

Odd Central Square Solitaire

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Abstract

The classical game of peg solitaire has uncertain origins, but was certainly popular by the time of Louis XIV, and was described by Leibniz in 1710. One of the classical problems concerning peg solitaire is the feasibility issue. An early tool used to show the infeasibility of various peg games is the *rule-of-three* [Suremain de Missery 1841], which can be slightly generalized by the *lattice criterion*. In the 1960s the description of the *solitaire cone* [Boardman and Conway] provides necessary conditions: valid inequalities over this cone, known as pagoda functions, were used to show the infeasibility of various peg games. In this paper, we recall these necessary conditions and give an explicit solution to the odd central square complementary game.

1 Introduction and Basic Definitions

1.1 Introduction

Peg solitaire is a peg game for one player which is played on a board containing a number of holes. The most common modern version uses a cross shaped board with 33 holes - see Fig. 1 - although a 37 hole board is common in France. Computer versions of the game now feature a wide variety of shapes, including rectangles and triangles. Initially the central hole is empty, the others contain pegs. If in some row (column respectively) two consecutive pegs are adjacent to an empty hole in the same row (column respectively), we may make a *move* by removing the two pegs and placing one peg in the empty hole. The objective of the game is to make moves until only one peg remains in the central hole. Variations of the original game, in addition to being played on different boards, also consider various alternate starting and finishing configurations.

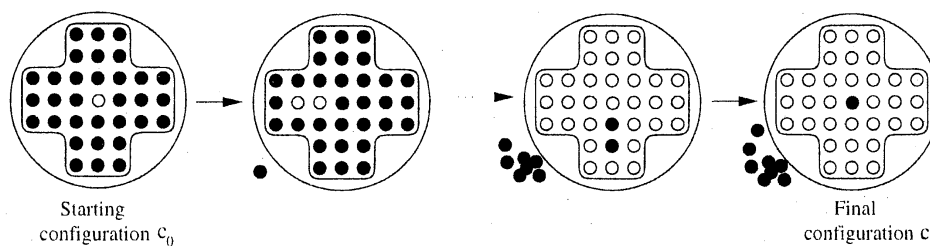


Figure 1: A feasible English solitaire peg game with possible first and last moves

The game itself has uncertain origins, and different legends attest to its discovery by various cultures. An authoritative account with a long annotated bibliography can be found in the comprehensive book of Beasley [4]. The book mentions an engraving of Beres, dated 1697, of a lady with a solitaire board. The book also contains a quotation of Leibniz [7] which was written for the Berlin Academy in 1710. Apparently the first theoretical study of the game that was published was done in 1841 by Suremain de Missery, and was reported in a paper by Vallot [9]. The modern mathematical study of the game dates to the 1960s at Cambridge University. The group was led by Conway who has written a chapter in [5] on various mathematical aspects of the subject. One of the problems studied by the Cambridge group is the following basic *feasibility* problem (see Definition 1.1 in the sequel for a formal definition):

Peg solitaire feasibility problem. Given a board B and a pair of configurations (c, c') on B , determine if (c, c') is *feasible*, that is, if there is a legal sequence of moves transforming c into c' .

The complexity of the feasibility problem for the game played on a n by n board was shown by Uehara and Iwata [8] to be NP-complete, so easily checked necessary and sufficient conditions for feasibility are unlikely to exist. In this paper, we recall constructions used to prove the infeasibility of some pair (c, c') : the *rule-of-three* in Section 2.1, the *solitaire cone* in Section 2.2, the *lattice criterion* in Section 2.3; and give an explicit solution to the *odd central square solitaire game* in Section 3.

