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逆凸計画問題に対する内部近似法

An Inner Approximation Method for a Reverse Convex Programming Problem

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Abstract

In this paper, we consider a reverse convex programming problem constrained by a convex set and a reverse convex set which is defined by the complement of the interior of a compact convex set X. We propose an inner approximation method to solve the problem in case X is not necessarily a polytope. The algorithm utilizes inner approximation of X by a sequence of polytopes to generate relaxed problems. It is shown that every accumulation point of the sequence of optimal solutions of relaxed problems is an optimal solution of the original problem.
Keywords: Global Optimization, Reverse Convex Programming Problem, Dual Problem, Inner Approximation Method, Penalty Function Method

1 Introduction

In this paper, we consider a reverse convex programming problem constrained by a convex set and a reverse convex set which is defined by the complement of the interior of a compact convex set X. In case when X is a polytope in the problem, a solution method using duality has been proposed (Horst and Tuy [4], Horst and Pardalos [5], Konno, Thach and Tuy [6], Tuy [8]). Duality is one of the most powerful tools in dealing with a global optimization problem like the problem described above. The dual problem to the problem is a quasi-convex maximization problem over a convex set and solving one of the original and the dual problems is equivalent to solving the other (Konno, Thach and Tuy [6], Tuy [8]). Since the feasible set of the dual problem is a polytope, there exists a vertex which solves the dual problem. Moreover, since the objective function of the dual problem is the quasi-conjugate function of the objective function of the original problem, for every vertex, the objective function value is obtained by solving a constrained convex minimization problem. Consequently, an optimal solution of the original problem is obtained by solving a finite number of constrained convex minimization problems.

We propose an inner approximation method to solve the reverse convex programming problem in case X is not necessarily a polytope. The algorithm utilizes inner approximation of X by a sequence of polytopes. That is, at every iteration of the algorithm, a relaxed problem in which X is replaced by a polytope contained in X is solved. Then, it is shown that every accumulation point of the sequence of optimal solutions of relaxed problems is an optimal solution of the original problem. Every relaxed problem can be solved through a finite number of constrained convex minimization problems. By using penalty functions, such constrained problems can be transformed into unconstrained convex minimization problems. Thus, the minimum of the optimal values of such unconstrained problems underestimates the optimal value of the relaxed problem.

The organization of this paper is as follows: In Section 2, we explain a reverse convex programming problem. Moreover, we describe an equivalent problem to the problem, and its dual problem, where equivalence is understood in the sense that the sets of optimal solutions coincide. In Section 3, we formulate an inner approximation algorithm for the problem, and establish the convergence of the algorithm. In Section 4, we propose another inner approximation algorithm for the problem which is incorporating a penalty function method.

Throughout this paper, we use the following notation: int X, bd X and co X denote the interior set of $X \subset R^n$, the boundary set of X and the convex hull of X, respectively. Let $\bar{R} = R \cup \{-\infty\} \cup \{+\infty\}$. Let for $a, b \in R^n$, $]a, b[= \{x \in R^n : x = a+\delta(b-a),\ 0 < \delta < 1,\ \delta \in R\}$ and $]a, b] = \{x \in R^n : x = a + \delta(b - a),\ 0 < \delta \leq 1,\ \delta \in R\}$. Given a convex polyhedral set (or polytope) $X \subset R^n$, $V(X)$ denotes the set of all vertices of X. For a subset $X \subset R^n$, $X^\circ = \{u \in R^n : \langle u, x\rangle \leq 1,\ \forall x \in X\}$ is called the polar set of X. For a nonempty closed set $X \subset R^n$, $N_X(y)$ denote the normal cone to X at $y \in X$. For a subset $X \subset R^n$, the indicator of X which is denoted by $\delta(\ \cdot\ |X)$ is an extended-real-valued function defined as follows:

$$\delta(x|X) = \begin{cases} 0 & \text{if } x \in X \\ +\infty & \text{if } x \notin X. \end{cases}$$

Given a function $f : R^n \to R \cup \{+\infty\}$, the quasi-conjugate of f is the function f^H defined as follows:

$$f^H(u) = \begin{cases} -\sup\{f(x) : x \in R^n\} & \text{if } u = 0 \\ -\inf\{f(x) : \langle u, x\rangle \geq 1\} & \text{if } u \neq 0. \end{cases}$$

The gradient of f at x is denoted by $\nabla f(x)$ and the subdifferential of f at x by $\partial f(x)$.

