is still an open problem. What we can do is to characterize the Hopf formulas as yielding the minimal supersolution of the equation.

§1 Classical Lax and Hopf formulas

In this section we examine the classical Lax and Hopf formulas. We thus assume that $H = H(p)$, i.e., H depends only on p, and that H and g are continuous.

Under the assumption that H is convex, the Lax formula asserts that the function

$$u(x,t) = \inf_{y\in\mathbf{R}^N}\left(g(y) + tH^*\left(\frac{x-y}{t}\right)\right) \tag{1.1}$$
$$= \inf_{z\in\mathbf{R}^N}\left(g(x-tz) + tH^*(z)\right) \tag{1.2}$$
$$= \inf_{y\in\mathbf{R}^N}\sup_{p\in\mathbf{R}^N}\left[(x-y)\cdot p + g(y) - tH(p)\right] \tag{1.3}$$

gives a "solution" of (IVP). However, these formulas do not always yield a continuous function on $\mathbf{R}^N \times [0,\infty)$, not to speak of a differentiable function. In respect of this breakdown of the classical notion of solution to (0.1), we adopt the notion of viscosity solution as that of weak solution to (0.1).

Now we assume that

$$H \text{ is convex and that } g \in \mathrm{UC}(\mathbf{R}^N), \tag{1.4}$$

and will see that this assumption on g is a sufficient condition for the Lax formula gives a unique continuous viscosity solution of (IVP).

Note that, by the convexity of H, there are $a \in \mathbf{R}^N$ and $b > 0$ such that

$$H(p) \geq a\cdot p - b \quad (p \in \mathbf{R}^N).$$

Accordingly, we have $H^*(a) \leq b$.

Since $H \in \mathrm{C}(\mathbf{R}^N)$, we have

$$\lim_{|p|\to\infty}\frac{H^*(p)}{|p|} = \infty.$$

Thus, according to (1.2), we have

$$\begin{aligned} g(x-ta) + tH^*(a) \geq u(x,t) &\geq \inf_{z\in\mathbf{R}^N}\left(tH^*(z) - C_1(1+|x|+t|z|)\right) \\ &\geq \inf_{z\in\mathbf{R}^N}\left(t(H^*(z) - |z|)\right) - C_1(1+|x|) \\ &\geq -C_2(1+|x|) \end{aligned}$$

for some constants $C_i > 0$, $i = 1,2$. Hence, the Lax formula gives a real-valued function on $\mathbf{R}^N \times [0,\infty)$.

Let $\omega \in \mathrm{C}([0,\infty))$ be the modulus of continuity of g, i.e., ω is defined by

$$\omega(r) = \sup\{|g(x) - g(y)| \mid x, y \in \mathbf{R}^N,\ |x-y| \leq r\} \quad (r \geq 0).$$

It is easily seen that formula (1.2) yields

$$|u(x,t) - u(y,t)| \leq \omega(|x-y|) \quad (x, y \in \mathbf{R}^N,\ t \geq 0). \tag{1.5}$$

Fix $\varepsilon \in (0,1)$ and select constants $A_\varepsilon > 0$ and $B_\varepsilon > 0$ so that

$$\omega(r) \leq \varepsilon + A_\varepsilon r \quad (r \geq 0);$$
$$H^*(z) \geq A_\varepsilon |p| - B_\varepsilon \quad (p \in \mathbf{R}^N).$$

These together with (1.2) yields

$$\begin{aligned} u(x,t) \geq & \inf_{z\in\mathbf{R}^N} [g(x) - \varepsilon - tA_\varepsilon |z| + t(A_\varepsilon|z| - B_\varepsilon)] \\ & \geq g(x) - \varepsilon - tB_\varepsilon \quad (x \in \mathbf{R}^N,\ t \geq 0). \end{aligned}$$

Hence, setting $\omega_1(r) = \inf_{\varepsilon\in(0,1)}(\varepsilon + rB_\varepsilon)$ for $r \geq 0$, we have

$$|u(x,t) - g(x)| \leq \max\{\omega_1(t),\ \omega(t|a|) + t|H^*(a)|\} \quad (x \in \mathbf{R}^N,\ t \geq 0). \tag{1.6}$$