1.2 Basic definitions

In this section we introduce some terminology used throughout this paper. The *board* of a peg solitaire game is a *finite* subset $B \subset \mathbb{Z}^2$. Thus, B stands for the set of locations (i, j) of holes of the board on which the game is played. For example, the classical 33-board is: $B = \{(i, j) : -1 \leq i \leq 1, -3 \leq j \leq 3\} \cup \{(i, j) : -3 \leq i \leq 3, -1 \leq j \leq 1\}$. A *configuration* c on the board is an integer vector $c \in \mathbb{Z}^B \subset \mathbb{R}^B$. It can be interpreted as a configuration of pegs on the board: in the usual game, all configurations c lie in $\{0, 1\}^B$, with the interpretation that hole $(i, j) \in B$ contains a peg if $c_{i,j} = 1$ and is empty if $c_{i,j} = 0$; extending this, we allow any integer (possibly negative) number $c_{i,j}$ of pegs to occupy any hole $(i, j) \in B$. The *complement* of a $\{0, 1\}$ -configuration $c \in \{0, 1\}^B$ is defined to be the configuration $\bar{c} := \mathbb{1} - c$ where $\mathbb{1} = (1, 1, \dots, 1) \in \mathbb{R}^B$ is the all-ones configuration. A *move* or a *jump* μ is a vector in \mathbb{R}^B which has 3 non-zero entries: two entries of -1 in the positions from which pegs are removed and one entry of 1 for the hole receiving the new peg. We can now make the peg solitaire feasibility problem precise.

Definition 1.1 Given a board B and an associated set of moves \mathcal{M} , a pair (c, c') of configurations is *feasible* if there is a sequence $\mu^1, \dots, \mu^k \in \mathcal{M}$ of moves on B such that

$$c' - c = \sum_{i=1}^k \mu^i \text{ and } c + \sum_{j=1}^i \mu^j \in \{0, 1\}^B \text{ for } i = 1, \dots, k$$

For instance, the English 33-board admits 76 moves (none over the 8 corners, 24 moves over the 12 holes next to a corner and 52 moves over the 13 remaining holes); see Fig. 1 for possible first and last moves in some sequence of moves transforming the initial configuration c_0 to its complementary c'_0 .

2 Necessary Conditions

2.1 The rule-of-three

In this section we recall the so-called rule-of-three (cf. [4, 5]), a classical construction used to test solitaire game feasibility. The rule-of-three can be used, for example, to show that on the cross shaped English 33-board, starting with the initial configuration c_0 of Fig. 1, the only reachable final configurations with *exactly one* peg are c'_0 (given in Fig. 1), c'_1 , c'_2 , c'_3 and c'_4 with, respectively, a final peg in position $(0, 0)$, $(-3, 0)$, $(0, 3)$, $(3, 0)$ and $(0, -3)$.

Let $\mathbb{Z}_2 := \{a, b, c, e\}$ be the Abelian group with identity e and addition table

$$a + a = b + b = c + c = e, \quad a + b = c, \quad a + c = b, \quad b + c = a.$$

Define the following two maps $g_1, g_2 : \mathbb{Z}^2 \rightarrow \mathbb{Z}_2$, which simply color the integer lattice \mathbb{Z}^2 by diagonals of a , b and c in either direction; see Fig. 2:

$$g_1(i, j) := \begin{cases} a & \text{if } (i + j) \equiv 0 \pmod{3} \\ b & \text{if } (i + j) \equiv 1 \pmod{3} \\ c & \text{if } (i + j) \equiv 2 \pmod{3} \end{cases} \quad , \quad g_2(i, j) := \begin{cases} a & \text{if } (i - j) \equiv 0 \pmod{3} \\ b & \text{if } (i - j) \equiv 1 \pmod{3} \\ c & \text{if } (i - j) \equiv 2 \pmod{3} \end{cases} \quad .$$

For each $(i, j) \in B \subset \mathbb{Z}^2$ let $e_{i,j}$ be the (i, j) th unit vector in \mathbb{R}^B , and define the *score map* to be the \mathbb{Z} -module homomorphism $\phi : \mathbb{Z}^B \rightarrow \mathbb{Z}_2^2$ with $\phi(e_{i,j}) := (g_1(i, j), g_2(i, j))$. Thus, the score of a configuration $c \in \mathbb{Z}^B$ is given by

$$\phi(c) = \sum_{(i,j) \in B} c_{i,j} \cdot (g_1(i, j), g_2(i, j)) \quad .$$

Since the board B under discussion will always be clear from the context, we use the notation ϕ for any board. For instance, the score of the configuration c'_0 of one peg in the center of the English 33-board is $\phi(c'_0) = (a, a)$, as is also the score of its complement c_0 : see Fig. 2. The score of the board B (all holes filled) is defined to be $\phi(B) = \phi(\mathbb{I})$. It is easy to verify that any feasible move μ on any board B has the identity score $\phi(\mu) = (e, e)$. This gives the following proposition.