2 A Reverse Convex Programming Problem

Let us consider the following reverse convex programming problem problem:

$$(RCP) \begin{cases} \text{minimize} & f(x), \\ \text{subject to} & x \in Y \backslash \text{int } X, \end{cases}$$

where $f : R^n \to R$ is a convex function, X is a compact convex set and Y is a closed convex set in R^n. In general, the feasible set of problem (RCP) is not convex. For problem (RCP), we shall assume the following throughout this paper:

(A1) $Y \backslash \text{int } X \neq \emptyset$.

(A2) For some $\alpha \in R$, $\{x \in R^n : f(x) \leq \alpha\}$ is nonempty and compact.

(A3) $X = \{x \in R^n : p_j(x) \leq 0,\ j = 1, \ldots, t_X\}$ and $Y = \{x \in R^n : r_j(x) \leq 0,\ j = 1, \ldots, t_Y\}$ where $p_j : R^n \to R$ $(j = 1, \ldots, t_X)$ and $r_j : R^n \to R$ $(j = 1, \ldots, t_Y)$ are convex functions. Moreover, there exists $x_X, x_Y \in R^n$ such that $p_j(x_X) < 0$ $(j = 1, \ldots, t_X)$ and $r_j(x_Y) < 0$ $(j = 1, \ldots, t_Y)$.

Let $p(x) = \max_{j=1,\dots,t_X} p_j(x)$ and $r(x) = \max_{j=1,\dots,t_Y} r_j(x)$. Then, from assumption (A3), $X = \{x \in R^n : p(x) \leq 0\}$, $Y = \{x \in R^n : r(x) \leq 0\}$, int $X = \{x \in R^n : p(x) < 0\}$ and int $Y = \{x \in R^n : r(x) < 0\}$. From assumption (A2), the minimal value of f over R^n exists. Moreover, for any $\beta \geq \min\{f(x) : x \in R^n\}$, $\{x \in R^n : f(x) \leq \beta\}$ is nonempty and compact. From assumption (A1), there exists a feasible solution x' of problem (RCP). Then, problem (RCP) is equivalent to minimize $f(x)$ subject to $x \in (Y \setminus \text{int } X) \cap \{x \in R^n : f(x) \leq f(x')\}$. Since $\{x \in R^n : f(x) \leq f(x')\}$ is compact, problem (RCP) has an optimal solution. Denote by $\min(RCP)$ the optimal value of problem (RCP). Then, we have $\min(RCP) < +\infty$. From assumptions (A1) and (A2), Y is nonempty and there exists a minimal solution x^0 of f over Y. Then, it is fairly easy to find x^0. In case $x^0 \in R^n \setminus \text{int } X$, x^0 solves problem (RCP). In the other case, we propose a solution method in this paper. Throughout this paper, without loss of generality, we may assume the following:

(A4) $p(0) < 0$ and $r(0) \leq 0$, that is, $0 \in \text{int } X$ and $0 \in Y$. Moreover, $0 \in R^n$ is a minimal solution of f over Y.

By using the indicator of Y, problem (RCP) can be reformulated as

$$(MP) \begin{cases} \text{minimize} & g(x) \\ \text{subject to} & x \in R^n \setminus \text{int } X \end{cases}$$

where $g(x) := f(x) + \delta(x|Y)$. The objective function $g : R^n \to \bar{R}$ is a quasi-convex function. From assumption (A4), we have $g(0) = \inf\{g(x) : x \in R^n\}$. The dual problem of problem (MP) is formulated as

$$(DP) \begin{cases} \text{maximize} & g^H(u) \\ \text{subject to} & u \in X^\circ. \end{cases}$$

Hence, by assumption (A4) and the principle of the duality, X° is a compact convex set. Furthermore, since g^H is a quasi-convex function (Konno, Thach and Tuy [6], Chapter 2), we note that problem (DP) is a quasi-convex maximization problem over a compact convex set in R^n. Denote by $\min(MP)$ and $\max(DP)$ the optimal values of (MP) and (DP), respectively. Since problem (MP) is equivalent to problem (RCP), we have $\min(MP) = \min(RCP) < +\infty$. Moreover, it follows from the duality relation between problems (MP) and (DP) that $\min(MP) = -\max(DP)$ (cf., Konno, Thach and Tuy [6], Chapter 4).

3 An Inner Approximation Method for Problem (MP)

3.1 Relaxed Problems for Problems (MP) and (DP)

One of the reasons for difficulty in solving problem (MP) is that X is not a polytope. If X is a polytope, then the feasible set of problem (MP) can be formulated as the union of finite halfspaces. In this case, problem (MP) is fairly easy to solve by minimizing g over every halfspace.