Proposition 1.1. *We have*

$$u(x,t+s) = \inf_{y\in\mathbf{R}^N} \left(u(y,t) + sH^*\left(\frac{x-y}{s}\right)\right) \quad (x \in \mathbf{R}^N,\ t,s \geq 0). \tag{1.7}$$

Formula (1.7) says that the family of maps $U(t) : g \to u(x,t)$, with $t \geq 0$, has the semigroup property, i.e., $U(t+s) = U(t) \circ U(s)$ for all $t, s \geq 0$.

Once we have Proposition at hand, we infer from (1.5) and (1.6) that u is uniformly continuous on $\mathbf{R}^N \times [0,\infty)$.

Proof of Proposition 1.1. First we compute that for $x \in \mathbf{R}^N$ and $t, s \geq 0$,

$$\begin{aligned} & \inf_{y\in\mathbf{R}^N} \left(u(y,t) + sH^*\left(\frac{x-y}{s}\right)\right) \\ & \quad = \inf_{z\in\mathbf{R}^N} (u(x-tz,s) + tH^*(z)) \\ & \quad = \inf_{y,z\in\mathbf{R}^N} (g(x-tz-sy) + sH^*(y) + tH^*(z)) \\ & \quad \leq \inf_{z\in\mathbf{R}^N} (g(x-(t+s)z) + (t+s)H^*(z)) = u(x,t+s). \end{aligned}$$

Next, let $x \in \mathbf{R}^N$ and $t, s > 0$. By the convexity of H^*, we have

$$\begin{aligned} H^*\left(\frac{x-z}{t+s}\right) &= H^*\left(\frac{s}{t+s}\frac{x-y}{s} + \frac{t}{t+s}\frac{y-z}{t}\right) \\ &\leq \frac{s}{t+s}H^*\left(\frac{x-y}{s}\right) + \frac{t}{t+s}H^*\left(\frac{y-z}{t}\right) \end{aligned}$$

for $y, z \in \mathbf{R}^N$. Hence, we get

$$\begin{aligned} & \inf_{y\in\mathbf{R}^N} \left(u(y,t) + sH^*\left(\frac{x-y}{s}\right)\right) \\ & \quad = \inf_{y,z\in\mathbf{R}^N} \left(g(z) + tH^*\left(\frac{y-z}{t}\right) + sH^*\left(\frac{x-y}{s}\right)\right) \\ & \quad \geq \inf_{y,z\in\mathbf{R}^N} \left(g(z) + (t+s)H^*\left(\frac{x-z}{t+s}\right)\right) = u(x,t+s). \end{aligned}$$

Combining these inequalities completes the proof. □

As we already mentioned, we have

Theorem 1.2. *The Lax formulas* (1.1), (1.2), *or* (1.3) *give a uniformly continuous function* u *on* $\mathbf{R}^N \times [0,\infty)$.

The main assertion concerning the classical Lax formulas is stated as follows.

Theorem 1.3. *The function* $u \in \mathrm{UC}(\mathbf{R}^N \times [0,\infty))$ *given either* (1.1), (1.2), *or* (1.5) *is a viscosity solution of* (0.1).

Remark 1.4. The problem (IVP) has at most one viscosity solution in the class $\mathrm{UC}(\mathbf{R}^N \times [0,\infty))$. See [2] for this uniqueness assertion.

Proof. Let $\varphi \in \mathrm{C}^1(\mathbf{R}^N \times (0,\infty))$ and $(x,t) \in \mathbf{R}^N \times (0,\infty)$. Assume that $u-\varphi$ has a maximum at (x,t) and the maximum value is zero. Let $\varepsilon \in [0,t)$.

