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A Higher Order Method for SDP and Monotone SDLCPs along Weighted Central Trajectory induced by AHO Search Directions

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Abstract

We develop a higher order infeasible-interior-point method for semidefinite programming and monotone semidefinite linear complementarity problems. Our algorithm is based on the predictor-corrector infeasible-interior-point algorithm using the Alizadeh-Haeberly-Overton search direction proposed by Kojima, Shida and Shindoh. Under the nondegenerate condition, the gap to solutions converges to zero arbitrary higher ($p+1$-th) order by our higher (p-th) order algorithm.

Keywords Higher order method, semidefinite programming, monotone semidefinite linear complementarity problems, interior-point algorithm, AHO search direction, weighted central trajectory

AMS subject classfications 90C33, 65K05, 90C05, 90C25

1 Introduction

For linear programming (LP) and monotone linear complementarity problems (LCPs), higher order interior-point methods have been practically and theoretically discussed by several researchers [3, 4, 13, 16, 24, 25, etc.]. Recently, Stoer, Wechs and Mizuno [24] showed higher order convergent property of higher order interior-point algorithms for sufficient LCPs without assuming the strict complementarity condition. For semidefinite programming (SDP) and monotone semidefinite linear complementarity problems (SDLCPs), while first order feasible/infeasible-interior-point algorithms have been extensively developed in the last 5 years [1, 7, 9, 10, 11, 12, 14, 15, 18, 19, 21, 22, 28, etc.], only computational experiments of higher order implementation such as "Mehrotra's higher order corrections" were reported by Alizadeh, Haeberly and Overton [2] and recently by Haeberly, Nayakkankuppam and Overton [6].

For SDP and monotone SDLCPs as in the case of LP and LCPs, the computations of the search direction and higher order derivatives at each iterate involve the inversion of a single coefficient (common for any order of derivatives). In contrast with LP and monotone LCPs, the inversion and/or the Schur complement of the coefficient are fully dense even if the coefficient itself is sparse. Hence, this computation is quite expensive in the whole computation cost of interior-point methods for SDP and monotone SDLCPs. Therefore, especially for SDP and monotone SDLCPs, it is desirable to use as much information as possible from one factorization. On the other hand, SDP and monotone SDLCPs have several difficulties to establish higher order methods. For example, there are many distinct search directions for SDP and monotone SDLCPs (see a survey paper [26]). In view of [10], each search direction is considerable as an inexact search direction of other search directions. Who can expect the inexact search direction enjoys higher order convergence ?

Our purpose in this paper is to establish higher order convergent property of higher order infeasible-interior-point method for SDP and monotone SDLCPs. Our higher order algorithm is based on the long-step predictor-corrector infeasible-interior-point algorithm using the Alizadeh-Haeberly-Overton search direction [1] proposed by Kojima, Shida and Shindoh [9]. In this paper, we shall show that the gap to solutions converges to zero arbitrarily higher (p+1-th) order by our higher (p-th) order method under the nondegeneracy condition. The result gives the theoretical background of the practical advantage of the AHO search direction and the numerical efficiency of higher order implementation "Mehrotra's higher order corrections" reported by Alizadeh, Haeberly and Overton [2] and Haeberly, Nayakkankuppam and Overton [6].

The paper is organized as follows. Section 2 is devoted to introducing several basic notions, such as the monotone SDLCP we are concerned, the AHO search direction, the weighted central trajectory, the nondegeneracy condition and the higher order approximation. In Section 3, we present our higher order algorithm for the monotone SDLCP, and show its higher order convergence. We make concluding remarks in Section 4.

In this paper, we only discuss the local convergence property of our higher order algorithm, and hence we assume that there exists a solution of monotone SDLCP. For more detailed discussion of the global convergence property and the infeasibility detection, see the paper [9].

2 Preliminaries

2.1 Monotone Semidefinite Linear Complementarity Problem

Let $\mathcal{S}$ ($\mathcal{S}_+$ or $\mathcal{S}_{++}$) denote the set of $n \times n$ symmetric (positive semidefinite or positive definite, resp.) matrices. Let $\mathcal{F}$ be a maximal monotone affine subspace of $\mathcal{S} \times \mathcal{S}$, *i.e.*, $n(n+1)/2$-dimensional and $\langle \boldsymbol{X} - \boldsymbol{X}', \boldsymbol{Y} - \boldsymbol{Y}' \rangle (\equiv \textbf{Tr } (\boldsymbol{X} - \boldsymbol{X}')(\boldsymbol{Y} - \boldsymbol{Y}')) \geq 0$ for any two pairs $(\boldsymbol{X}, \boldsymbol{Y}), (\boldsymbol{X}', \boldsymbol{Y}')$ in $\mathcal{F}$. We are concerned with the monotone SDLCP:

$$\textbf{SDLCP} \quad \begin{array}{ll} \text{Find} & (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_+ \times \mathcal{S}_+ \\ \text{such that} & (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}, \langle \boldsymbol{X}, \boldsymbol{Y} \rangle (\equiv \textbf{Tr } \boldsymbol{XY}) = 0. \end{array} \tag{1}$$

The monotone SDLCP (1) was introduced by Kojima, Shindoh and Hara [12], and is a broad mathematical framework which contains LP, monotone LCPs and SDP.

Lemma 2.1. *Let $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_+ \times \mathcal{S}_+$. The following three statements are equivalent.*

(1). $\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = 0$,
(2). $\boldsymbol{XY} = \boldsymbol{O}$,
(3). $\boldsymbol{XY} + \boldsymbol{YX} = \boldsymbol{O}$.

