数理解析研究所講究録
1111 巻 1999 年 157-164

A Note on Invariant Three-Point Curvature Approximations

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Abstract

In this paper we consider Euclidean invariant three-point finite difference approximations of the curvature of a plane curve and study asymptotic properties of such approximations. The paper reveals interesting relations between certain three-point curvature approximations and mechanical and wooden splines.

Discrete approximations of the curvature

Consider a smooth function $f(x)$. A well-known three-point finite difference approximation of the second derivative of the function and asymptotic expansion of the approximation have the following form

$$\begin{aligned} f''(x) &\approx \frac{2f(x-a)}{a(a+b)} - \frac{2f(x)}{ab} + \frac{2f(x+b)}{b(a+b)} = \\ &= f''(x) + \frac{b-a}{3} f'''(x) + \frac{a^2-ab+b^2}{12} f''''(x) + \dots \\ &\qquad \dots + \frac{2\left(b^{n-1} - (-a)^{n-1}\right)}{(a+b)n!} f^{(n)}(x) + O\left(a,b\right)^{n+1} \end{aligned} \tag{1}$$

The goal of this paper is to analyze asymptotic properties of various three-point finite difference approximations of the curvature. Since the curvature is a second-order differential function, it turns out that asymptotic expansions of the three-point curvature approximations unaffected by the rigid motions are similar to (1).

Let us consider a smooth curve $\mathbf{r}(s)$ parameterized by arc length s. Let A, O, B be three successive points on the curve. The distances $a = |OA|$, $b = |OB|$ are assumed to be small and, therefore, so is $c = |AB|$. Below we introduce discrete analogs of the curvature at the central point O based on three points approximations.

We study the approximations expanding them into Taylor series in a and b.

Let

$$\mathbf{t} = \mathbf{r}', \quad \mathbf{n} = \mathbf{t}^{\perp}$$

be the Frenet basis at O. According to the Frenet formulas

$$\mathbf{t}' = k\,\mathbf{n}, \quad \mathbf{n}' = -k\,\mathbf{t}.$$

Approximation via derivatives. The first derivative of $\mathbf{r}(s)$ at O can be approximated as follows

$$\mathbf{r}' = \mathbf{t} \approx \frac{\mathbf{r}(B) - \mathbf{r}(O)}{b} + \frac{\mathbf{r}(O) - \mathbf{r}(A)}{a} - \frac{\mathbf{r}(B) - \mathbf{r}(A)}{a+b}$$

Expanding the right-hand side of of this formula into Taylor series expansion in a and b and using the Frenet formulas we obtain

$$\mathbf{t} \approx \frac{\mathbf{r}(B) - \mathbf{r}(O)}{b} + \frac{\mathbf{r}(O) - \mathbf{r}(A)}{a} - \frac{\mathbf{r}(B) - \mathbf{r}(A)}{a+b} =$$
$$= \left(1 - \frac{ab}{8}k^2 - \frac{ab(b-a)}{12}k\,k' + O(a,b)^3\right)\mathbf{t} + \left(\frac{ab}{6}k' + \frac{ab(b-a)}{24}k'' + O(a,b)^3\right)\mathbf{n}. \quad (2)$$

See Appendix A of this paper for details.

The second derivative of $\mathbf{r}(s)$ at O can be approximated as follows

$$\mathbf{r}'' = \mathbf{t}' = k\,\mathbf{n} \approx \frac{2\mathbf{r}(A)}{a(a+b)} - \frac{2\mathbf{r}(O)}{ab} + \frac{2\mathbf{r}(B)}{(a+b)b} \quad (3)$$