Using Proposition 1.1, we observe that

$$\begin{aligned}\varphi(x,t) = u(x,t) &= \inf_{y\in\mathbf{R}^N} (u(x-\varepsilon y, t-\varepsilon) + \varepsilon H^*(y))\\ &\leq \varphi(x-\varepsilon t, t-\varepsilon) + \varepsilon H^*(z) \quad (z \in \mathbf{R}^N).\end{aligned}$$

That is, for each $z \in \mathbf{R}^N$ the function

$$\varepsilon \mapsto \varphi(x-\varepsilon t, t-\varepsilon) + \varepsilon H^*(z)$$

attains a minimum over $[0,t)$ at $\varepsilon = 0$. Hence, the derivative of this function at $\varepsilon = 0$ must be nonnegative. That is to say,

$$-z\cdot D_x\varphi(x,t) - \varphi_t(x,t) + H^*(z) \geq 0 \quad (z \in \mathbf{R}^N).$$

Therefore, we conclude that

$$\varphi_t(x,t) + H(D_x\varphi(x,t)) \leq 0,$$

proving that u is a viscosity subsolution of (0.1).

Next, select a family $\{g_\varepsilon \mid \varepsilon \in (0,1)\} \subset \mathrm{C}^1(\mathbf{R}^N) \cap \mathrm{UC}(\mathbf{R}^N)$ so that

$$g(x) = \inf_{0<\varepsilon<1} g_\varepsilon(x) \quad (x \in \mathbf{R}^N).$$

The formula (1.2) reads

$$u(x,t) = \inf_{z\in\mathbf{R}^N,\ 0<\varepsilon<1} (g_\varepsilon(x-tz) + tH^*(z)) \quad (x \in \mathbf{R}^N,\ t \geq 0).$$

Note that if we fix $z \in \mathbf{R}^N$ and set

$$u^\varepsilon(x,t) = g_\varepsilon(x-tz) + tH^*(z) \quad (x \in \mathbf{R}^N,\ t \geq 0),$$

then we have

$$\begin{aligned}
&u_t^\varepsilon(x,t) + H(D_x u^\varepsilon(x,t)) \\
&= -z \cdot Dg_\varepsilon(x - tz) + H^*(z) + H(Dg_\varepsilon(x - tz)) \\
&\geq -z \cdot Dg_\varepsilon(x - tz) + H^*(z) + z \cdot Dg_\varepsilon(x - tz)) - H^*(z) = 0 \quad (x \in \mathbf{R}^N,\ t > 0).
\end{aligned}$$

Since the pointwise infimum of a family of viscosity supersolutions is a viscosity supersolution, we conclude that u is a viscosity supersolution of (0.1). □

Next we turn to the classical Hopf formula, and assume, in addition to the hypotheses that $g, H \in \mathrm{C}(\mathbf{R}^N)$, that

$$g \text{ is convex.} \tag{1.8}$$

First we note that (0.4) reads

$$\begin{aligned}
u(x,t) &= (g^* + tH)^*(x) && (1.9) \\
&= \sup_{p \in \mathbf{R}^N} [x \cdot p - (g^*(p) + tH(p))] && (1.10) \\
&= \sup_{p \in \mathbf{R}^N} \inf_{y \in \mathbf{R}^N} [(x - y) \cdot p + g(y) - tH(p)]\,. && (1.11)
\end{aligned}$$

It is clear that $u(x,0) = g(x)$ for all $x \in \mathbf{R}^N$, due to the duality for convex conjugates.

The formulas above actually define a convex function on $\mathbf{R}^N \times \mathbf{R}$. Therefore, the function u is the restriction of a convex function on $\mathbf{R}^N \times \mathbf{R}$ to $\mathbf{R}^N \times [0, \infty)$.

As a sufficient condition for u given above to have finite values, we assume that there is a constant $C > 0$ such that

$$H(p) \geq -C(|p| + 1) \quad (p \in \mathbf{R}^N). \tag{1.12}$$

Indeed, since $g \in \mathrm{C}(\mathbf{R}^N)$ is convex, we have

$$\lim_{|p| \to \infty} \frac{g^*(p)}{|p|} = \infty,$$

and therefore,

$$\lim_{|p| \to \infty} \frac{g^*(p) + tH(p)}{|p|} = \infty \quad (t \geq 0).$$

This combined with (1.10) yields that $u(x,t) < \infty$ for all $x \in \mathbf{R}^N$ and $t \geq 0$. Thus we see that $u(x,t) \in \mathbf{R}$ for all $x \in \mathbf{R}^N$ and $t \geq 0$.