Proof: Since the parts [(2)⇒(3)] and [(3)⇒(1)] are trivial, we shall show the non-trivial part [(1)⇒(2)]. Since $\textbf{Tr } \boldsymbol{X}^{1/2}\boldsymbol{Y}\boldsymbol{X}^{1/2} = \textbf{Tr } \boldsymbol{XY} = \langle \boldsymbol{X}, \boldsymbol{Y} \rangle = 0$ and $\boldsymbol{X}^{1/2}\boldsymbol{Y}\boldsymbol{X}^{1/2}$ is positive semidefinite, we have that $\boldsymbol{X}^{1/2}\boldsymbol{Y}\boldsymbol{X}^{1/2} = \boldsymbol{O}$. Since the set of eigenvalues of $\boldsymbol{XY}$ corresponds to that of $\boldsymbol{X}^{1/2}\boldsymbol{Y}\boldsymbol{X}^{1/2}$, we conclude the assertion. ∎

2.2 Neighborhood of Infeasible Central Trajectory

Let $(\boldsymbol{X}^0, \boldsymbol{Y}^0) = (\sqrt{\mu^0}\boldsymbol{I}, \sqrt{\mu^0}\boldsymbol{I})$ be an initial iterate with a positive constant μ^0, and $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})$ denotes an arbitrary pair of matrices in $\mathcal{F}$. Let

$$\begin{aligned} \mathcal{F}(\theta) &:= \left\{(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{F} + \theta\left((\boldsymbol{X}^0,\boldsymbol{Y}^0) - (\bar{\boldsymbol{X}} - \bar{\boldsymbol{Y}})\right)\right\}, \\ \mathcal{F}_0 &:= \left\{(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{F} - (\bar{\boldsymbol{X}},\bar{\boldsymbol{Y}})\right\} \ : \text{linearity subspace of } \mathcal{F}. \end{aligned}$$

Note that $\mathcal{F}(\theta)$ and $\mathcal{F}_0$ do not depend on the choice of the pair $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})$ in $\mathcal{F}$. For each $\gamma \in (0,1)$, $\theta \in (0,1]$ and $\zeta > 1/n$, let

$$\widetilde{\mathcal{N}}(\gamma,\theta) = \left\{(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{F}(\theta) \cap (\mathcal{S}_{++} \times \mathcal{S}_{++}) : \begin{array}{l} \boldsymbol{XY} + \boldsymbol{YX} \succeq 2(1-\gamma)\theta\mu^0\boldsymbol{I}, \\ \langle \boldsymbol{X},\boldsymbol{Y}\rangle \leq n(1+\zeta\gamma)\theta\mu^0 \end{array}\right\}$$

(the condition $\zeta > 1/n$ is needed to ensure the global convergence [9, Lemmas 3.7 and 3.8]). We confine all iterates $\{(\boldsymbol{X}^k, \boldsymbol{Y}^k)\}$ in the set $\left\{(\boldsymbol{X},\boldsymbol{Y}) \in \widetilde{\mathcal{N}}(\gamma,\theta) : \theta > 0\right\}$ which forms a neighborhood of the infeasible central trajectory

$$\begin{aligned} &\{(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{F}(\theta) \cap (\mathcal{S}_{++} \times \mathcal{S}_{++}) : \boldsymbol{XY} = \theta\mu^0\boldsymbol{I}, \theta > 0\} \\ &= \{(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{F}(\theta) \cap (\mathcal{S}_{++} \times \mathcal{S}_{++}) : \boldsymbol{XY} + \boldsymbol{YX} = 2\theta\mu^0\boldsymbol{I}, \theta > 0\}. \end{aligned}$$

θ serves as a gap to solutions of the SDLCP (1) in the following sense:

Lemma 2.2. *Let $(\boldsymbol{X},\boldsymbol{Y}) \in \widetilde{\mathcal{N}}(\gamma,\theta)$ for some $\theta > 0$ and $\gamma \in (0,1)$. Then $(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{F}(\theta)$ and*

$$2n(1-\gamma)\theta\mu^0 \leq \|\boldsymbol{XY} + \boldsymbol{YX}\|_F \leq 2\langle \boldsymbol{X},\boldsymbol{Y}\rangle \leq 2n(1+\zeta\gamma)\theta\mu^0,$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

Proof: Let $\nu_i (i = 1, \cdots, n)$ denote the eigenvalues of $\boldsymbol{XY} + \boldsymbol{YX}$. Then $\nu_i \geq 2(1-\gamma)\theta\mu^0 \geq 0$. Since the matrix $\boldsymbol{XY} + \boldsymbol{YX}$ is symmetric and positive definite, we see that

$$2n(1-\gamma)\theta\mu^0 \leq n\min\{\nu_i\} \leq \|\boldsymbol{XY}+\boldsymbol{YX}\|_F = \left(\sum_{i=1}^{n}(\nu_i)^2\right)^{1/2} \leq \sum_{i=1}^{n}\nu_i = 2\langle \boldsymbol{X},\boldsymbol{Y}\rangle \leq 2n(1+\zeta\gamma)\theta\mu^0.$$

∎

2.3 AHO Search Direction

For the path-following type interior-point algorithms for the SDP and monotone SDLCPs in the literature, several search directions have been proposed, such as the AHO, the HRVW/KSH/M, the NT, the MT search direction and etc. (see a survey paper [26]). In this paper, we utilize the AHO search direction [1].

Let $(\boldsymbol{X}^k, \boldsymbol{Y}^k) \in \widetilde{\mathcal{N}}(\gamma^{(k)}, \theta^k)$ denote an iterate. The AHO search direction $(d\boldsymbol{X}, d\boldsymbol{Y})$ is defined as a solution of the system of equations;

$$\left.\begin{array}{c} \boldsymbol{X}^k d\boldsymbol{Y} + d\boldsymbol{X}\boldsymbol{Y}^k + \boldsymbol{Y}^k d\boldsymbol{X} + d\boldsymbol{Y}\boldsymbol{X}^k = 2\beta\theta^k\mu^0\boldsymbol{I} - \boldsymbol{X}^k\boldsymbol{Y}^k - \boldsymbol{Y}^k\boldsymbol{X}^k, \\ (\boldsymbol{X}^k + d\boldsymbol{X}, \boldsymbol{Y}^k + d\boldsymbol{Y}) \in \mathcal{F}(\beta\theta^k). \end{array}\right\} \qquad (2)$$