Expanding the right-hand side of (3) into Taylor series with respect to a and b and using the Frenet formulas we arrive at

$$k\,\mathbf{n} \approx \frac{2}{a+b}\left[\frac{\mathbf{r}(B) - \mathbf{r}(O)}{b} + \frac{\mathbf{r}(A) - \mathbf{r}(O)}{a}\right] =$$
$$= \left(\frac{a-b}{4}k^2 - \frac{a^2 - ab + b^2}{6}kk' + O(a,b)^3\right)\mathbf{t} +$$
$$+ \left(k + \frac{b-a}{3}k' + \frac{a^2 - ab + b^2}{12}k'' + O(a,b)^3\right)\mathbf{n}. \quad (4)$$

See Appendix A for details. Since $k = k\,\mathbf{n} \cdot \mathbf{t}^{\perp}$, we can approximate the curvature at O by

$$k \approx \overline{k} = \frac{2}{a+b}\left[\frac{\mathbf{r}(B) - \mathbf{r}(O)}{b} + \frac{\mathbf{r}(A) - \mathbf{r}(O)}{a}\right] \cdot$$
$$\cdot \left[\frac{\mathbf{r}(B) - \mathbf{r}(O)}{b} + \frac{\mathbf{r}(O) - \mathbf{r}(A)}{a} - \frac{\mathbf{r}(B) - \mathbf{r}(A)}{a+b}\right]^{\perp} \quad (5)$$

Now (4), (2), and (5) yield

$$\overline{k} = k + \frac{b-a}{3}k' + \left[\frac{2\,(b^3 + a^3)}{4!(a+b)}k'' - \frac{ab}{8}k^3\right] + O(a,b)^3.$$

It follows from (1) that

$$\bar{k} = k + \frac{b-a}{3}k' + \left[\frac{2\,(b^3+a^3)}{4!(a+b)}k'' - \frac{ab}{8}k^3\right] + \ldots + \left[\frac{2\,(b^{n+1}-(-a)^{n+1})}{(a+b)(n+2)!}k^{(n)} + \ldots\right] + O(a,b)^{n+1} \tag{6}$$

Circle approximation. Let us use the circle passing through the points A, O, B as an approximation to the osculating circle to the curve at O, see Fig. 1. Then the inverse

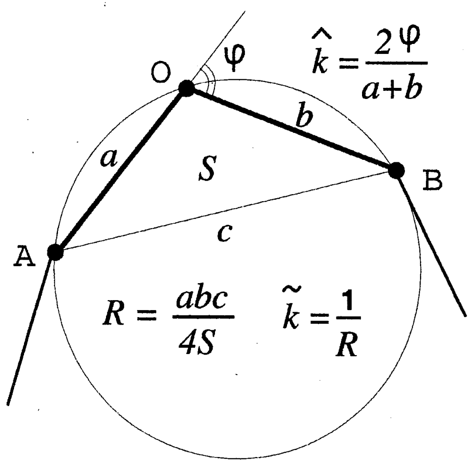

Figure 1:

value of the radius can serve as an approximation of the curvature at O. This curvature approximation was considered in [5, 3, 2]. Let S denote the area of the triangle AOB. The discrete curvature at O is given by

$$\tilde{k} = \frac{4S}{abc}. \tag{7}$$

Using Taylor series manipulations it can be shown that

$$\begin{aligned}\tilde{k} = k + \frac{b-a}{3}k' + \frac{2\,(b^3+a^3)}{4!(a+b)}k'' + \left[\frac{2\,(b^4-a^4)}{5!(a+b)}k''' + \frac{3b^3-2ab^2-2a^2b-3a^3}{120}k^2k'\right] + \\ + \left[\frac{2\,(b^5+a^5)}{6!(a+b)}k'''' + \frac{16a^4+16b^4-28a^2b^2-a^3b-ab^3}{1440}k''k^2 + \right. \\ \left. + \frac{8a^4+8b^4-7a^2b^2+2a^3b+2ab^3}{360}k'^2k\right] + O(a,b)^5.\end{aligned} \tag{8}$$

This expansion without the fourth order terms was obtained in [2][Theorem 2.4] (see also [3] for details). Our derivation of the expansion is similar to that proposed in [2, 3]. It is outlined in Appendix A of this paper.