It is worth noting that for the function u is given by (1.9),

$$u(x,t) < \infty \quad (x \in \mathbf{R}^N,\ t > 0)$$

if and only if

$$\liminf_{|p| \to \infty} \frac{H(p)}{g^*(p)} \geq 0.$$

The following observation for the Hopf formula corresponds to Proposition 1.1 for the Lax formula.

Proposition 1.5. *We have*

$$u(x,t+s) = \left(u^*(\cdot,t) + sH\right)^*(x) \quad (x \in \mathbf{R}^N,\ t,s \geq 0). \tag{1.13}$$

Proof. Fix $x \in \mathbf{R}^N$ and $t,s \geq 0$, and let R denote the right hand side of (1.13).

We note that

$$\begin{aligned} R &= \sup_{p\in\mathbf{R}^N} \inf_{y\in\mathbf{R}^N} \sup_{q\in\mathbf{R}^N} \inf_{z\in\mathbf{R}^N} \left((x-y)\cdot p + (y-z)\cdot q + g(z) - sH(p) - tH(q)\right) & (1.14)\\ &= \sup_{p\in\mathbf{R}^N} \inf_{\xi\in\mathbf{R}^N} \sup_{q\in\mathbf{R}^N} \inf_{\eta\in\mathbf{R}^N} \left(\xi\cdot p + \eta\cdot q + g(x-\xi-\eta) - sH(p) - tH(q)\right). & (1.15)\end{aligned}$$

From (1.14) we see that

$$R \geq \sup_{p\in\mathbf{R}^N} \inf_{y\in\mathbf{R}^N} \inf_{z\in\mathbf{R}^N} \left((x-y)\cdot p + (y-z)\cdot p + g(z) - sH(p) - tH(p)\right) = u(x,t+s).$$

We may assume without loss of generality that $s,t > 0$. From (1.15), we have

$$\begin{aligned} R = &\sup_{p\in\mathbf{R}^N} \inf_{\xi\in\mathbf{R}^N} \sup_{q\in\mathbf{R}^N} \inf_{\eta\in\mathbf{R}^N} \Big(\frac{s}{t+s}\xi\cdot p + \frac{t}{t+s}\eta\cdot q \\ &+ g\Big(x - \frac{s}{t+s}\xi - \frac{t}{t+s}\eta\Big) - sH(p) - tH(q)\Big) \\ \leq &\frac{s}{t+s} \sup_{p\in\mathbf{R}^N} \inf_{\xi\in\mathbf{R}^N} \left(\xi\cdot p + g(x-\xi) - (s+t)H(p)\right) \\ &+ \frac{t}{t+s} \sup_{q\in\mathbf{R}^N} \inf_{\eta\in\mathbf{R}^N} \left(\eta\cdot q + g(x-\eta) - (s+t)H(q)\right) \\ = &\left(\frac{s}{t+s} + \frac{t}{t+s}\right) u(x,s+t) = u(x,s+t). \end{aligned}$$

□

Since the function u given by (1.9) is convex and finite on $\mathbf{R}^N \times [0,\infty)$, it is locally Lipschitz continuous in $\mathbf{R}^N \times (0,\infty)$. The following proposition guarantees the continuity of u for $t = 0$.

Theorem 1.6. $u \in \mathrm{C}(\mathbf{R}^N \times [0,\infty))$.

Proof. As we noted above, we only need to show that u is continuous for $t = 0$.

It follows from (1.10) that the function u is lower semicontinuous on $\mathbf{R}^N \times [0,\infty)$. It remains to show that u is upper semicontinuous at $(x,0)$ for all $x \in \mathbf{R}^N$.