where $\beta \in [0,1]$. The system (2) has the unique solution $(d\boldsymbol{X}, d\boldsymbol{Y})$ in $\mathcal{S} \times \mathcal{S}$ whenever $\boldsymbol{X}^k \succ \boldsymbol{O},\ \boldsymbol{Y}^k \succ \boldsymbol{O}, \boldsymbol{X}^k\boldsymbol{Y}^k + \boldsymbol{Y}^k\boldsymbol{X}^k \succeq \boldsymbol{O}$ ([23, 27], see also [20]). It should be noted that the existence of the AHO search direction is not guaranteed on the whole set $\mathcal{S}_{++} \times \mathcal{S}_{++}$ of pair of positive definite matrices. Our algorithm generates a sequence $\{(\boldsymbol{X}^k, \boldsymbol{Y}^k)\}$ in the neighborhood $\{(\boldsymbol{X}, \boldsymbol{Y}) \in \widetilde{\mathcal{N}}(\gamma, \theta) : \theta > 0\}$ (for some $\gamma \in (0,1)$) of the infeasible central trajectory, so that the AHO search direction is well-defined at each iterate $(\boldsymbol{X}^k, \boldsymbol{Y}^k)$.

Remark 2.3. Among the proposed search directions for SDP and monotone SDLCPs, the AHO search direction has the following good properties:

- the AHO search direction (2) with $\beta = 0$ (the affine scaling AHO search direction) is a pure Newton direction towards the set $\{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F} \cap (\mathcal{S}_+ \times \mathcal{S}_+) : \boldsymbol{XY} + \boldsymbol{YX} = \boldsymbol{O}\}$, which is an equivalent system of the solution set of the monotone SDLCP (1) (see Lemma 2.1).
- the (first order) predictor-corrector algorithm using the AHO search direction proposed in [9] possesses the locally quadratically convergent property under a mild assumption (the strict complementarity condition and/or the nondegeneracy condition, see Subsection 2.5 for definitions). ∎

2.4 Infeasible Weighted Central Trajectory induced by AHO Search Directions

For LP and monotone (L)CPs, the weighted central trajectory, which is induced by the so-called affine scaling search directions, is a fundamental guide to lead solutions of problems [5, 8]. The trajectory is a background of the stability aspects of practical interior-point algorithms.

For SDP and monotone SD(L)CPs, Monteiro and Pang [17] studied the fundamental interior-point mapping which leads to a family of continuous trajectories. Let $(\boldsymbol{X}^k, \boldsymbol{Y}^k) \in \widetilde{\mathcal{N}}(\gamma^{(k)}, \theta^k)$ be a current iterate. By using their interior-point mapping, we define the infeasible weighted central trajectory

$$\{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}(\tau\theta^k) \cap (\mathcal{S}_{++} \times \mathcal{S}_{++}) : \boldsymbol{XY} + \boldsymbol{YX} = \tau(\boldsymbol{X}^k\boldsymbol{Y}^k + \boldsymbol{Y}^k\boldsymbol{X}^k), \tau > 0\},$$

through the current iterate $(\boldsymbol{X}^k, \boldsymbol{Y}^k)$. From the definition, the AHO search direction (2) with $\beta = 0$ (the affine scaling AHO search direction) is a tangent direction of the infeasible weighted central trajectory, in other words, the infeasible weighted central trajectory is induced by the affine scaling AHO search directions. It should be noted that, as the existence of the AHO search direction, the infeasible weighted central trajectory is well-defined on the neighborhood $\{(\boldsymbol{X}, \boldsymbol{Y}) \in \widetilde{\mathcal{N}}(\gamma, \theta) : \theta > 0\}$, but not on the whole set $\mathcal{S}_{++} \times \mathcal{S}_{++}$ of pairs of positive definite matrices, in contrast with the cases of LP and monotone (L)CPs.

2.5 Nondegeneracy Condition

We introduce two generic assumptions.

Condition 2.4. (Strict Complementarity Condition) There exists a solution pair $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ of the monotone SDLCP (1) such that $\boldsymbol{X}^* + \boldsymbol{Y}^* \in \mathcal{S}_{++}$. ∎

Under the strict complementarity condition, Kojima, Shida and Shindoh [9] showed that the gap θ to solutions of the monotone SDLCP (1) quadratically converges to zero by their first order algorithm. For higher order convergence, we impose a stronger condition, "the nondegeneracy condition":

Condition 2.5. (Nondegeneracy Condition) Let $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ be a solution of the monotone SDLCP (1). Then, $(\boldsymbol{U}, \boldsymbol{V}) = (\boldsymbol{O}, \boldsymbol{O})$ if

$$\boldsymbol{X}^*\boldsymbol{V} + \boldsymbol{U}\boldsymbol{Y}^* + \boldsymbol{V}\boldsymbol{X}^* + \boldsymbol{Y}^*\boldsymbol{U} = \boldsymbol{O} \quad \text{and} \quad (\boldsymbol{U}, \boldsymbol{V}) \in \mathcal{F}_0.$$

∎

It is easy to see that the nondegeneracy condition is stronger than the strict complementarity condition. Moreover, the condition implies the uniqueness of the solution $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ of the monotone SDLCP (1) (see [11, Section 5]). Under the nondegeneracy condition, the limiting system of the AHO search direction (2) at the unique solution $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ is nonsingular.

Remark 2.6. The nondegeneracy condition (Condition 2.5) may seem to be rather strong or artificial for the AHO search direction. However, the condition is generic and natural. In [9, 11], Kojima, Shida and Shindoh used a different form of the nondegeneracy condition;

$$(\boldsymbol{U}, \boldsymbol{V}) = (\boldsymbol{O}, \boldsymbol{O}) \text{ if } [\boldsymbol{X}^*\boldsymbol{V} + \boldsymbol{U}\boldsymbol{Y}^* = \boldsymbol{O} \text{ and } (\boldsymbol{U}, \boldsymbol{V}) \in \mathcal{F}], \tag{3}$$

which is equivalent to Condition 2.5 (see [9, Lemma 6.3]). Note that the condition (3) only ensures that the set $\{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_+ \times \mathcal{S}_+ : \boldsymbol{X}\boldsymbol{Y} = \boldsymbol{O}\}$ of complementary pairs of positive semidefinite matrices transversally intersects the feasibility affine subspace $\mathcal{F}$ at the (unique) solution $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$.