In this subsection, we discuss the following problem:

$$(P) \begin{cases} \text{minimize} & g(x), \\ \text{subject to} & x \in R^n \setminus \text{int } S, \end{cases}$$

where S is a polytope such that $S \subset X$ and $0 \in \text{int } S$. Then, we get $R^n \backslash \text{int } S \supset R^n \backslash \text{int } X$. Therefore, problem (P) is a relaxed problem for problem (MP). From the definition of g, we note that problem (P) is equivalent to minimize $f(x)$ subject to $x \in Y \backslash \text{int } S$. Since $(Y \backslash \text{int } S) \supset (Y \backslash \text{int } X) \neq \emptyset$, by assumption (A2), a minimal solution of f on $Y \backslash \text{int } S$ exists and solves problem (P). Denote by $\min(P)$ the optimal value of problem (P). Then, we have $\min(P) \leq \min(MP) < +\infty$.

The dual problem of problem (P) is formulated as

$$(D) \begin{cases} \text{maximize} & g^H(u), \\ \text{subject to} & u \in S^\circ. \end{cases}$$

Since $S \subset X$, the feasible set of problem (D) includes X°. Therefore, problem (D) is a relaxed problem of (DP). We note that the feasible set S° is a polytope because S is a polytope and $0 \in \text{int } S$. Hence, problem (D) is a quasi-convex maximization over a polytope S°. There exists an optimal solution of problem (D) over the set of all vertices of S°. Denote by $\max(D)$ the optimal value of problem (D). Since problem (D) is the dual problem of problem (P) and a relaxed problem of problem (DP), we obtain $\max(D) = -\min(P) \geq -\min(MP) = \max(DP) > -\infty$ (Konno, Thach and Tuy [6], Chapter 4). Consequently, we can choose an optimal solution of problem (D) from $V(S^\circ)$. Since $0 \in \text{int } S$, from the principle of duality, we have

$$S^\circ = \{u \in R^n : \langle u, z \rangle \leq 1, \ \forall z \in V(S)\} \text{ and } S = \{x \in R^n : \langle v, x \rangle \leq 1, \ \forall v \in V(S^\circ)\}. \quad (1)$$

Hence, we obtain $0 \notin V(S^\circ)$.

For any $v \in V(S^\circ)$, we have $g^H(v) = -\inf\{g(x) : \langle v, x \rangle \geq 1\}$. From the definition of g, for any $v \in V(S^\circ)$,

$$g^H(v) = \begin{cases} -\infty, & \text{if } Y \cap \{x \in R^n : \langle v, x \rangle \geq 1\} = \emptyset, \\ -\inf\{f(x) : \langle v, x \rangle \geq 1, \ x \in Y\} & \text{otherwise.} \end{cases}$$

This implies that $v \in V(S^\circ)$ is not optimal to problem (D) if $Y \cap \{x \in R^n : \langle v, x \rangle \geq 1\} = \emptyset$.

Lemma 3.1 *There exists $v \in V(S^\circ)$ such that $Y \cap \{x \in R^n : \langle v, x \rangle \geq 1\} \neq \emptyset$.*

Denote by Γ the set of all $v \in V(S^\circ)$ such that $Y \cap \{x \in R^n : \langle v, x \rangle \geq 1\} \neq \emptyset$. From Lemma 3.1, $\Gamma \neq \emptyset$. For every $v \in \Gamma$, we consider the following convex minimization problem:

$$(SP(v)) \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \in Y \cap \{x \in R^n : \langle v, x \rangle \geq 1\}. \end{cases}$$

From assumption (A2), for every $v \in \Gamma$, problem $(SP(v))$ has an optimal solution x^v. Then, we have $g^H(v) = -\min(SP(v)) = -f(x^v)$, where $\min(SP(v))$ is the optimal value of problem $(SP(v))$. Hence, $\hat{v} \in \Gamma$ is an optimal solution of problem (D) if $f(x^{\hat{v}}) = \min\{f(x^v) : v \in V(S^\circ)\}$. Moreover, $x^{\hat{v}}$ is optimal to problem (P) (Konno, Thach and Tuy [6], Proposition 4.3). However, it is hard to examine whether $Y \cap \{x \in R^n : \langle v, x \rangle \geq 1\}$ is empty. This examination is not necessary to execute the inner approximation algorithm proposed in Section 4.

3.2 An Inner Approximation Algorithm

From the discussion in Subsection 3.1, we notice that inner approximation of X by a sequence of polytopes is applicable in solving problem (MP).

We propose an inner approximation algorithm for problem (MP) as follows:

Algorithm IA

Initialization. Generate a finite set V_1 such that $V_1 \subset X$ and that $0 \in \text{int}\,(\text{co}\,V_1)$. Let $S_1 = \text{co}\,V_1$. Compute the vertex set $V((S_1)^\circ)$. For convenience, let $V((S_0)^\circ) = \emptyset$. Set $k \leftarrow 1$ and go to Step 1.