Proposition 2.1 [The rule-of-three]

A necessary condition for a pair of configuration (c, c') to be feasible is that $\phi(c' - c) = (e, e)$, namely, $c' - c \in \text{Ker}(\phi)$.

Using Proposition 2.1, we can show that, besides the configuration c'_0 given in Fig. 1, the only final configuration c' with exactly one non-zero entry $c'_{i,j} = 1$ forming a feasible pair (c_0, c') are the 4 configurations c'_1, c'_2, c'_3 and c'_4 . Fig. 2 shows that $\phi(c') = (a, a) = \phi(c_0)$ if c' is one of $c'_0, c'_1 \dots c'_4$, whereas $\phi(c') \neq (a, a)$ otherwise.

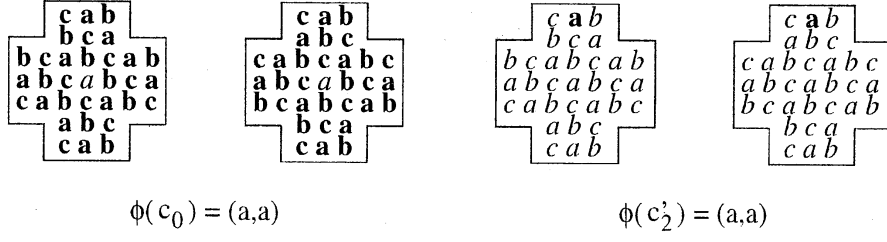


Figure 2: The score of a final configuration with only one peg remaining

With $\bar{c} = \mathbb{I} - c$ the complement of c , the rule-of-three implies that (c, \bar{c}) is feasible only if $\phi(\bar{c}) = \phi(c)$, which is equivalent to $\phi(B) = \phi(c) + \phi(\bar{c}) = \phi(c) + \phi(c) = (e, e)$. In other words, a necessary condition for the configurations pair (c, \bar{c}) to be feasible is that the board score is $\phi(B) = (e, e)$. Such a board is called a *null-class* board in [4]. For example, the score of the English 33-board is $\phi(B) = \phi(c_0) + \phi(c'_0) = (a, a) + (a, a) = (e, e)$.

2.2 Solitaire cone and pagoda functions

A first relaxation of the feasibility problem is to allow any integer (positive or negative) number of pegs to occupy any hole for any *intermediate* configurations. We call this game the *integer game*, and call the original game the *0-1 game*. Note that in a 0-1 game we require that for each intermediate configuration of the game a hole is either empty or contains a single peg. Clearly:

$$c', c \text{ is integer feasible} \iff c' - c \in IC_B = \left\{ \sum_{\mu \in \mathcal{M}} \lambda_\mu \mu \quad : \lambda_\mu \in \mathbb{N} \right\}$$

where the *integer solitaire cone* IC_B is the set of all *non-negative integer* linear combinations of moves. Unfortunately deciding if $c' - c$ can be expressed as the sum of move seems to be a hard computational problem. We get the following necessary criterion:

Proposition 2.2 [The integer cone criterion]

A necessary condition for a pair of configurations (c, c') to be feasible is that $c' - c \in IC_B$.