Fix $x \in \mathbf{R}^N$, $t \in (0,1)$, and $\varepsilon > 0$, and choose $\hat{p} \in \mathbf{R}^N$ so that

$$u(x,t) < \varepsilon + x\cdot\hat{p} - g^*(\hat{p}) - tH(\hat{p}).$$

Now, we have

$$u(x,t) < \varepsilon + (1-t)\left(\frac{x}{1-t}\cdot \hat{p} - g^*(\hat{p})\right) - t\big(g^*(\hat{p}) + H(\hat{p})\big)$$
$$\le \varepsilon + (1-t)g\Big(\frac{x}{1-t}\Big) + tC_1,$$

where $C_1 > 0$ is a constant such that $\inf_{\mathbf{R}^N}(g^* + H) \ge -C_1$. This shows that u is upper semicontinuous for $t = 0$. □

Theorem 1.7. *The function u is a viscosity solution of* (0.1).

Proof. If we set

$$u(x,t;p) = x\cdot p - g^*(p) - tH(p) \quad (x, p \in \mathbf{R}^N,\ t \ge 0),$$

then

$$u_t(x,t;p) + H(D_x u(x,t;p)) = 0$$

in the classical sense and $u(x,t) = \sup_{p\in\mathbf{R}^N} u(x,t;p)$. Since the pointwise supremum of a family of viscosity subsolutions is a viscosity subsolution, we see that u is a viscosity subsolution of (0.1).

Next we show that u is a viscosity supersolution of (0.1). Let $\varphi \in \mathrm{C}^1(\mathbf{R}^N \times (0,\infty))$ and assume that $u - \varphi$ attains a maximum at $(\hat{x},\hat{t}) \in \mathbf{R}^N \times (0,\infty)$. We may assume that $u(\hat{x},\hat{t}) = \varphi(\hat{x},\hat{t})$, so that $u \le \varphi$. Since

$$\liminf_{|p|\to\infty} \frac{g^*(p) + \hat{t}H(p)}{|p|} = \infty,$$

in view of (1.10) we find $\hat{p} \in \mathbf{R}^N$ such that

$$u(\hat{x},\hat{t}) = \hat{x}\cdot\hat{p} - g^*(\hat{p}) - \hat{t}H(\hat{p}).$$

Since

$$x\cdot\hat{p} - g^*(\hat{p}) - tH(\hat{p}) \le u(x,t) \le \varphi(x,t) \quad (x \in \mathbf{R}^N,\ t > 0)$$

and

$$\hat{x}\cdot\hat{p} - g^*(\hat{p}) - \hat{t}H(\hat{p}) = \varphi(\hat{x},\hat{t}),$$

the function

$$x\cdot\hat{p} - g^*(\hat{p}) - tH(\hat{p}) - \varphi(x,t)$$

of (x,t) attains a maximum at $(\hat{x},\hat{t})$. Hence,

$$\hat{p} = D_x\varphi(\hat{x},\hat{t}) \quad \text{and} \quad -H(\hat{p}) = \varphi_t(\hat{x},\hat{t}),$$

and therefore,

$$\varphi_t(\hat{x},\hat{t}) + H(D_x\varphi(\hat{x},\hat{t})) = 0,$$

concluding the proof. □

We have the following comparison theorem.

Theorem 1.8. *Let $u \in C(\mathbf{R}^N \times [0,\infty))$ and $v \in LSC(\mathbf{R}^N \times [0,\infty))$ be a viscosity subsolution and a viscosity supersolution of* (0.1)*, respectively. Assume that u is convex in $\mathbf{R}^N \times [0,\infty)$, that for each $T > 0$ there is a constant $C_T > 0$ such that*

$$v(x,t) \geq -C_T(|x|+1) \quad (x \in \mathbf{R}^N,\ 0 \leq t \leq T). \tag{1.16}$$

and that $u(x,0) \leq v(x,0)$ for all $x \in \mathbf{R}^N$. Then $u \leq v$ in $\mathbf{R}^N \times [0,\infty)$.

Immediate consequences of the theorem above are that the Hopf formula (1.9) gives the unique viscosity solution of (IVP) in the class of convex continuous functions in $\mathbf{R}^N \times [0,\infty)$ and that (1.9) gives the pointwise minimum of viscosity supersolutions $v \in LSC(\mathbf{R}^N \times [0,\infty))$ of (0.1) which satisfy (1.16) and that $v(x,0) \geq g(x)$ for all $x \in \mathbf{R}^N$.