Angle approximation. Another idea to define the curvature of a polygonal line is based on the definition of the curvature as the rate of change of the angle between the tangent and the positive direction of the x-axis when we proceed along the curve. Let φ denote the turn angle at O (see Fig. 1). Following [1, 6] let us define the discrete curvature at O as

$$\widehat{k} = \frac{2\varphi}{a+b}. \tag{9}$$

Applying Taylor series expansions we get

$$\begin{aligned}\widehat{k} = k + \frac{b-a}{3}k' + \frac{2\left(b^3+a^3\right)}{4!(a+b)}\left(k'' + \frac{k^3}{2}\right) + \frac{2\left(b^4-a^4\right)}{5!(a+b)}\left(k''' + 4k^2k'\right) + \\ + \frac{2\left(b^5+a^5\right)}{6!(a+b)}\left(k'''' + \frac{31}{4}k''k^2 + 13k'^2k + \frac{27}{16}k^5\right) + O(a,b)^5.\end{aligned} \tag{10}$$

The derivation is outlined in Appendix A.

General three-point curvature approximation. Note a similarity between (6), (8), (10). Let us consider a symmetric three-point curvature approximation and denote its Taylor series expansion with respect to a and b by $(*)$. Like (6), (8), (10) expansion $(*)$ must be scale invariant. It is clear that we can estimate the curvature using three curve points but cannot estimate the curvature derivatives. Thus $(*)$ must start from

$$k + \frac{b-a}{3}k' + \dots .$$

Similarly, if we set $a = b$ then $(*)$ starts from

$$k + \frac{a^2}{12}\left(k'' + C\,k^3\right) + \dots ,$$

where C is a constant.

We can conclude that approximating a smooth plane curve by a polygonal line with equal links is preferable for accurate curvature estimation.

The nth term in $(*)$ has the form

$$P_n(a,b)\,k^{(n)} + Q_n(a,b)\,k^{(n-1)}k + \dots + R_n(a,b)\,k^{n+1}$$

where $P_n(a,b)$, $Q_n(a,b)$, $\dots$, $R_n(a,b)$ are homogeneous polynomials of degree n in a and b. They are symmetric with respect to a and b if n is even and antisymmetric if n is odd. Since changing the orientation alters the signs in $(*)$, then $Q_n(a,b) \equiv 0$.

The curvature is a second-order differential function which is unaffected by the rigid motions. Therefore, any numerical approximation of the curvature requires six numbers, for example, six coordinates of three points. Similarly, to estimate the nth derivative of the curvature, $k^{(n)}$, one needs $n+6$ numbers. Thus, $k^{(n)}$ cannot be estimated from $\mathbf{r}(A)$, $\mathbf{r}(O)$, $\mathbf{r}(B)$ and $k'(O)$, $k''(O)$, $k'''(O)$, $\dots$, $k^{(n-1)}(O)$. Therefore, two different three-point curvature approximations have the same coefficients $P_n(a, b)$ in their expansions in a and b. From (6) it follows that

$$P_n(a, b) = \frac{2\,(b^{n+1} - (-a)^{n+1})}{(n+2)!(a+b)} \tag{11}$$

and we have proved the following theorem.

Theorem 1 *Let A, O, B be three successive points on a smooth curve, with $|AO| = a$ and $|OB| = b$. Consider a symmetric three-point curvature approximation unaffected by the rigid motions. Then the following expansion is valid*

$$k + \frac{b-a}{3} k' + \left[\frac{2\,(b^3 + a^3)}{4!(a+b)} k'' + R_2(a,b)\, k^3 \right] + \ldots + \left[\frac{2\,(b^{n+1} - (-a)^{n+1})}{(n+2)!(a+b)} k^{(n)} + \ldots \right] + O(a,b)^{n+1},$$

where $R_2(a, b)$ is a homogeneous symmetric quadratic polynomial in a and b.

