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On Several Properties of Spiral-like Functions

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1 Introduction

We consider an analytic function f on the unit disk $\mathbb{D}$ normalized so that $f(0)=f'(0)-1=0$. For a constant $\beta\in(-\pi/2,\pi/2)$, such a function f is called *β-spiral-like* if f is univalent on $\mathbb{D}$ and for any $z\in\mathbb{D}$, the β-logarithmic spiral $\{f(z)\exp(-e^{i\beta}t);t\geq 0\}$ is contained in $f(\mathbb{D})$. It is equivalent to the analytic condition that $\Re(e^{-i\beta}zf'(z)/f(z))>0$ in $\mathbb{D}$. We denote by $SP(\beta)$ the set of β-spiral-like functions. We call $f_{\beta}(z):=z(1-z)^{-2e^{i\beta}\cos\beta}\in SP(\beta)$ the β-spiral Koebe function. Note that $SP(0)$ is the set of starlike functions and that $f_{0}(z)=z(1-z)^{-2}$ is the Koebe function. The β-spiral Koebe function conformally maps the unit disk onto the complement of the β-logarithmic spiral $\{f_{\beta}(-e^{-2i\beta})\exp(-e^{i\beta}t);t\leq 0\}$ in $\mathbb{C}$. For the known results about these classes of the functions, see, for example, [1].

2 Norm estimates

For a locally univalent holomorphic function f, we define

$$T_{f}=\frac{f''}{f'}\quad\text{and}\quad S_{f}=(T_{f})'-\frac{1}{2}(T_{f})^{2},$$

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which are said to be the *pre-Schwarzian derivative* (or nonlinearity) and the *Schwarzian derivative* of f, respectively. For a locally univalent function f in $\mathbb{D}$, we define the norms of T_f and S_f by

$$\|T_f\|_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)|T_f(z)| \quad \text{and} \quad \|S_f\|_2 = \sup_{z\in\mathbb{D}}(1-|z|^2)^2|S_f(z)|,$$

respectively.

As well as $\|S_f\|_2$, the norm $\|T_f\|_1$ has a significant meaning in the theory of Teichmüller spaces. For example, see [8], [2] and [13].

We shall give the best possible estimate of the norms of pre-Schwarzian derivatives for the class $SP(\beta)$.

Main Theorem 1 ([9]). *For any $f \in SP(\beta)$, where $\beta \in (-\pi/2, \pi/2)$, we have the following.*

I) *In the case $|\beta| \le \pi/3$, we have*

$$\|T_f\|_1 \le \|T_{f_\beta}\|_1 = 2|2+e^{2i\beta}|. \tag{1}$$

II) *In the case $|\beta| > \pi/3$, we have $\|T_f\|_1 \le \|T_{f_\beta}\|_1$, where*

$$\|T_{f_\beta}\|_1 = \max_{0\le m\le \frac{4}{3}\sin|\beta|} 2m\cos\beta\left(1+\sqrt{\frac{m^2+4-4m\sin|\beta|}{m^2+1-2m\sin|\beta|}}\right) \quad \textit{and} \tag{2}$$

$$2|2+e^{2i\beta}| < \|T_{f_\beta}\|_1 < 2\left(1+\frac{4}{3}\sin 2|\beta|\right). \tag{3}$$

In particular, $\|T_{f_\beta}\|_1 \to 2$ as $|\beta| \to \pi/2$.

In both cases, the equality $\|T_f\|_1 = \|T_{f_\beta}\|_1$ holds if and only if f is a rotation of the β-spiral Koebe function, i.e., $f(z) = (1/\varepsilon)f_\beta(\varepsilon z)$ for some $|\varepsilon| = 1$.

The proof of Main Theorem 1 is in [9]. From the proof, if $|\beta| \le \pi/3$, the function $(1-|z|^2)|T_{f_\beta}(z)|$ does not attain its supremum in $\mathbb{D}$. However if $|\beta| > \pi/3$, it does since

$$\max_{\partial\mathbb{D}\ni z_0} \limsup_{\mathbb{D}\ni z\to z_0}(1-|z|^2)|T_{f_\beta}(z)| = 2|2+e^{2i\beta}| < \|T_{f_\beta}\|_1.$$

This phenomenon of *phase transition* seems to be quite interesting.