Proof. Fix any $(\hat{x},\hat{t}) \in \mathbf{R}^N \times (0,\infty)$. Since u is locally Lipschitz continuous, there is a $(p,a) \in D^-u(\hat{x},\hat{t}) \cap \overline{D}^+u(\hat{x},\hat{t})$. Define $w : \mathbf{R}^N \times [0,\infty) \to \mathbf{R}$ by

$$w(x,t) = u(\hat{x},\hat{t}) + p\cdot(x-\hat{x}) + a\cdot(t-\hat{t}).$$

Then w is a classical subsolution of (0.1) and satisfies that $w(x,0) \leq u(x,0)$ for all $x \in \mathbf{R}^N$. By the standard comparison theorem, we see that $w \leq v$ in $\mathbf{R}^N \times [0,2\hat{t}]$, from which we have

$$u(\hat{x},\hat{t}) = w(\hat{x},\hat{t}) \leq v(\hat{x},\hat{t}).$$

Because $(\hat{x},\hat{t})$ is arbitrary, we conclude that $u \leq v$ in $\mathbf{R}^N \times [0,\infty)$. □

§2 The Lax and Hopf formulas for lsc data

We continue in this section to assume that $H = H(p)$ is a function of the variable p only.

In the first half of this section we turn our attention to the Lax formula for initial data which is lsc. The Lax formula requires that the Hamiltonian is convex and therefore we may use the theory of lsc viscosity solutions (see [5, 6]) to characterize the solution.

In what follows we will use the assumption

There is a constant $C > 0$ such that $g(x) \geq -C(|x|+1)$ for all $x \in \mathbf{R}^N$. (2.1)

We say that a function $u : \mathbf{R}^N \times [0,\infty) \to (-\infty,\infty]$ is *bounded from below by a function of linear growth* if for each $T > 0$ there is a constant $C_T > 0$ such that

$$u(x,t) \geq -C_T(|x|+1) \quad (x \in \mathbf{R}^N,\ 0 \leq t \leq T).$$

Theorem 2.1. *Let $g : \mathbf{R}^N \to (-\infty,\infty]$ be a proper lsc function and satisfy* (2.1)*. Let H be continuous, finite, and convex. The function $u : \mathbf{R}^N \times [0,\infty) \to (-\infty,\infty]$ given by* *the Lax formula* (1.1) *is the unique lsc viscosity solution of* (IVP) *that is bounded from below by a function of linear growth.*

The precise definition of lsc viscosity solution of (IVP) is as follows: a function $u : \mathbf{R}^N \times [0,\infty) \to (-\infty, \infty]$ is called a lsc viscosity solution of (IVP) if it is a lsc viscosity solution of (0.1) and

$$g(x) = \lim_{r\downarrow 0} \inf\{u(y,s) \mid |y-x| < r,\ 0 < s < r\} \quad (x \in \mathbf{R}^N). \tag{2.2}$$

We refer to [1] for the main part of the proof of the theorem above, and just check that the function u given by the Lax formula satisfies the initial condition in the sense of (2.2).

We fix $\hat{x} \in \mathbf{R}^N$. We first observe that for any $r \geq 0$,

$$v(x,t) := tH^*\left(\frac{x}{t}\right) = \sup_{p\in\mathbf{R}^N} (p\cdot x - tH(p)) \geq r|x| - t\max_{|p|\leq r} H(p) \quad (x \in \mathbf{R}^N,\ t > 0).$$