For SDP, Haeberly pointed out that the nondegeneracy condition is equivalent to the combination of the primal-dual nondegeneracy condition introduced by Alizadeh, Haeberly and Overton [2] and the strict complementarity condition. ∎

Lemma 2.7 below is a key to estimate the norms of the AHO search direction and the higher order derivatives.

Lemma 2.7. *Assume that the nondegeneracy condition holds. Let $(\boldsymbol{X}, \boldsymbol{Y})$ be near $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$, and $\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}$ are symmetric matrices such that*

$$\max\left\{\|\boldsymbol{C}\|_F, \inf_{(D', E') \in \mathcal{F}_0} \{\|(\boldsymbol{D}, \boldsymbol{E}) - (\boldsymbol{D}', \boldsymbol{E}')\|_F\}\right\} = \tau.$$

Then the system

$$\begin{aligned} \boldsymbol{X}\boldsymbol{V} + \boldsymbol{U}\boldsymbol{Y} + \boldsymbol{V}\boldsymbol{X} + \boldsymbol{Y}\boldsymbol{U} &= \boldsymbol{C} \\ (\boldsymbol{U}, \boldsymbol{V}) &\in \mathcal{F}_0 + (\boldsymbol{D}, \boldsymbol{E}), \end{aligned} \tag{4}$$

has a solution $(\boldsymbol{U}, \boldsymbol{V})$, which is unique and $\|(\boldsymbol{U}, \boldsymbol{V})\|_F = \Theta(\tau)$.

Proof: It is shown by the nonsingularity of the system (4) at the solution $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ (the nondegeneracy condition) and the continuity of data. ∎

2.6 Higher Order Approximation

Let $(\boldsymbol{X}^k, \boldsymbol{Y}^k) \in \widetilde{\mathcal{N}}(\gamma^{(k)}, \theta^k)$ be a current iterate. We consider the higher order approximation of the solution system

$$\boldsymbol{XY} + \boldsymbol{YX} = \boldsymbol{O}, (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F} \tag{5}$$

of the monotone SDLCP (1) along the infeasible weighted central trajectory

$$\{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}(\tau\theta^k) \cap (\mathcal{S}_{++} \times \mathcal{S}_{++}) : \boldsymbol{XY} + \boldsymbol{YX} = \tau(\boldsymbol{X}^k\boldsymbol{Y}^k + \boldsymbol{Y}^k\boldsymbol{X}^k), \tau > 0\}$$

at $(\boldsymbol{X}^k, \boldsymbol{Y}^k)$.

By differentiating the system

$$\left.\begin{array}{c} \boldsymbol{XY} + \boldsymbol{YX} = (1-\alpha)(\boldsymbol{X}^k\boldsymbol{Y}^k + \boldsymbol{Y}^k\boldsymbol{X}^k), \\ (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}((1-\alpha)\theta^k), \end{array}\right\} \tag{6}$$

we obtain the system (2) of the AHO search direction $(d\boldsymbol{X}, d\boldsymbol{Y})$ with $\beta = 0$. Let

$$(\boldsymbol{X}^k(\alpha), \boldsymbol{Y}^k(\alpha))_1 = (\boldsymbol{X}^k, \boldsymbol{Y}^k) + \alpha(d\boldsymbol{X}, d\boldsymbol{Y})$$

denote a linear approximation of the solution system (5) of the monotone SDLCP (1) along the infeasible weighted central trajectory at $(\boldsymbol{X}^k, \boldsymbol{Y}^k)$.

Let p be an arbitrary positive integer greater than 1. Upto p-times differentiating the equation (6), the p-th order derivative $(d^p\boldsymbol{X}, d^p\boldsymbol{Y})$ is described by

$$\left.\begin{array}{c} \boldsymbol{X}^k d^p\boldsymbol{Y} + d^p\boldsymbol{X}\boldsymbol{Y}^k + \boldsymbol{Y}^k d^p\boldsymbol{Y}\boldsymbol{X}^k + \boldsymbol{Y}^k d^p\boldsymbol{X}^k = -\boldsymbol{R}_p(d^i\boldsymbol{X}d^{p-i}\boldsymbol{Y}, d^i\boldsymbol{Y}d^{p-i}\boldsymbol{X}) \\ (d^p\boldsymbol{X}, d^p\boldsymbol{Y}) \in \mathcal{F}_0, \end{array}\right\} \tag{7}$$

where $\boldsymbol{R}_p$ is an appropriate $n \times n$-symmetric matrix valued function which has $(2^{p+1} - 4)$ terms of $d^i\boldsymbol{X}d^{p-i}\boldsymbol{Y}$ and $d^i\boldsymbol{Y}d^{p-i}\boldsymbol{X}$ for $i = 1, \cdots, p-1$. For example,

$$\begin{array}{rcl} \boldsymbol{R}_2 & \equiv & 2(d\boldsymbol{X}d\boldsymbol{Y} + d\boldsymbol{Y}d\boldsymbol{X}) \\ \boldsymbol{R}_3 & \equiv & 3(d^2\boldsymbol{X}d\boldsymbol{Y} + d\boldsymbol{X}d^2\boldsymbol{Y} + d^2\boldsymbol{Y}d\boldsymbol{X} + d\boldsymbol{Y}d^2\boldsymbol{X}), \\ \boldsymbol{R}_4 & \equiv & 6(d^2\boldsymbol{X}d^2\boldsymbol{Y} + d^2\boldsymbol{Y}d^2\boldsymbol{X}) + 4(d^3\boldsymbol{X}d\boldsymbol{Y} + d\boldsymbol{X}d^3\boldsymbol{Y} + d^3\boldsymbol{Y}d\boldsymbol{X} + d\boldsymbol{Y}d^3\boldsymbol{X}). \end{array}$$