A further relaxation of the game leads to a more tractable condition. In the *fractional game* we allow any fractional (positive or negative) number of pegs to occupy any hole for any *intermediate* configurations. A *fractional move* is obtained by multiplying a move by any positive scalar and is defined to correspond to the process of adding a move to a given configuration. For example, let $c = [1 \ 1 \ 1]$, $c' = [1 \ 0 \ 1]$. Then $c' - c = [0 \ -1 \ 0] = \frac{1}{2}[-1 \ -1 \ 1] + \frac{1}{2}[1 \ -1 \ -1]$ is a feasible fractional game and can be expressed as the sum of two fractional moves, but is not feasible as a 0-1 or integer game. Clearly:

$$c', c \text{ is fractional feasible} \iff c' - c \in C_B = \left\{ \sum_{\mu \in \mathcal{M}} \lambda_\mu \mu \quad : \lambda_\mu \in \mathbb{R}^+ \right\}$$

where the *solitaire cone* C_B is the set of all *non-negative* linear combinations of moves. We get the weaker, but useful, following necessary criterion:

Proposition 2.3 [The cone criterion]

A necessary condition for a pair of configurations (c, c') to be feasible is that $c' - c \in C_B$.

The condition $c' - c \in C_B$ provides a certificate for the infeasibility of certain games. The certificate of infeasibility is any inequality valid for C_B which is violated by $c' - c$. According to [4], page 71, these inequalities “were developed by J.H. Conway and J.M. Boardman in 1961, and were called *pagoda functions* by Conway...”. They are also known as *resource counts*, and are discussed in some detail in Conway [5]. The strongest such inequalities are induced by the facets of C_B . For example, the facet (given by Beasley) of Fig. 4 induces an inequality $a \cdot x \leq 0$ that is violated by $c' - c$ with (c, c') given in Fig. 3: $(c' - c) \cdot a = 2 > 0$. This implies that this game is not feasible even as a fractional game and, therefore, not feasible as an integer game or classical 0-1 game either.

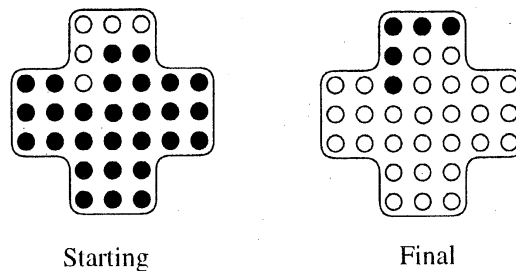


Figure 3: An infeasible classical solitaire peg game

		11	8	3		
		7	5	2		
4	0	4	3	1	2	-1
3	0	3	2	1	1	0
1	0	1	1	0	1	-1
		2	1	1		
		-1	0	-1		

Figure 4: A facet of the English solitaire cone

2.3 The lattice criterion

For the fractional game, we relaxed the integrality while keeping the non-negativity condition. Another relaxation of the integer game is to drop the non-negativity while keeping the integrality. It amounts, besides allowing any integer (positive or negative) number of pegs for any intermediate configurations, to allow *additive moves*. The configuration of an additive move μ^+ (jumping over an empty hole and putting a peg in) is $c_\mu^+ = -c_\mu$ where c_μ is the configuration of an ordinary (subtractive) move μ . We call this game the *lattice game*. Clearly:

$$c', c \text{ is lattice feasible} \iff c' - c \in L_B = \left\{ \sum_{\mu \in \mathcal{M}} \lambda_\mu \mu : \lambda_\mu \in \mathbb{Z} \right\}$$

where the *solitaire lattice* L_B is the set of all *integer* linear combinations of moves. It gives the following criterion (weaker than Proposition 2.2):

Proposition 2.4 [The lattice criterion]

A necessary condition for a pair of configurations (c, c') to be feasible is that $c' - c \in L_B$.

Since the score ϕ is a homomorphism of \mathbb{Z} -modules which maps each lattice generator $\mu \in \mathcal{M}$ to (e, e) , it follows that $\phi(v) = (e, e)$ for any $v \in L_B$; i.e., for any board B and any pair (c, c') on B if $c' - c \in L_B$ then $c' - c \in \text{Ker}(\phi)$. In other words, as stated in the following proposition, the lattice criterion is generally stronger than the rule-of-three.