Since

$$u(x,t) = \inf_{y\in\mathbf{R}^N} (g(y) + v(x-y,t)),$$

we have from (2.1)

$$\begin{aligned} u(x,t) &\geq \inf_{y\in\mathbf{R}^N} (g(y) + r|x-y| - t\max_{|p|\leq r} H) \\ &\geq \inf_{y\in\mathbf{R}^N} (-C|y| - C + r|x-y| - t\max_{|p|\leq r} H) \\ &\geq \inf_{y\in\mathbf{R}^N} ((r-C)|x-y| - C|x| - C - t\max_{|p|\leq r} H), \end{aligned}$$

for any $r \geq 0$. Fix $\varepsilon > 0$. Since g is lsc, there is $\delta > 0$ such that if $|y - \hat{x}| \leq 2\delta$, then

$$g(y) \geq g(\hat{x}) - \varepsilon.$$

Choose $r > C$ so that

$$(r-C)\delta - C(|\hat{x}| + \delta) - C \geq g(\hat{x}),$$

and then $\tau > 0$ so that

$$\tau \max_{|p|\leq r} |H| \leq \varepsilon.$$

We see that if $0 < t \leq \tau$ and $x \in B(\hat{x}, \delta)$, then

$$(r-C)|x-y| - C|x| - C - t\max_{|p|\leq r} H \geq g(\hat{x}) - \varepsilon \quad (|y - \hat{x}| > 2\delta)$$

and so for any $d \geq 0$,

$$u(x,t) \geq \min\{g(\hat{x}) - \varepsilon,\ \inf_{y\in B(\hat{x},2\delta)} (g(y) + d|x-y| - t\max_{|p|\leq d} H)\}.$$

In particular, we have

$$u(x,t) \geq \min\{g(\hat{x}) - \varepsilon, \inf_{y \in B(\hat{x},2\delta)} (g(y) - \varepsilon - tH(0))\} \quad (x \in B(\hat{x},\delta), 0 < t \leq \tau).$$

We may assume by replacing τ by a smaller one that $\tau|H(0)| \leq \varepsilon$. Then we have

$$u(x,t) \geq g(\hat{x}) - 2\varepsilon \quad (x \in B(\hat{x},\delta), 0 < t \leq \tau).$$

This shows that

$$\liminf_{r\downarrow 0} \{u(x,t) \mid |x - \hat{x}| < r,\ 0 < t < r\} \geq g(\hat{x}).$$

Next we observe that since $H(p) \geq -a \cdot p - b$ for some $a \in \mathbf{R}^N$ and $b \in \mathbf{R}$,

$$v(x,t) = \sup_{p \in \mathbf{R}^N} (p \cdot x - tH(p)) \leq \sup_{p \in \mathbf{R}^N} p \cdot (x + ta) + bt.$$

Therefore we have

$$u(\hat{x} - ta, t) \leq \inf_{y \in \mathbf{R}^N} \sup_{p \in \mathbf{R}^N} (g(y) + p \cdot (\hat{x} - at - y + at) + bt) \leq g(\hat{x}) + bt \quad \forall\, t \geq 0,$$

and hence

$$\liminf_{r\downarrow 0} \{u(x,t) \mid |x - \hat{x}| < r,\ 0 < t < r\} \leq g(\hat{x}).$$

Thus we conclude that (2.2) holds.

Next we turn to extending the Hopf formula for lsc convex data.

First we assume that H is convex, so that we can apply the theory of lsc viscosity solutions.

We note that the Hopf formula gives a lsc convex function in (x,t). The function has values in $(-\infty,\infty]$ because $g^* \not\equiv +\infty$ and H is finite. If g is proper, u is proper but, of course, it may happen that $u(x,t) = +\infty$ for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$.

Theorem 2.2. *Let $g : \mathbf{R}^N \to (-\infty,\infty]$ be a lower semicontinuous, proper, and convex function. Assume H is convex and continuous. Then the function $u : \mathbf{R}^N \times [0,\infty) \to (-\infty,\infty]$, defined by $u(x,t) = (g^* + tH)^*(x)$, is the unique lower semicontinuous viscosity solution of* (IVP) *that is bounded from below by a function of linear growth.*

We refer the reader to [1] for the proof of this theorem.

The next theorem generalizes this observation to the case where the Hamiltonian is not convex and not bounded from below. The argument is direct. Of course, the notion of lsc solution is not applicable here, so we can only say that the Hopf formula gives the pointwise minimum of supersolutions.