Remark. Clearly, the β-spiral Koebe function f_β converges to $id_{\mathbb{D}}$ (which is bounded) locally uniformly on $\mathbb{D}$ as $|\beta| \to \pi/2$ but does not converge to it with respect to the norm $\|\cdot\|_1$ since $\lim_{|\beta|\to\pi/2}\|T_{f_\beta}\|_1 = 2$. On the other hand, it is known that a normalized analytic function f is bounded if $\|T_f\|_1 < 2$. In fact, the value 2 is the least one of the norms of unbounded normalized analytic functions.

We would also like to mention the related works about norm estimates of pre-Schwarzian derivatives in other classes by Shinji Yamashita [11] and Toshiyuki Sugawa [10].

Theorem 2.1. *Let* $0 \leq \alpha < 1$ *and* f *be a normalized analytic function.*

If f *is starlike of order* α*, i.e.,* $\Re(zf'(z)/f(z)) > \alpha$*, then* $\|T_f\|_1 \leq 6-4\alpha$*.*

If f *is convex of order* α*, i.e.,* $\Re(1+zf''(z)/f'(z)) > \alpha$*, then* $\|T_f\|_1 \leq 4(1-\alpha)$*.*

If f *is strongly starlike of order* α*, i.e.,* $\arg(zf'(z)/f(z)) < \pi\alpha/2$*, then* $\|T_f\|_1 \leq M(\alpha)+2\alpha$*, where* $M(\alpha)$ *is a specified constant depending only on* α *satisfying* $2\alpha < M(\alpha) < 2\alpha(1+\alpha)$*.*

All of the bounds are sharp.

On the other hand, we also obtain the estimate of the norms of Schwarzian derivatives of β-spiral-like functions.

Main Theorem 2 ([9]). *Assume* $|\beta| < \pi/2$*. For any* $f \in SP(\beta)$*,* $\|S_f\|_2 \leq \|S_{f_\beta}\|_2 = 6$*.*

Proof. From direct calculation, it follows that

$$\begin{aligned} S_{f_\beta} =& (T_{f_\beta})' - \frac{1}{2}(T_{f_\beta})^2 \\ =& -c\frac{e^{2i\beta}\{e^{2i\beta}(e^{2i\beta}-1)z^2+4(e^{2i\beta}-1)z+6\}}{2(1-z)^2(1+ze^{2i\beta})^2} \end{aligned}$$

and that

$$(1-|z|^2)^2|S_{f_\beta}(z)| = |c|\frac{(1-|z|^2)^2|e^{2i\beta}(e^{2i\beta}-1)z^2+4(e^{2i\beta}-1)z+6|}{2|1-z|^2|1+ze^{2i\beta}|^2}.$$

We can easily see that $(1-|z|^2)^2|S_{f_\beta}(z)| \to 6$ as $z \to -e^{-2i\beta}$ radially. By the Kraus-Nehari theorem, we obtain $\|S_{f_\beta}\|_2 = 6$ and the extremality of f_β in $SP(\beta)$ for any $|\beta| < \pi/2$. □

3 Order estimates of the coefficients

Knowing the norm $\|T_f\|_1$ enables us to estimate the growth of coefficients of f. For example, the following holds.

Theorem 3.1 (cf. [7]). *Let* $(3/2) < \lambda \leq 3$*. For a normalized analytic function* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ *such that* $\|T_f\|_1 \leq 2\lambda$*, it holds that* $a_n = O(n^{\lambda-2})$ *as* $n \to +\infty$*. This order estimate is best possible.*

However the sharp estimate of coefficients of $f \in SP(\beta)$ has been already obtained by Zamorski [12] in 1960. We would like to remark that we can derive the sharp growth estimate of coefficients of $f \in SP(\beta)$ from this.