Proposition 2.5 *For any board B , we have $L_B \subseteq \text{Ker}(\phi)$.*

Fig. 5 provides an example of a null-class board and a game on it whose associated pair (c, \bar{c}) satisfies $\bar{c} - c \in \text{Ker}(\phi)$ but $\bar{c} - c \notin L_B$. This shows that the lattice criterion may be strictly stronger than the rule-of-three and therefore could be more useful in proving infeasibility.

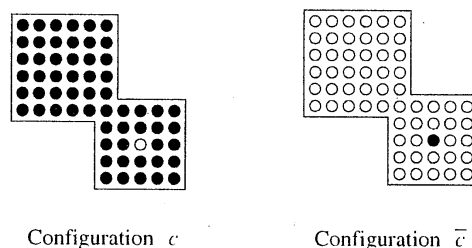


Figure 5: An infeasible game satisfying the rule-of-three but not the solitaire lattice criterion

Checking membership in the lattice L_B is usually easy (once we have a basis) and checking membership in the cone C_B amounts to solve a linear program in polynomial time. Combining these two criteria, that is, checking membership in $C_B \cap L_B$, is usually efficient in proving infeasibility. For example, while the game in Fig. 3 satisfies $c' - c \in L_B$ but $c' - c \notin C_B$, the central game (see Fig. 1) played on a French board (an English board with 4 additional holes in positions $(\pm 2, \pm 2)$) satisfies $c' - c \in C_B$ but $c' - c \notin L_B$. Note that for both the French and English boards, we have $L_B = \text{ker}(\phi)$; therefore checking the membership in L_B can be easily done using the rule-of-three. The membership in C_B of the central game played on a French board can be shown by moving all pegs on the boundary to the inner part of the board, then moving one peg in the center and finally remove the other pegs using the fractional move given after Proposition 2.2. Clearly we have $C_B \cap L_B \subset IC_B$ but this inclusion is strict as illustrated by Fig. 6. A further step could be to find a relatively small generating set (Hilbert basis) for the integer cone IC_B of some interesting classes of boards B .