Remark. Let $\psi_A(a)$ be the angle between the segment OA and the tangent at O and $\psi_B(b)$ be the angle between the segment OB and the tangent at O. If $\varphi(a, b)$ is the angle between AO and OB, then $\varphi(a, b) = \psi_A(a) + \psi_B(b)$ and $\psi_A(a) = -\psi_B(-a)$. Hence, $\varphi(a, b) = \psi_B(b) - \psi_B(-a)$. From this decomposition it follows that all homogeneous polynomials in a and b occurring in the Taylor series expansion of (9) are given by (11).

Mechanical and wooden splines

There are interesting relations between curvature approximations (7) and (9) and the mechanical and wooden splines which are important for CAGD purposes [4].

A mechanical spline called also an elastica curve minimizes the bending energy

$$\frac{1}{2} \int k^2\, ds,$$

where k is the curvature and s is arclength. Along the mechanical splines

$$k'' + \frac{k^3}{2} = 0.$$

A wooden spline called also a Cornu spiral, or Euler spiral, or clothoid is characterized by the equation

$$k'' = 0,$$

that means the curvature varies linearly with respect to arclength along the spline.

The next theorem follows from (8) and (10).

Theorem 2 *Consider an approximation of a smooth plane curve by a polygonal line with small equal-length links.*

- *Let us define the discrete curvature at a vertex of the polygonal line as the reciprocal of the radius of the circle passing through the vertex and two its neighboring vertices. Consider a polygonal approximation of a wooden spline. The first error term in the approximation of the curve curvature by the discrete curvature is of fourth order with respect to the length of the link.*

- *Let us define the discrete curvature at a vertex of the polygonal line as the ratio of the turn angle at the vertex and length of the link. Consider a polygonal approximation of a mechanical spline. The first error term in the approximation of the curve curvature by the discrete curvature is of fourth order with respect to the length of the link.*

This theorem indicates that the relations between the four-parametric families of mechanical and wooden splines and approximations (8) and (10) respectively are similar to the relation between the four-parametric family of cubic polynomials and (1).

Acknowledgments

It is my pleasure to acknowledge fruitful discussions with I. A. Bogaevski. Thanks also to Maple for its help in performing formal computations presented in the paper.

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Appendix A

Let $\mathbf{r}(s)$ be a plane curve parameterized by arclength parameter s. Consider three points on the curve

$$A = \mathbf{r}(s-\alpha), \quad O = \mathbf{r}(s), \quad B = \mathbf{r}(s+\beta)$$

with distances $a = |OA|$ and $b = |OB|$ between them. Let $\mathbf{r}' = \mathbf{t}$ and $\mathbf{n} = \mathbf{t}^{\perp}$ compose the Frenet frame at O. Denote by φ the angle between $\overrightarrow{AO}$ and $\overrightarrow{OB}$. See Fig. 2.

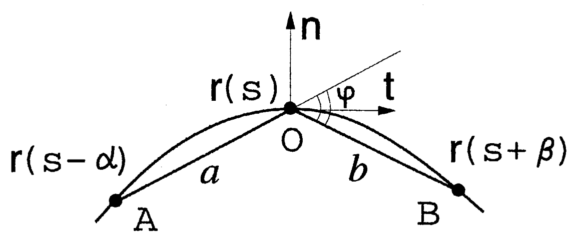

Figure 2:

According to the Frenet formulas

$$\mathbf{t}' = k\,\mathbf{n}, \quad \mathbf{n}' = -k\,\mathbf{t}$$

we have

$$\begin{aligned} &\mathbf{r}' = \mathbf{t}, \\ &\mathbf{r}'' = \mathbf{t}' = k\,\mathbf{n}, \\ &\mathbf{r}''' = (k\mathbf{n})' = k'\mathbf{n} - k^2\mathbf{t}, \\ &\mathbf{r}'''' = (k'\mathbf{n} - k^2\mathbf{t})' = (k'' - k^3)\mathbf{n} - 3k\,k'\mathbf{t}, \\ &\dots\ . \end{aligned}$$