Note that, for any order of derivatives, the system (7) has the same coefficient, which is nonsingular in $\{(\boldsymbol{X}, \boldsymbol{Y}) \in \widetilde{\mathcal{N}}(\gamma, \theta) : \theta > 0\}$. By using the derivatives, we define the p-th order approximation $(\boldsymbol{X}^k(\alpha), \boldsymbol{Y}^k(\alpha))_p$ of solution system (5) of the monotone SDLCP (1) along the infeasible weighted central trajectory at $(\boldsymbol{X}^k, \boldsymbol{Y}^k)$ by

$$\begin{array}{l} (\boldsymbol{X}^k(\alpha), \boldsymbol{Y}^k(\alpha))_p = (\boldsymbol{X}^k(\alpha)_p, \boldsymbol{Y}^k(\alpha)_p) \\ = (\boldsymbol{X}^k, \boldsymbol{Y}^k) + \alpha(d\boldsymbol{X}, d\boldsymbol{Y}) + \dfrac{\alpha^2}{2}(d^2\boldsymbol{X}, d^2\boldsymbol{Y}) + \cdots + \dfrac{\alpha^p}{p!}(d^p\boldsymbol{X}, d^p\boldsymbol{Y}). \end{array} \tag{8}$$

By Lemma 2.8 below, we have

$$\boldsymbol{X}^k(\alpha)_p\boldsymbol{Y}^k(\alpha)_p + \boldsymbol{Y}^k(\alpha)_p\boldsymbol{X}^k(\alpha)_p = (1-\alpha)(\boldsymbol{X}^k\boldsymbol{Y}^k + \boldsymbol{Y}^k\boldsymbol{X}^k) + \boldsymbol{O}((\alpha\theta^k)^{p+1}), \tag{9}$$

where $\boldsymbol{O}(\tau)$ is a symmetric matrix whose Frobenius norm converges to zero with the same order as $\tau \searrow 0$.

Lemma 2.8. *Assume that the nondegeneracy condition holds. Let* $(\boldsymbol{X}^k, \boldsymbol{Y}^k) \in \widetilde{\mathcal{N}}(\gamma, \theta^k)$. *Then, we have that* $\|(d^p\boldsymbol{X}, d^p\boldsymbol{Y})\|_F = O((\theta^k)^p)$ *for all* $p > 1$.

Proof: It is easily shown by the induction with Lemmas 2.2 and 2.7. ∎

In this paper, the nondegeneracy condition is needed only to show Lemma 2.8. However, the condition might be relaxed to the strict complementarity condition, since Kojima, Shida and Shindoh showed Lemma 2.8 for $p = 1$ under the strict complementarity condition only ([9, Section 5]).

3 Higher Order Infeasible-Interior-Point Algorithm

Our higher order algorithm is completely based on the long-step predictor-corrector infeasible-interior-point algorithm proposed by Kojima, Shida and Shindoh [9]. Therefore, the globally convergent property and the locally quadratically convergent property (under the strict complementarity condition) were guaranteed by [9]. In Steps 4 and 5 of Algorithm 3.1 below, we use higher order approximation.

Starting from $(\boldsymbol{X}^0, \boldsymbol{Y}^0, \theta^0, \gamma^0) = (\sqrt{\mu^0}\boldsymbol{I}, \sqrt{\mu^0}\boldsymbol{I}, 1, 0)$, our algorithm generates a sequence $\{(\boldsymbol{X}^k, \boldsymbol{Y}^k, \boldsymbol{X}^k_c, \boldsymbol{Y}^k_c, \theta^k, \gamma^k)\}$ such that for every $k = 1, 2\cdots$,

$$\left.\begin{array}{l} 1 = \theta^0 > \theta^k >, \theta^{k+1} \geq 0, \\ 1 > \gamma > \gamma^k \geq 0, \\ (\boldsymbol{X}^k, \boldsymbol{Y}^k) \in \widetilde{\mathcal{N}}(\gamma^k, \theta^k), \\ (\boldsymbol{X}^k_c, \boldsymbol{Y}^k_c) \in \widetilde{\mathcal{N}}(\gamma, \theta^k). \end{array}\right\} \tag{10}$$

Algorithm 3.1. (Higher Order Infeasible-Interior-Point Algorithm)

Step 1: Choose an accuracy parameter $\epsilon \geq 0$, a neighborhood parameter $\gamma \in (0, 1)$ and an initial point $(\boldsymbol{X}^0, \boldsymbol{Y}^0) = (\sqrt{\mu^0}\boldsymbol{I}, \sqrt{\mu^0}\boldsymbol{I})$ with some $\mu^0 > 0$. Let $\theta^0 = 1$, $0 < \theta^* \ll 1$ and $\gamma^0 = 0$.

Step 2 (Predictor Step): Compute the AHO search direction $(d\boldsymbol{X}, d\boldsymbol{Y})$ by solving the system (2) with $\beta = 0$. Let

$$\left.\begin{array}{rcl} \delta^k & = & \dfrac{\|d\boldsymbol{X}\|_F\|d\boldsymbol{Y}\|_F}{\theta^k\mu^0}, \\ \hat{\alpha}^k & = & \dfrac{2}{\sqrt{1+4\delta^k/(\gamma-\gamma^k)}+1}, \\ \check{\alpha}^k & = & \max\left\{\alpha' \in [0,1] : \begin{array}{l} (\boldsymbol{X}^k(\alpha), \boldsymbol{Y}^k(\alpha))_1 \in \widetilde{\mathcal{N}}(\gamma, (1-\alpha)\theta^k) \\ \text{for every } \alpha \in [0, \alpha'] \end{array}\right\}. \end{array}\right\} \tag{11}$$

Choose a step length $\alpha^k_1 \in [\hat{\alpha}^k, \check{\alpha}^k]$ (the relation $0 < \hat{\alpha}^k \leq \check{\alpha}^k \leq 1$ was shown in [9, Lemma3.7] .