Figure 6: A lattice and fractional feasible but integer infeasible game

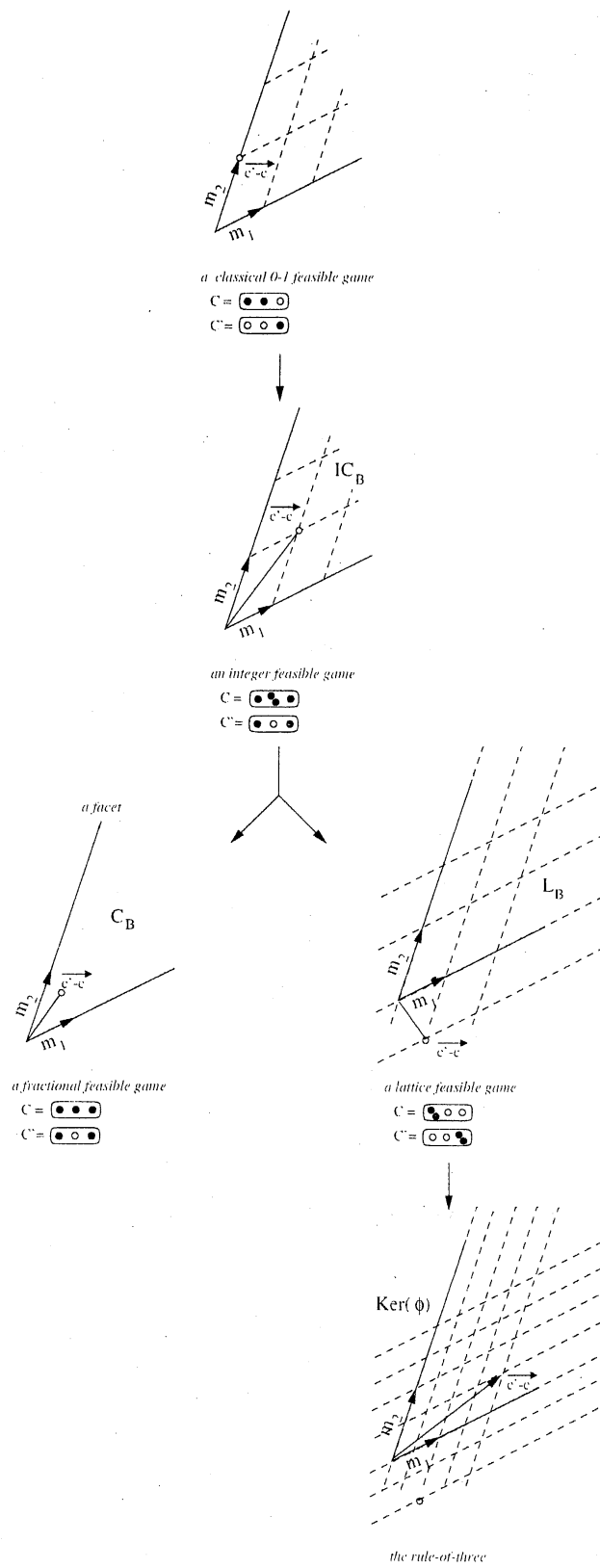


Figure 7: The rule-of-three and the integer, fractional and lattice relaxations

3 Odd Central Square Solitaire

In this Section, we consider the feasibility of the central odd square solitaire game: Given odd n , is it possible, on the $n \times n$ board, to start with pegs in all holes but the center and finish with a single peg in the center? In other words, with $n = 2k + 1$, $k \in \mathbb{N}$, $B_n = \{(i, j) : -k \leq i, j \leq k\}$ the centrally symmetric square $n \times n$ board and $\bar{c} = e_{0,0}$ the final configuration with exactly one peg at the origin, is the pair $\{c, \bar{c}\}$ feasible on B_n ?

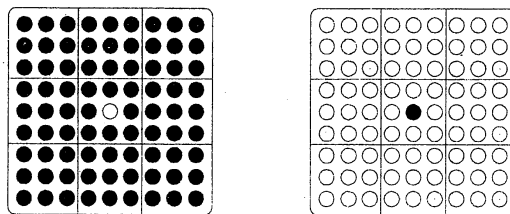


Figure 8: The odd central square solitaire on B_9

The central odd square game is trivially infeasible on the board B_3 . For $n \geq 5$, let first check if $\{c, \bar{c}\}$ satisfies the necessary conditions given in Section 2. Since for any $m \times n$ board with $m \geq 4$ or $n \geq 4$, we have $L_B = \ker(\phi)$, checking the membership of $\bar{c} - c$ in the lattice L_B can be done using the rule-of-three. It gives the necessary condition that n should be a multiple of 3. For $n \equiv 1 \pmod{2}$, $n \equiv 0 \pmod{3}$ and $n \geq 5$, one can easily check that $\bar{c} - c$ can be written as a non-negative linear combinations of moves μ_i ; that is, $\bar{c} - c \in C_{B_n}$.

We now show that the odd central square game is feasible for $n \equiv 1 \pmod{2}$, $n \equiv 0 \pmod{3}$ and $n \geq 5$ by giving an explicit sequence of moves from \bar{c} to c on the board B_n . The basic sub-sequences or moves are the following two *purges*: the 3-purge (respectively 6-purge) is a sub-sequence of 3 (respectively 6) moves. See Fig. 9 where, besides black (respectively white) hole representing a peg (respectively an empty hole), a pair of \otimes represents two holes such that one is empty and the other contains a peg. For other purges and *packages* (useful short sequences of moves), see [5] where, in particular, an elegant solution to the English central complementary game, see Fig. 1. is given.



Figure 9: A 3-purge and a 6-purge

Fig. 10 illustrates the sequences of 3- and 6-purges needed to reach \bar{c} from c on the B_{15} board. The 3-purges (respectively 6) correspond to light (respectively dark) grey colored holes. This solution can be easily generalized to any B_n with $n \equiv 1 \pmod{2}$, $n \equiv 0 \pmod{3}$ and $n \geq 5$.