Step 1. Let Γ_k be the set of all $v \in V((S_k)^\circ)$ satisfying $Y \cap \{x \in R^n : \langle v, x\rangle \geq 1\} \neq \emptyset$. For every $v \in \Gamma_k \backslash V((S_{k-1})^\circ)$, let x^v be an optimal solution of problem $(SP(v))$. Choose $v^k \in \Gamma_k$ satisfying $f(x^{v^k}) = \min\{f(x^v) : v \in \Gamma_k\}$. Let $x(k) = x^{v^k}$.

Step 2.

a. If $p(x(k)) \geq 0$, then stop; $x(k)$ solves problem (MP) and the optimal value of problem $(SP(v^k))$ is the optimal value of problem (MP).

b. Otherwise, solve the following convex minimization problem:

$$\begin{cases} \text{minimize} & \phi(x; v^k) = \max\{p(x), h(x, v^k)\} \\ \text{subject to} & x \in R^n \end{cases} \tag{2}$$

where $h(x, v^k) = -\langle v^k, x\rangle + 1$. Let z^k denote an optimal solution of problem (2). It will be proved later in Lemmas 3.2 and 3.3 that problem (2) has an optimal solution and that $z^k \in X$, respectively. Set $V_{k+1} = V_k \cup \{z^k\}$. Let $S_{k+1} = \text{co}\,V_{k+1}$. Compute the vertex set $V((S_{k+1})^\circ)$. Set $k \leftarrow k+1$ and return to Step 1.

Note that S_k, $k = 1, 2, \ldots$, are polytopes. Since $0 \in \text{int}\,(\text{co}\,V_1) = \text{int}\,S_1$, S_k, $k = 1, 2, \ldots$, satisfy that $0 \in \text{int}\,S_k$. It follows from the following theorems that at every iteration of the algorithm, problem (2) has an optimal solution and S_k is contained in X.

Lemma 3.2 *For any $v \in R^n$, the function $\phi(x; v)$ attains its minimum over R^n.*

Lemma 3.3 *At iteration k of Algorithm IA, assume that $S_k \subset X$. Then*

(i) $v \notin \text{int}\,X^\circ$ *for any* $v \in V((S_k)^\circ)$.

(ii) $\phi(z^k; v^k) \leq 0$,

(iii) $z^k \in X$.

From Lemma 3.3 and the definition of S_1, we have

- $S_1 \subset S_2 \subset \ldots \subset S_k \subset \ldots \subset X$,
- $(S_1)^\circ \supset (S_2)^\circ \supset \ldots \supset (S_k)^\circ \supset \ldots \supset X^\circ$.

Hence, for every iteration k of the algorithm, the following problems (P_k) and (D_k) are relaxed problems of (MP) and (DP), respectively.

$$(P_k)\begin{cases} \text{minimize} & g(x) \\ \text{subject to} & x \in R^n \setminus \text{ int } S_k, \end{cases}$$

$$(D_k)\begin{cases} \text{maximize} & g^H(u) \\ \text{subject to} & u \in (S_k)^\circ. \end{cases}$$

From the discussion in Subsection 3.1, $x(k)$ and v^k obtained in Step 1 of the algorithm solve problems (P_k) and (D_k), respectively. Moreover, we note that $\max(D_{k-1}) \geq \max(D_k)$ for any $k \geq 2$, that is,

$$g^H(v^1) \geq g^H(v^2) \geq \cdots \geq g^H(v^k) \geq \cdots \geq \max(DP), \tag{3}$$

and that $\min(P_{k-1}) \leq \min(P_k)$ for any $k \geq 2$, that is,

$$g(x(1)) \leq g(x(2)) \leq \cdots \leq g(x(k)) \leq \cdots \leq \min(MP). \tag{4}$$

Since $g(x) = +\infty$ for any $x \notin Y$, $x(k)$ belongs to Y. It follows from the following theorem that $x(k)$ solves problem (MP) if $p(x(k)) \geq 0$.

Lemma 3.4 *At iteration k of the algorithm, $x(k)$ solves problem (MP) if $p(x(k)) \geq 0$.*

For any k, the following assertions are valid.

- $V(S_k) \subset V_k$.
- $(S_k)^\circ = \{u \in R^n : \langle u, z\rangle \leq 1 \ \forall z \in V_k\}$.
- $(S_{k+1})^\circ = (S_k)^\circ \cap \{u \in R^n : \langle u, z^k\rangle \leq 1\}$.

Moreover, the following lemma holds.