Step 3: If $\theta^k > \theta^*$, let $(\boldsymbol{X}^k_c, \boldsymbol{Y}^k_c) = (\boldsymbol{X}^k(\alpha^k_1), \boldsymbol{Y}^k(\alpha^k_1))_1$ and $\theta^{k+1} = (1-\alpha^k_1)\theta^k$, and goto Step 6. Otherwise goto Step 4.

Step 4 (Higher Order Predictor Step): Compute the derivatives upto p-th order along the infeasible weighted central trajectory at $(\boldsymbol{X}^k, \boldsymbol{Y}^k)$. Let

$$\alpha_p^k = \max\left\{\alpha' \in [0,1] : \begin{array}{l} (\boldsymbol{X}^k(\alpha), \boldsymbol{Y}^k(\alpha))_p \in \widetilde{\mathcal{N}}(\gamma, (1-\alpha)\theta^k) \\ \text{for every } \alpha \in [0, \alpha'] \end{array}\right\}.$$

Step 5: If $\alpha_1^k \geq \alpha_p^k$, let $\alpha^k = \alpha_1^k$, $(\boldsymbol{X}_c^k, \boldsymbol{Y}_c^k) = (\boldsymbol{X}^k(\alpha^k), \boldsymbol{Y}^k(\alpha^k))_1$ and $\theta^{k+1} = (1-\alpha^k)\theta^k$. Otherwise, let $\alpha^k = \alpha_p^k$, $(\boldsymbol{X}_c^k, \boldsymbol{Y}_c^k) = (\boldsymbol{X}^k(\alpha^k), \boldsymbol{Y}^k(\alpha^k))_p$ and $\theta^{k+1} = (1-\alpha^k)\theta^k$.

Step 6: If $\theta^{k+1} < \epsilon$, then stop. Otherwise goto Step 7.

Step 7 (Corrector Step): Compute the AHO search direction $(d\boldsymbol{X}, d\boldsymbol{Y})$ by solving (2) with $\beta = 1$. Let

$$\left.\begin{array}{rcl} \delta_c^k &=& \dfrac{\|d\boldsymbol{X}_c^k\|_F \|d\boldsymbol{Y}_c^k\|_F}{\theta^{k+1}\mu^0}, \\ (\hat{\alpha}_c^k, \check{\gamma}^{k+1}) &=& \begin{cases} \left(\dfrac{\gamma}{2\delta_c^k}, \gamma\left(1 - \dfrac{\gamma}{4\delta_c^k}\right)\right) & \text{if } \gamma \leq 2\delta_c^k, \\ (1, \delta_c^k) & \text{if } \gamma > 2\delta_c^k, \end{cases} \\ \hat{\gamma}^{k+1} &=& \min\left\{\gamma' \in [0,1] : \begin{array}{l} (\boldsymbol{X}_c^k + \alpha d\boldsymbol{X}, \boldsymbol{Y}_c^k + \alpha d\boldsymbol{Y}) \in \widetilde{\mathcal{N}}(\gamma', \theta^{k+1}), \\ \text{for } \alpha \in [0,1] \end{array}\right\}. \end{array}\right\} \tag{12}$$

Choose a step length $\alpha_c^k \in [0,1]$ and γ^{k+1} such that

$$\left.\begin{array}{l} \hat{\gamma}^{k+1} \leq \gamma^{k+1} \leq \check{\gamma}^{k+1}, \\ (\boldsymbol{X}_c^k + \alpha_c^k d\boldsymbol{X}, \boldsymbol{Y}_c^k + \alpha_c^k d\boldsymbol{Y}) \in \widetilde{\mathcal{N}}(\gamma^{k+1}, \theta^{k+1}). \end{array}\right\} \tag{13}$$

(By [9, Lemma 3.8], the pair $(\alpha_c^k, \gamma^{k+1}) = (\hat{\alpha}_c^k, \check{\gamma}^{k+1})$ satisfies the relations (13)) . Let $(\boldsymbol{X}^{k+1}, \boldsymbol{Y}^{k+1}) = (\boldsymbol{X}_c^k, \boldsymbol{Y}_c^k) + \alpha_c^k(d\boldsymbol{X}, d\boldsymbol{Y})$.

Step 8: Replace k by $k+1$. Goto Step 1. ∎

Note that Steps 4 and 5 do not affect the following properties shown in [9]:

- Algorithm 3.1 consistently generates a sequence $\{(\boldsymbol{X}^k, \boldsymbol{Y}^k, \boldsymbol{X}_c^k, \boldsymbol{Y}_c^k, \theta^k, \gamma^k)\}$ satisfying (10)[9, Theorem 2.3],
- for $\epsilon > 0$, Algorithm 3.1 stops in a finite number of iterates in Step 6 [9, Theorem 2.3],
- (for $\epsilon = 0$) under the strict complementarity condition, the gap θ to solutions of the monotone SDLCP (1) converges to zero quadratically [9, Theorem 5.1].
- under the nondegeneracy condition, we can take the unit step length $\alpha_c^k = 1$ in Step 6 (Corrector Step) for every sufficiently large k [9, Theorem 6.2].

Theorem 3.2. *Suppose that the nondegeneracy condition holds. Then the gap θ to solutions of the monotone SDLCP (1) converges zero Q-superlinearly with order $p+1$ by Algorithm 3.1 (for $\epsilon = 0$).*

To show Theorem 3.2, we need the following Lemmas.