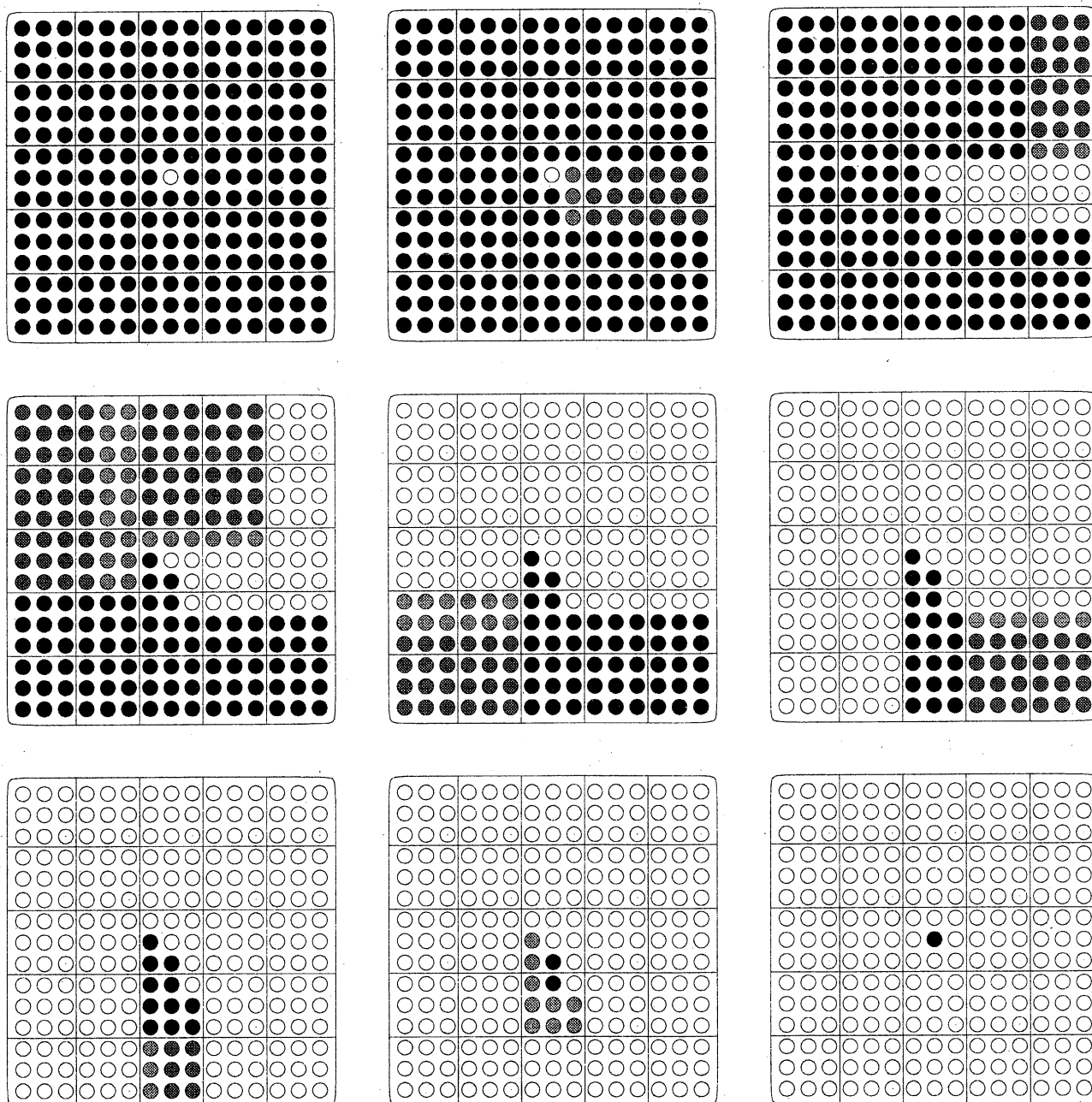


Figure 10: A solution to the 15×15 central square complementary game

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