Lemma 3.5 *At iteration k of Algorithm IA, if $p(x(k)) < 0$, then $\langle v^k, z^k\rangle > 1$.*

From Lemma 3.5, $S_{k+1} = \text{co }(S_k \cup \{z^k\}) \neq S_k$ because $S_k \subset \{x \in R^n : \langle v^k, x\rangle \leq 1\}$ and $\langle v^k, z^k\rangle > 1$. Moreover, since $V(S_{k+1}) \subset V(S_k) \cup \{z^k\}$, we have

$$(S_{k+1})^\circ = (S_k)^\circ \cap \{u \in R^n : \langle u, z^k\rangle \leq 1\} \neq (S_k)^\circ \tag{5}$$

3.3 Convergence of Algorithm IA

Algorithm IA doesn't necessarily terminate after finitely many iterations. In this subsection, we consider the case that an infinite sequence $\{v^k\}$ is generated by the algorithm.

It follows from the following theorem that every accumulation point of $\{v^k\}$ belongs to the feasible set of problem (DP).

Theorem 3.1 *Assume that $\{v^k\}$ is an infinite sequence such that for all k, v^k is an optimal solution of (D_k) at iteration k of Algorithm IA and that $\bar{v}$ is an accumulation point of $\{v^k\}$. Then $\bar{v}$ belongs to X°.*

Corollary 3.1 *Assume that $\{v^k\}$ is an infinite sequence such that for all k, v^k is an optimal solution of problem (D_k) at iteration k of Algorithm IA and that $\bar{v}$ is an accumulation point of $\{v^k\}$. Then $\bar{v} \notin \operatorname{int} X^\circ$.*

Moreover, from the following theorem, every accumulation point of $\{v^k\}$ solves problem (DP).

Lemma 3.6 *At iteration k of Algorithm IA, let $v^k \in V((S_k)^\circ)$ be an optimal solution for problem (D_k). Then, $v^k \notin \operatorname{int} Y^\circ$.*

Lemma 3.7 *Assume that $\{x(k)\}$ is an infinite sequence such that for all k, $x(k)$ is an optimal solution of problem (P_k) at iteration k of Algorithm IA. Then, $\{x(k)\}$ has an accumulation point.*

Theorem 3.2 *Assume that $\{v^k\}$ is an infinite sequence such that for all k, v^k is an optimal solution of (D_k) at iteration k of Algorithm IA and that $\bar{v}$ is an accumulation point of $\{v^k\}$. Then $\bar{v}$ solves problem (DP). Furthermore, $\lim_{k\to\infty} g^H(v^k) = \max(DP)$.*

By Theorems 3.1 and 3.2, we get that every accumulation point of $\{v^k\}$ belongs to the feasible set of problem (DP) and solves problem (DP).

Theorem 3.3 *Assume that $\{x(k)\}$ is an infinite sequence such that for all k, $x(k)$ is an optimal solution of problem (P_k) at iteration k of Algorithm IA and that $\bar{x}$ is an accumulation point of $\{x(k)\}$. Then $\bar{x}$ belongs to $R^n \setminus \operatorname{int} X$ and solves problem (MP). Furthermore, $\lim_{k\to\infty} g(x(k)) = \min(MP)$.*

From Theorem 3.3, we note that $\liminf_{k\to\infty} p(x(k)) \geq 0$. Hence, in order to terminate Algorithm IA after finitely many iterations, using admissible tolerance $\gamma > 0$, we propose the following stopping criterion:

If $p(x(k)) \geq -\gamma$, then stop; $x(k)$ is an approximate solution of problem (MP).

4 An Inner Approximation Method Incorporating a Penalty Function Method

4.1 Underestimation of the Optimal Value of Relaxed Problems by Using Penalty Functions

In order to obtain an optimal solution of problem (P_k), problem $(SP(v))$ has been solved for each $v \in \Gamma_k \setminus V((S_{k-1})^\circ)$ at every iteration of Algorithm IA discussed in Section 3. In Subsection 3.1, we remarked that problem $(SP(v))$ is a convex minimization problem with convex constraints. In this section, we propose another inner approximation algorithm which is incorporating a penalty function method. By using penalty functions, problem $(SP(v))$ can be transformed into an unconstrained convex minimization problem. That is, without solving problem $(SP(v))$ at every iteration, the algorithm guarantees the global convergence to an optimal solution of problem (MP). Furthermore, the problem is solvable for every

$v \in V((S_k)^\circ)$. Hence, by incorporating a penalty function method, the inner approximation algorithm does not need to generate Γ_k at every iteration.