Lemma 3.3. *Suppose that the nondegeneracy condition holds. Then there exists a positive constant $\bar{\gamma}$ such that*

$$0 \leq \gamma^k \leq \bar{\gamma} < \gamma \text{ for every } k = 0, 1, \cdots.$$

Proof: By [9, Lemma 5.7], $\delta_c^k = O(1)$. The assertion is derived by the definition (12) of $\check{\gamma}^k$. ∎

Lemma 3.4. *For any positive c, the polynomial $p(x) = (1-x) - cx^n$ has only one positive real root x_+, which is less than 1. Moreover, $x_+ \in (1-c, 1)$ for $c \in (0,1)$. ($1 - x_+$ converges to 0 with the same order as $c \searrow 0$.)*

Proof: Since $Dp(x) = -1 - cnx^{n-1} < 0$ for any nonnegative x, the function p is strictly decreasing on $[0, \infty)$. Therefore, the function p have only one positive real root x_+, which is in $(0,1)$ together with the fact that $p(0) = 1 > 0$ and $p(1) = -c < 0$. For $c \in (0,1)$, we have that $p(1-c) = c(1-(1-c)^n) > 0$ and $p(1) = -c < 0$. Hence, we have $x_+ \in (1-c, 1)$. ($1 - x_+ = cx_+^{p+1} = \Theta(c)$.) ∎

Proof of Theorem 3.2: To show Theorem 3.2, we have only to show that $1 - \alpha^k = O((\theta^k)^p)$ for any sufficiently large k, since $\theta^{k+1} = (1-\alpha^k)\theta^k$. For the purpose, we shall show that $1 - \alpha_p^k = O((\theta^k)^p)$. By Lemmas 2.8 and 3.3, we have that

$$\begin{aligned} &\boldsymbol{X}^k(\alpha)_p \boldsymbol{Y}^k(\alpha)_p + \boldsymbol{Y}^k(\alpha)_p \boldsymbol{X}^k(\alpha)_p - 2(1-\gamma)(1-\alpha)\theta^k\mu^0 \boldsymbol{I} \\ &= (1-\alpha)(\boldsymbol{X}^k\boldsymbol{Y}^k + \boldsymbol{Y}^k\boldsymbol{X}^k) + \boldsymbol{O}((\alpha\theta^k)^{p+1}) - 2(1-\gamma)(1-\alpha)\theta^k\mu^0 \boldsymbol{I} \\ &\succeq 2(1-\alpha)(\gamma - \bar{\gamma})\theta^k\mu^0 \boldsymbol{I} - \boldsymbol{O}((\alpha\theta^k)^{p+1}), \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\langle \boldsymbol{X}^k(\alpha)_p, \boldsymbol{Y}^k(\alpha)_p \rangle - n(1-\alpha)(1+\zeta\gamma)\theta\mu^0 \\ &\leq n(1-\alpha)(1+\zeta\gamma^k)\theta\mu^0 + O((\alpha\theta^k)^{p+1}) - n(1-\alpha)(1+\zeta\gamma)\theta\mu^0 \\ &\leq -n(1-\alpha)\zeta(\gamma - \bar{\gamma})\theta\mu^0 + O((\alpha\theta^k)^{p+1}). \end{aligned} \tag{15}$$

Therefore, by Lemma 3.4, we can take $(1 - \alpha_p^k) = O((\theta^k)^p)$ to keep the left hand side matrix of (14) positive definite and the left hand side of (15) negative, *i.e.*, to keep the p-th order approximation $(\boldsymbol{X}^k(\alpha), \boldsymbol{Y}^k(\alpha))_p$ in $\widetilde{\mathcal{N}}(\gamma, (1-\alpha)\theta^k)$. ∎

In practice, an exact computation of α_p^k is not possible. Since we only use the higher order approximation near the solution $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$, we can guess the upperbound of the coefficient of $(\alpha\theta)^{p+1}$ in (9) (and therefore (14) and (15)) from the current point $(\boldsymbol{X}^k, \boldsymbol{Y}^k)$, and therefore we can choose an inexact α_p^k so that $1 - \alpha_p^k = O((\theta^k)^p)$ by using Lemma 3.4 with $x_+ = \alpha_p^k$.

4 Concluding Remarks

In this paper, we develop the higher order predictor-corrector infeasible-interior-point algorithm for the monotone SDLCP (1) along the infeasible weighted central trajectory induced by the AHO search directions. The result gives the theoretical background of the practical advantage of the AHO search direction and the numerical good performance of higher order implementation "Mehrotra's higher order corrections" reported by Alizadeh, Haeberly and Overton [2] and Haeberly, Nayakkankuppam and Overton [6].

To establish the higher order convergence of our higher order method, we need to impose the nondegeneracy condition. However, Kojima, Shida and Shindoh [9] proved the quadratically convergent property of their first order algorithm under the strict complementarity condition only by showing that the norm of the AHO search direction $(d\boldsymbol{X}, d\boldsymbol{Z})$ converges to zero with the same order as $\theta \searrow 0$ (see Lemma 2.8). Hence, the higher order convergence may be shown under the strict complementarity condition only.

We close the paper by listing two other open topics in the field:

- globally polynomial-time first order algorithm (using the AHO search direction) with the locally quadratically convergent property under the strict complementarity condition or the nondegeneracy condition. Globally polynomial-time feasible interior-point algorithms with families of search directions which contains the AHO search direction were proposed by Monteiro [15] and Kojima, Shida and Shindoh [10], while the infeasible interior-point algorithm using the AHO search direction proposed by Kojima, Shida and Shindoh [9] possesses the locally quadratically convergent property (our algorithm is based on the algorithm).

- p-th order algorithm using non-AHO search direction with $(p+1)$-th order convergence under the nondegeneracy condition or the strict complementarity condition. The case $p = 1$ is still open, in my knowledge. The superlinear convergence of first order algorithms using the HRVW/KSH/M search direction [7, 12, 14] was discussed in [11, 22]. To ensure the superlinear convergence, the additional condition "tangential convergence of sequence $\{(\boldsymbol{X}^k, \boldsymbol{Y}^k)\}$ to the central surface $\{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_{++} \times \mathcal{S}_{++} : \boldsymbol{XY} = \mu\boldsymbol{I}, \mu > 0\}$" is needed ([11, Section 6.2]). But the condition implies that we shall asymptotically take the AHO search direction (note that almost all the proposed search directions correspond if the current point lies on the central surface, see [10, 23]).