Expanding $\mathbf{r}(s+\beta)$ into the Taylor series with respect to β yields

$$\begin{aligned} \overrightarrow{OB} &= \mathbf{r}(s+\beta) - \mathbf{r}(s) = \beta\mathbf{r}' + \frac{\beta^2}{2}\mathbf{r}'' + \frac{\beta^3}{6}\mathbf{r}''' + \frac{\beta^4}{24}\mathbf{r}'''' + O\left(\beta^5\right) = \\ &= \mathbf{t}\left[\beta - \frac{\beta^3}{6}k^2 - \frac{\beta^4}{8}k\,k' + O\left(\beta^5\right)\right] + \mathbf{n}\left[\frac{\beta^2}{2}k + \frac{\beta^3}{6}k' + \frac{\beta^4}{24}(k'' - k^3) + O\left(\beta^5\right)\right]. \end{aligned}$$

Thus

$$b^2 = \beta^2 - \frac{\beta^4}{12}k^2 - \frac{\beta^5}{12}k\,k' + O\left(\beta^6\right), \qquad b = \beta - \frac{\beta^3}{24}k^2 - \frac{\beta^4}{24}k\,k' + O\left(\beta^5\right).$$

Inverting the Taylor series for b we obtain

$$\beta = b + \frac{b^3}{24}k^2 + \frac{b^4}{24}k\,k' + O\left(b^5\right).$$

It gives

$$\frac{\overrightarrow{OB}}{b} = \mathbf{t}\left[1 - \frac{b^2}{8}k^2 - \frac{b^3}{12}k\,k' + O\left(b^4\right)\right] + \mathbf{n}\left[\frac{b}{2}k + \frac{b^2}{6}k' + \frac{b^3}{24}k'' + O\left(b^4\right)\right].$$

Performing similar calculations we arrive at

$$\frac{\overrightarrow{OA}}{a} = \mathbf{t}\left[-1+\frac{a^2}{8}k^2-\frac{a^3}{12}k\,k'+O\left(a^4\right)\right]++\mathbf{n}\left[\frac{a}{2}k-\frac{a^2}{6}k'+\frac{a^3}{24}k''+O\left(a^4\right)\right].$$

Since

$$\mathbf{t}\times\mathbf{n}=1, \qquad \mathbf{n}\times\mathbf{t}=-1,$$

where $\times$ stands for the cross product, then

$$\sin\varphi = -\frac{\overrightarrow{OA}}{a}\times\frac{\overrightarrow{OB}}{b}=\frac{\overrightarrow{OB}}{b}\times\frac{\overrightarrow{OA}}{a}=$$
$$=\frac{a+b}{2}k+\frac{b^2-a^2}{6}k'+\frac{a^3+b^3}{24}k''-\frac{a^2b+ab^2}{16}k^3+O(a,b)^4$$

and finally

$$\varphi=\frac{a+b}{2}k+\frac{b^2-a^2}{6}k'+\frac{a^3+b^3}{24}\left(k''+\frac{k^3}{2}\right)+O(a,b)^4.$$

We can define the discrete curvature at O as

$$\widehat{k}=\frac{2\varphi}{a+b}.$$

The Taylor series expansion of $\widehat{k}$ with respect to a and b has the form

$$\widehat{k}=k+\frac{b-a}{3}k'+\frac{(b-a)^2+ab}{12}\left(k''+\frac{k^3}{2}\right)+O(a,b)^3.$$

Next terms can be obtained with help of computer algebra systems like MAPLE and MATHEMATICA.

Similar calculations can be done for $\widetilde{k}$, the discrete curvature defined as the inverse value of the radius of the circle passing through the points A, O, B.