Let $S \subset X$ be a polytope satisfying $0 \in \text{int } S$. For any $v \in V(S^\circ)$, we consider the following problem:

$$(SP1(v,\mu)) \begin{cases} \text{minimize} & F_{v,\mu}(x) = f(x) + \mu\theta_v(x), \\ \text{subject to} & x \in R^n, \end{cases}$$

where $\theta_v(x) = \sum_{j=1}^{t_Y}[\max\{0, r_j(x)\}]^s + [\{\max\{0, h(x,v)\}]^s$, $s \geq 1$ and $\mu > 0$. We know that the objective function $F_{v,\mu}$ of problem $(SP1(v,\mu))$ is convex (Bazaraa, Sherali and Shetty [1], Chapter 9). It follows from the following lemma that problem $(SP1(v,\mu))$ is solvable for every $v \in V(S^\circ)$.

Lemma 4.1 *For every $v \in R^n$ and $\mu > 0$, the function $F_{v,\mu}$ attains its minimum over R^n.*

Denote by $\min(SP1(v,\mu))$ the optimal value of problem $(SP1(v,\mu))$. From the definition of g, $\min(SP1(v,\mu)) < -g^H(v) = +\infty$ if $v \notin \Gamma$. In case $v \in \Gamma$, since $F_{v,\mu}(x) = f(x)$ for any $x \in Y \cap \{x \in R^n : \langle v, x\rangle \geq 1\}$,

$$\begin{aligned} \min(SP1(v,\mu)) &= \min\{F_{v,\mu}(x) : x \in R^n\} \\ &\leq \min\{F_{v,\mu}(x) : \langle v, x\rangle \geq 1,\ x \in Y\} \\ &= \min\{f(x) : \langle v, x\rangle \geq 1,\ x \in Y\} \\ &= \min(SP(v)) \\ &= -g^H(v). \end{aligned} \tag{6}$$

Hence, we have the following relations between problem $(SP1(v,\mu))$ and relaxed problems (P) and (D) described in Subsection 3.1:

$$\begin{aligned} \min(P) &= \min\{\min(SP(v)) : v \in \Gamma\} \\ &\geq \min\{\min(SP1(v,\mu)) : v \in \Gamma\} \\ &\geq \min\{\min(SP1(v,\mu)) : v \in V(S^\circ)\}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \max(D) &= \max\{g^H(v) : v \in V(S^\circ)\} \\ &\leq \max\{-\min(SP1(v,\mu)) : v \in V(S^\circ)\}. \end{aligned} \tag{8}$$

4.2 An Inner Approximation Algorithm Using Penalty Functions

An inner approximation algorithm for problem (MP) incorporating an exterior penalty method is as follows:

Algorithm IA-P

Initialization. Choose a penalty parameter $\mu_1 > 0$, a scalar $B > 1$ and $s \geq 1$. Generate a polytope V_1 such that $V_1 \subset X$ and that $0 \in$ int (co V_1). Let $S_1 =$ co V_1. Compute the vertex set $V((S_1)^\circ)$. Set $k \leftarrow 1$ and go to Step 1.

Step 1. For every $v \in V((S_{k+1})^\circ)$, let A_v and x^v be the optimal value and an optimal solution of problem $(SP1(v,\mu_k))$, respectively. Choose $v^k \in V((S_k)^\circ)$ satisfying $A_{v^k} = \min\{A_v : v \in V((S_k)^\circ)\}$. Let $x(k) = x^{v^k}$.

Step 2.

a. If $p(x(k)) \geq 0$ and $r(x(k)) \leq 0$, then stop; $x(k)$ are optimal solutions of problem (MP).

b. Otherwise, for v^k, solve problem (2). Let z^k and ω_k denote an optimal solution and the optimal value of problem (2), respectively. Let

$$V_{k+1} = \begin{cases} V_k \cup \{z^k\} & \text{if } \omega_k < 0, \\ V_k & \text{if } \omega_k = 0, \end{cases}$$

and let

$$\mu_{k+1} = \begin{cases} B\mu_k & \text{if } \theta_{v^k}(x(k)) > 0, \\ \mu_k & \text{if } \theta_{v^k}(x(k)) = 0. \end{cases}$$

Let $S_{k+1} = \text{co } V_{k+1}$. Compute the vertex set $V((S_{k+1})^\circ)$. Replace k by $k+1$, and return to Step 1.

From the discussion of Subsection 4.1, at every iteration k of the algorithm, we have

$$f(x(k)) \leq F_{v^k,\mu_k}(x(k)) = A_{v^k} \leq \min(P_k) \leq \min(MP). \tag{9}$$

Lemma 4.2 *At iteration k of algorithm IA-P, if $p(x(k)) \geq 0$ and $r(x(k)) \leq 0$, then $x(k)$ solves problem (MP).*

4.3 Convergence of Algorithm IA-P

In this subsection, the convergence of Algorithm IA-P will be verified.