References

[1] F. Alizadeh, J.-P.A. Haeberly, and M.L. Overton. Primal-dual interior-point methods for semidefinite programming. Technical Report 659, Computer Science Department, Courant Institute of Mathematical Sciences, New York University, 1994.

[2] F. Alizadeh, J.-P.A. Haeberly, and M.L. Overton. Complementarity and nondegeneracy in semidefinite programming. *SIAM Journal on Optimization, 8*, pages 746–768, 1998.

[3] T.J. Carpenter, I.J. Lustig, J.M. Mulvcey, and D.F. Shanno. Higher-order predictor-corrector interior-point methods with application to quadratic objectives. *SIAM Journal on Optimization, 3*, pages 696–725, 1993.

[4] J. Gonzio. Multiple centrality corrections in a primal-dual method for linear programming. *Computational Optimization and Applications, 6*, pages 137–156, 1996.

[5] J. Ding and T.Y. Li. An algorithm based on weighted logarithmic barrier functions for linear complementarity problems. *Arabian Journal for Science and Engineering, 15*, pages 679–685, 1990.

[6] J.-P. Haeberly, M.V. Nayakkankuppam, and M.L. Overton. Extending Mehrotra and Gonzio higher order methods to mixed semidefinite-quadratic-linear programming. to appear in Optimization Methods and Software, 1998.

[7] C. Helmberg, F. Rendl, R.J. Vanderbei, and H. Wolkowicz. An interior-point method for semidefinite programming. *SIAM Journal on Optimization, 6*, pages 342–361, 1996.

[8] D. den Hertog, C. Roos, and T. Terlaky. A sufficient condition for self-concordance, with application to some class of structured convex programming problems. *Journal of Information and Optimization Sciences, 12*, pages 187–205, 1991.

[9] M. Kojima, M. Shida, and S. Shindoh. A predictor-corrector interior-point algorithm for the semidefinite linear complementarity problem using the Alizadeh-Haeberly-Overton search direction. *SIAM Journal on Optimization, 9*, pages 444–465, 1999.

[10] M. Kojima, M. Shida, and S. Shindoh. Search directions in the SDP and the monotone SDLCP: Generalization and inexact computation. *Mathematical Programming, 85*, pages 51-80, 1999.

[11] M. Kojima, M. Shida, and S. Shindoh. Local convergence of predictor-corrector infeasible-interior-point algorithms for the SDPs and the monotone SDLCPs. *Mathematical Programming, 80*, pages 129–160, 1998.

[12] M. Kojima, S. Shindoh, and S. Hara. Interior-point methods for the monotone semidefinite linear complementarity problem in symmetric matrices. *SIAM Journal on Optimization, 7*, pages 86–125, 1997.

[13] S. Mehrotra. On the implementation of a primal-dual interior point method. *SIAM Journal on Optimization, 2*, pages 575–601, 1992.

[14] R.D.C. Monteiro. Primal-dual path-following algorithms for semidefinite programming. *SIAM Journal on Optimization, 7*, pages 663–678, 1997.

[15] R.D.C. Monteiro. Polynomial convergence of primal-dual algorithms for semidefinite programming based on Monteiro and Zhang family of directions. *SIAM Journal on Optimization, 8*, pages 797–812, 1998.

[16] R.D.C. Monteiro, I. Adler, and M.G.C. Resende. A polynomial-time primal-dual affine scaling algorithm for linear and convex quadratic programming and its power series extension. *Mathematics of Operations Research, 15*, pages 191–214, 1990.

[17] R.D.C. Monteiro and Jong-Shi Pang. On two interior-point mappings for nonlinear semidefinite complementarity problems. *Mathematics of Operations Research, 23*, pages 39–60, 1998.

[18] R.D.C. Monteiro and T. Tsuchiya. Polynomial convergence of a new family of primal-dual algorithms for semidefinite programming. Technical Report 627, The Institute of Statistical Mathematics, Tokyo, to appear in SIAM Journal on Optimization, 1996.

[19] R.D.C. Monteiro and T. Tsuchiya. Polynomiality of primal-dual algorithms for semidefinite linear complementarity problems based on the Kojima-Shindoh-Hara family of directions. *Mathematical Programming, 84*, pages 39–53, 1999.

[20] R.D.C. Monteiro and P. Zanjacomo. A note on the existence of the Alizadeh-Haeberly-Overton direction for semidefinite programming. *Mathematical Programming, 78*, pages 393–396, 1996.

[21] Yu.E. Nesterov and M.J. Todd. Primal-dual interior-point methods for self-scaled cones. *SIAM Journal on Optimization, 8*, 1998.

[22] F.A. Potra and R. Sheng. A superlinearly convergent primal-dual infeasible-interior-point algorithm for semidefinite programming. *SIAM Journal on Optimization, 8*, pages 1007–1028, 1998.

[23] M. Shida, S. Shindoh, and M. Kojima. Existence and uniqueness of search directions in interior-point-algorithms for the SDP and the monotone SDLCP. *SIAM Journal on Optimization, 8*, pages 387–396, 1998.

[24] J. Stoer, M. Wechs, and S. Mizuno. High order infeasible-interior-point methods for solving sufficient linear complementarity problems. *Mathematics of Operations research, 23*, pages 832–862, 1998.

[25] J.F. Sturm. Superlinear convergence of an algorithm for monotone linear complementarity problems, when no strictly complementarity solution exists. Technical Report 9656/A, Econometric Institute, Erasmus University Rotterdam, The Netherlands, 1995.

[26] M.J. Todd. A Study of Search Directions in Primal-Dual Interior-Point Methods for Semidefinite Programming. Technical report, School of Operations Research and Industrial Engineering, Cornell University, USA, 1997.

[27] M.J. Todd, K.C. Toh, and R.H. Tütüncü. On the Nesterov-Todd direction in semidefinite programming. *SIAM Journal on Optimization, 8*, pages 769–796, 1998.

[28] Y. Zhang. On the convergence of a class of infeasible interior-point algorithms for the horizontal linear complementarity problem. *SIAM Journal on Optimization, 4*, pages 208–227, 1994.