Theorem 3.2 (Zamorski). *If $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ is in $SP(\beta)$ and $|\beta| < \pi/2$, then*

$$|a_n| \leq \prod_{k=1}^{n-1} \left|1 + \frac{e^{2i\beta}}{k}\right| \tag{4}$$

for any $n \geq 2$. The equality in (4) holds for some $n \geq 2$ if and only if f is a rotation of the β-spiral Koebe function f_β.

Remark. This is also shown in terms of generalized spiral-like functions by C. Burniak, J. Stankiewicz and Z. Stankiewicz [4](1980).

Corollary 3.1. *Let $|\beta| < \pi/2$ and $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be a β-spiral-like function. Then it holds that*

$$a_n = O(n^{\cos 2\beta}) \quad (n \to +\infty). \tag{5}$$

This order estimate is sharp.

Proof. From the inequality (4), we have that for $|\beta| < \pi/2$,

$$\begin{aligned}
\log|a_n| \leq & \frac{1}{2}\sum_{k=1}^{n-1} \log\left(1 + \frac{2\cos 2\beta}{k} + \frac{1}{k^2}\right) \\
= & \frac{1}{2}\sum_{k=1}^{n-1} \left(\frac{2\cos 2\beta}{k}\right) + O(1) \\
= & \cos 2\beta \log n + O(1)
\end{aligned}$$

as $n \to +\infty$. Therefore we obtain the estimate (5). □

Remark. In the case $|\beta| < \pi/4$, this is shown by Basgöze and Keogh in [3](1970).

4 Strongly normalized univalent functions are not always holomorphic.

The following is known.

Theorem 4.1. *For a holomorphic function ϕ on a simply connected domain A, there exists a locally univalent meromorphic function f on A such that*

$$S_f = \phi.$$

The solution is unique up to postcomposition of an arbitrary Möbius transformation.

We assume $A = \mathbb{D}$. Nehari showed that if $\|\phi\|_2 = \sup_{z\in\mathbb{D}} |\phi(z)|(1-|z|^2)^2 \leq 2$, then f is univalent (meromorphic) on $\mathbb{D}$. It is well-known that if $\|\phi\|_2 = \sup_{z\in\mathbb{D}} |\phi(z)|(1-|z|^2)^2 \leq 2$ and f is the *strongly normalized* solution, i.e., $f(0) = f'(0) - 1 = f''(0) = 0$, then f is holomorphic on $\mathbb{D}$. Since for a normalized analytic function $f(z) = z + a_2 z^2 + \cdots$, $g := f/(a_2 f + 1)$ is strongly normalized and $\|S_f\|_2 = \|S_g\|_2$, we have the following.

Proposition 4.1 ([6] and [5] Corollary 2.). *If a normalized analytic function $f(z) = z + a_2 z^2 + \cdots$ satisfies $\|S_f\|_2 \leq 2$, then f is univalent and $a_2 f + 1 \neq 0$ on $\mathbb{D}$.*

In [5] Chuaqui and Osgood remark that a strongly normalized univalent function f is not always holomorphic if $\|S_f\|_2 > 2$. Spiral-like functions are examples for this fact.

Theorem 4.2. *If $|\beta|$ is sufficiently close to $\pi/2$, the β-spiral-Koebe function $f_\beta(z) = z + a_2 z^2 + \cdots$ satisfies $a_2 f_\beta(z) + 1 = 0$ for some $z \in \mathbb{D}$.*

Proof. By direct calculation, we have $a_2 = f''_\beta(0)/2 = e^{2i\beta} + 1$. The β-logarithmic spiral $\{f_\beta(-e^{-2i\beta}) \exp(e^{i\beta} t); t \geq 0\}$ is the complement of $f_\beta(\mathbb{D})$ in $\mathbb{C}$. Thus $a_2 f_\beta(z) + 1 \neq 0$ on $\mathbb{D}$ if and only if this spiral contains $-1/a_2$. We can see that if $f_\beta(-e^{-2i\beta}) \exp(e^{i\beta} t) = -1/a_2$, then

$$t = e^{i\beta} \log(1 + e^{-2i\beta}). \tag{6}$$

and that the imaginary part of the right side of (6) tends to $-\infty$ (resp. $+\infty$) if β tends to $+\pi/2$ (resp. $-\pi/2$). □

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