Let $\{x(k)\}$ and $\{v^k\}$ be an infinite sequence generated by Algorithm IA-P. By Theorem 3.1, every accumulation point of $\{v^k\}$ belongs to the feasible set X° of problem (DP). It follows from the following theorems that every accumulation point is contained in $R^n \backslash \text{int } X$ and solves problem (RCP).

Lemma 4.3 *Let $\{x(k)\}$ and $\{v^k\}$ be infinite sequences generated by Algorithm IA-P. Then, $\theta_{v^k}(x(k)) \to 0$ as $k \to \infty$.*

Theorem 4.1 *Let $\{x(k)\}$ be an infinite sequence generated by Algorithm IA-P. Then, every accumulation point $\bar{x}$ of $\{x(k)\}$ belongs to the feasible set $R^n \backslash \text{int } X$ of problem (MP). Furthermore, $\bar{x}$ is contained in the feasible set $Y \backslash \text{int } X$ of problem (RCP).*

Theorem 4.2 *Let $\{x(k)\}$ be an infinite sequence generated by Algorithm IA-P. Then, every accumulation point $\bar{x}$ of $\{x(k)\}$ solves problem (MP).*

Theorem 4.3 *Let $\{v^k\}$ be an infinite sequence generated by Algorithm IA-P. Then, every accumulation point $\bar{v}$ of $\{v^k\}$ solves problem (DP).*

From Theorem 4.1, we have $\liminf_{k\to\infty} p(x(k)) \geq 0$. Moreover, from Lemma 4.3, we get $\lim_{k\to\infty} \theta_{v^k}(x(k)) = 0$, so that $\limsup_{k\to\infty} r(x(k)) \leq 0$. Hence, in order to terminate Algorithm IA-P after finitely many iterations, using admissible tolerances γ_1, $\gamma_2 > 0$, we propose the following stopping criterion:

If $p(x(k)) \geq -\gamma_1$ and $r(x(k)) \leq \gamma_2$, then stop; $x(k)$ is an approximate solution of problem (MP).

4.4 A Relationship between Problems $(SP(v^k))$ and $(SP1(v^k,\mu_k))$

In this subsection, we assume that

(A5) For any $x \in (\text{bd } X) \cap Y$ and $w \in \partial p(x)$, $\{y \in R^n : \langle w, y-x\rangle \geq 0\} \cap \text{int } Y \neq \emptyset$.

Let $s = 1$ at Initialization of Algorithm IA-P. Then, we shall show that there exists $\bar{k}$ such that for each $k \geq \bar{k}$, every optimal solution of problem $(SP1(v^k,\mu_k))$ solves problem $(SP(v^k))$.

Let Ω_M and Ω_D be the optimal solution sets of problems (MP) and (DP), respectively.

Lemma 4.4 *Assume that* $\nabla f(x') \neq 0$ *for some* $x' \in \Omega_M$. *Then, for any* $u \in \Omega_D$, $\{y \in R^n : \langle u, y\rangle \geq 1\} \cap \text{int } Y \neq \emptyset$.

For any $u \in (S_1)^\circ \setminus \text{int } Y^\circ$, let $\Omega_{(SP(u))}$ be the optimal solution set of problem $(SP(u))$.

Lemma 4.5 $\Omega_M = \bigcup_{u \in \Omega_D} \Omega_{(SP(u))}$.

Lemma 4.6 *Assume that* $\nabla f(x') \neq 0$ *for some* $x' \in \Omega_M$. *Then, for any* $u \in \Omega_D$, $\Omega_{(SP(u))} \subset \{x \in R^n : \langle u, x\rangle = 1\}$.

For any $u \in (S_1)^\circ \setminus \text{int } Y^\circ$, let $Y(u) = \{x \in Y : -\langle u, x\rangle + 1 \leq 0\}$. Then, $Y(u)$ is the feasible set of problem $(SP(u))$. Moreover, let $r(u,x) = \max\{r_j(u,x) : j = 1, \ldots, t_Y + 1\}$ where $r_j(u,x) = r_j(x)$, $j = 1, \ldots, t_Y$ and $r_{t_Y+1}(u,x) = -\langle u, x\rangle + 1$, and let $\partial_x r(u,x) = \text{co }(\nabla r_1(x), \ldots, \nabla r_{t_Y}(x), -u)$. Note that $Y(u) = \{x \in R^n : r_u(x) \leq 0\}$.

Lemma 4.7 *For any* $u \in \Omega_D$ *and* $x \in \text{bd } Y(u)$, $0 \notin \partial_x r(u,x)$.