Theorem 2.3. *Let* $g : \mathbf{R}^N \to (-\infty, \infty]$ *be a proper, lsc, convex function and* $H : \mathbf{R}^N \to \mathbf{R}$ *be a continuous function. The function* $u : \mathbf{R}^N \times [0, \infty) \to (-\infty, \infty]$ *defined by the Hopf formula*

$$u(x,t) = [g^* + tH]^* (x)$$

is the pointwise minimum of viscosity supersolutions v *of* (0.1) *that are lsc on* $\mathbf{R}^N \times [0, \infty)$ *and satisfy* $v(x, 0) \geq g(x)$ *for all* $x \in \mathbf{R}^N$ *and that are bounded from below by a function with linear growth.*

In particular, it is a viscosity solution in the interior of the domain of u.

We refer the reader to [1] for the proof of this theorem.

§3 Hamilton-Jacobi equation with the Hamiltonian depending on u

We begin this section with the Lax formula for (IVP) with the Hamiltonian H depending on u.

Theorem 3.1. *Let* $g : \mathbf{R}^N \to (-\infty, \infty]$ *be a proper lsc function. Let* $H : \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ *be a continuous function such that* $H(r,p)$ *is nondecreasing in* r *and convex, positively homogeneous of degree 1 in* p.

Then the function given by

$$u(x,t) := \inf_{y \in \mathbf{R}^N} g(y) \vee H^{\#} \left(\frac{x-y}{t} \right), \tag{3.1}$$

is the pointwise minimum of lsc supersolutions of (IVP).

We call (3.1) the quasiconvex Lax formula. We refer for the proof of Theorem 3.1 to [1].

Next we present the quasiconvex Hopf formula.

Theorem 3.2. *Let* $g : \mathbf{R}^N \to (-\infty, \infty]$ *be a lsc quasiconvex function. Let* $H : \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ *be a continuous function such that* $H(r,p)$ *is nondecreasing in* r *and positively homogeneous of degree 1 in* p.

Then the function given by quasiconvex Hopf formula

$$u(x,t) = \left(g^{\#} + tH\right)^{\#} (x)$$

is the pointwise minimum of lsc supersolutions of (IVP).

Again, we refer the reader for the proof of the theorem above to [1].

References

1. O. Alvarez, E. N. Barron and H. Ishii, Hopf-Lax formulas for semicontinuous data, preprint.
2. M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton Jacobi Bellman Equations*, Birkhauser Boston, Cambridge, Ma., 1997.

3. M. Bardi and L.C. Evans, On Hopf's formulas for solutions of Hamilton Jacobi equations, Nonlinear Anal., TMA, 8(1984), pp. 1373–1381.
4. G. Barles, *Solutions de Viscosité des Équations de Hamilton-Jacobi*, Mathematiques & Applications 17, Springer Verlag, New York, 1994.
5. E. N. Barron and R. Jensen, Semicontinuous viscosity solutions for Hamilton Jacobi equations with convex Hamiltonians, Comm. PDE, 15(1990), pp. 1713–1742.
6. E. N. Barron and R. Jensen, Optimal control and semicontinuous viscosity solutions, Proc. AMS, 113(1991), pp. 397–402.
7. E. N. Barron, R. Jensen, and W. Liu, Hopf-Lax formula for $u_t + H(u, Du) = 0$, J. Diff. Eqs., 126(1996), pp. 48–61.
8. E. N. Barron, R. Jensen, and W. Liu, Hopf-Lax formula for $u_t + H(u, Du) = 0$, II, Comm. PDE, 22(1997), pp. 1141–1160.
9. E. N. Barron, R. Jensen, and W. Liu, Explicit solution of some first order pde's, J. Dynamical and Control Systems, 3(1997), pp.1–16.
10. E. N. Barron, R. Jensen, and W. Liu, Applications of the Hopf-Lax formula for $u_t + H(u, Du) = 0$, SIAM J. Math. Anal. 29 (1998), no. 4, pp. 1022–1039.
11. E. Hopf, Generalized solutions of nonlinear equations of first order, J. Math. Mech., 14(1965), pp. 951–973.
12. P.-L. Lions and J.-C. Rochet, Hopf formula and multitime Hamilton Jacobi equations, Proc. AMS, 96(1986), pp.79–84.