Lemma 4.8 *Let* $\Gamma \subset R^n$ *satisfy that* $(\text{int } Y) \cap \{x \in R^n : \langle u, x\rangle \geq 1\} \neq \emptyset$ *for all* $u \in \Gamma$. *Then,* $\cup_{x \in \Omega_{(SP(u))}} \partial_x r(u,x)$ *is upper semicontinuous over* Γ.

Lemma 4.9 *Assume that* $\nabla f(x) \neq 0$ *for some* $x' \in \Omega_M$. *Then, the following assertion holds:*

$$\inf \left\{ \|w\| : w \in \bigcup_{u \in \Omega_D} \Big(\bigcup_{x \in \Omega(SP(u))} \partial_x r(u,x) \Big) \right\} > 0.$$

Theorem 4.4 *There exists* $\Lambda \geq 0$ *such that for any* $u \in \Omega_D$ *and* $x \in \Omega_{(SP(v))}$, *there exists* $\lambda(u,x)_j \geq 0$ $(j = 1, \ldots, t_Y + 1)$ *satisfying*

$$\max_{j=1,\ldots,t_Y+1} \lambda(u,x)_j \leq \Lambda, \tag{10}$$

$$\nabla f(x) + \sum_{j=1}^{t_Y} \lambda(u,x)_j \nabla r_j(x) - \lambda(u,x)_{t_Y+1} u = 0, \tag{11}$$

$$\lambda(u,x)_j r_j(y) = 0 \ (j = 1, \ldots, t_Y) \text{ and } \lambda(u,x)_{t_Y+1}(-\langle u, x\rangle + 1) = 0. \tag{12}$$

Lemma 4.10 *Let* $\{v^k\}$ *be generated by Algorithm IA-P. Assume that* $\nabla f(x') \neq 0$ *for some* $x' \in \Omega_M$. *Then, there exists* $\bar{k}$ *such that* $\text{int } Y \cap \{x \in R^n : \langle v^k, x\rangle \geq 1\} \neq \emptyset$ *for all* $k \geq \bar{k}$.

Theorem 4.5 *Assume that* $s = 1$ *at Initialization of Algorithm IA-P. Then, there exists* $\bar{k}$ *such that* $\theta_{v^k}(x(k)) = 0$ *for all* $k \geq \bar{k}$. *Furthermore, for all* $k \geq \bar{k}$, *an optimal solution* $x(k)$ *of problem* $(SP1(v^k,\mu_k))$ *solves problem* $(SP(v^k))$.

5 Conclusion

In this paper, instead of solving problem (RCP) directly, we have presented two inner approximation algorithms for problem (MP).

To execute the algorithms, a convex minimization problem (2) is solved at each iteration. However, we note that it is not necessary to obtain an optimal solution for problem (2) at each step. At iteration k of the algorithms, it suffices to get a point which is contained in X and is not contained in S_k. That is, at each step, we can compromise solving problem (2) by getting a point z^k satisfying $\phi(z^k; v^k) < 0$, because z^k belongs to $X \backslash S_k$ if $\phi(z^k; v^k) < 0$.

From the discussion of Section 3, by solving two kinds of convex minimization problems $(SP(v))$ and (2) successively, it is possible to obtain an optimal solution of problem (RCP). In Section 4, the proposed method using penalty functions transforms problem $(SP(v))$ into the unconstrained problem $(SP1(v, \mu))$. These unconstrained convex minimization problems are fairly easy to solve and therefore the proposed algorithm is practically useful.

References

[1] Bazaraa, M.S., H.D. Sherali and C.M. Shetty, *Nonlinear Programming: Theory and Algorithms*, 2nd ed., John Wiley, New York (1993).

[2] Hestenes, M.R., *Optimization Theory: The Finite Dimensional Case*, John Wiley, New York (1975).

[3] Hogan, W.W., "Point-to-Set Maps in Mathematical Programming", *SIAM Review*, Vol. 15, No. 3 (1973), pp.591–603.

[4] Horst, R. and H. Tuy, *Global Optimization: Deterministic Approaches, Third, Revised and Enlarged Edition*, Springer-Verlag, Berlin (1996).

[5] Horst, R. and P.M. Pardalos, *Handbook of Global Optimization*, Kluwer Academic Publishers, Dordrecht (1995).

[6] Konno, H., P.T. Thach and H. Tuy, *Optimization on Low Rank Nonconvex Structures* Kluwer Academic Publishers, Dordrecht (1997).

[7] Rockafellar, R.T., *Convex Analysis* Princeton University Press, Princeton, N.J. (1970).

[8] Tuy, H., *Convex Analysis and Global Optimization*, Kluwer Academic Publishers, Dordrecht (1998).