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Characterisations of Node–Search Antimatroids of Directed and Undirected Graphs

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Abstract

An antimatroid arises from various kinds of 'shellings' and 'searches': typical examples are poset shelling, node/edge shelling of a tree, node search of a directed/undirected graph etc. We shall present the forbidden-minor characterizations of node-search antimatroids of directed and undirected graphs. It is shown that an antimatroid is given as a node-search antimatroid on a directed graph if and only if it contains no minor isomorphic to a lattice D_5 where D_5 is a lattice of five elements $\emptyset$, $\{x\}$, $\{y\}$, $\{x,y\}$, $\{x,y,z\}$. It is also shown that an antimatroid is a node-search antimatroid of an undirected graph if and only if it does not contain D_5 nor S_{10} as a minor.

1 Introduction

Let E denote a nonempty finite set and $\mathbb{F}$ a family of subsets of E. $\mathbb{F}$ is called an *antimatroid* if it satisfies

(A1) $\emptyset \in \mathbb{F}$, [nonemptiness]

(A2) if $X \in \mathbb{F}$ and $X \neq \emptyset$, then $X \setminus e \in \mathbb{F}$ for some $e \in X$, [accessibility]

(A3) if $X, Y \in \mathbb{F}$ then $X \cup Y \in \mathbb{F}$. [closed under union]

The sets in $\mathbb{F}$ are called *feasible sets.* As is easily seen, $\mathbb{F}$ constitutes a semimodular lattice with respect to inclusion relation.

A chain of sets $A_0 \subset A_1 \subset \cdots \subset A_k$ is called *elementary* if every difference set is a singleton, i.e. $|A_i| = |A_{i-1}| + 1$ for $i = 1, \ldots, k$.

The condition (A2) of the axiom set is eqiuvalent to $(A2')$.

(A2') for any $X \in \mathbb{F}$, there exists an elementary chain of feasible sets from $\emptyset$ to X.

For a feasible set $X \in \mathbb{F}$, take an elementary chain $\emptyset = X_0 \subset X_1 \subset \ldots \subset X_k = X$ as in $(A2')$, and let $\{x_j\} = X_j \setminus X_{j-1}$ for $j = 1, \ldots, k$. Then the sequence $x_1 x_2 \cdots x_k$ of the elements of X is called a *feasible ordering.* In general, a feasible set may have a multiple number of feasible orderings.

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Take a feasible set $A \in \mathbb{F}$. Then $\mathbb{F}|A = \{X \subseteq A : X \in \mathbb{F}\}$ is an antimatroid on A, called a restriction to A, and $\mathbb{F}/A = \{X - A : X \in \mathbb{F}, A \subseteq X\}$ is an antimatroid on $E \setminus A$, called a contraction of A. And for $A, B \in \mathbb{F}$ with $A \subseteq B$,

$$(\mathbb{F}|B)/A = \{X \subseteq B \setminus A \ : \ A \cup X \in \mathbb{F}\}$$

is called a *minor* of $\mathbb{F}$.

If a class of antimatroids is closed under taking minors, we can characterize it by counting up all its forbidden minimal minors. For instance, an antimatroid is a poset shelling antimatroid if and only if it does not contain S_7 as a minor where $S_7 = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

A feasible set $X \in \mathbb{F}$ is called a *path set* if there exists uniquely an element $e \in X$ such that $X \setminus e \in \mathbb{F}$. In terms of lattice theory, a path set is equal to a join-irreducible element of the lattice $\mathbb{F}$.

Lemma 1 Suppose X to be a path set of an antimatroid $\mathbb{F}$. Let $A \in \mathbb{F}$ and $A \subseteq X$. Then $X \setminus A$ is a path set of $\mathbb{F}/A$.

(Proof) It follows from $(A2')$. □

2 Node-search Antimatroids of Directed Graphs

Let $G = (V \cup \{r\}, E)$ be a directed graph with a distinguished node r ($\notin V$) called a *root*. We shall call it a *rooted graph*. A node is called an *atom* if there is an edge from the root. An *r-path* of G is an elementary directed path which starts from the root. An r-path in an undiredted graph is similarly defined.

For an r-path $P = rv_1 \cdots v_k$ where $v_i \in V$ and $(v_{i-1}, v_i) \in E$ for $i = 1, \ldots, k$, we let $\partial P = \{v_1, \ldots, v_k\}$. The family of sets given by

$$\mathbb{F} = \{X \subseteq V \ : \ X = \bigcup_{j=1}^{m} \partial P_j \text{ and } \{P_1, \ldots, P_m\} \text{ is an arbitrary family of r-paths of } G\ \} \tag{1}$$

$$= \{X \subseteq V \ : \ \text{There exists a directed tree rooted at } r \text{ whose vertex set is } X \cup r\ \} \tag{2}$$

constitutes an antimatroid on V, called a *node-search antimatroid* of a directed graph G. The node-search antimatroid of an undirected graph is similarly defined replacing 'directed' with 'undirected' in the above.

Let us denote by $\mathfrak{NS}_D$ the class of node-search antimatroids of directed graphs, and by $\mathfrak{NS}_{UND}$ the class of those of undirected graphs. Both classes of $\mathfrak{NS}_D$ and $\mathfrak{NS}_{UND}$ are closed under taking minor.

In a rooted directed gaph G, an edge is called *redundant* if there is no r-path which contains it and is free of short-cuts. G is called *nonredundant* if it has no redundant edges. Actually, redundant edges are of no use in defining node-search antimatroids of graphs. Obviously, if a rooted graph is nonredundant, there is no in-edge to the root, and every atom has a unique in-edge which comes from the root.

Let $G = (V \cup \{r\}, E)$ be a rooted diredted graph, and $\mathbb{F}$ be its node-search antimatroid. For $A, B \in \mathbb{F}$ with $A \subseteq B$, we define an *r-minor* graph of G as follows: First delete nodes in $V \setminus B$ from G, and shrink the node set $A \cup \{r\}$ to a new root r'. Then delete the in-edges to r' and the in-edges to atoms which comes from nodes other than r'. We denote by $G[A, B]$ the resultant rooted directed graph, and call it an r-minor of G. An r-minor graph is necessarily nonredundant. Clearly, the node-search antimatroid of $G[A, B]$ is equal to the minor $(\mathbb{F}|B)/A$.

Furthermore, suppose G' to be another rooted directed graph and $\mathbb{F}'$ to be its node-search antimatroid. Then $\mathbb{F}$ contains a minor isomorphic to $\mathbb{F}'$ if and only if there is an r-minor graph of G which is isomorohic to G' under a isomorphism mapping a root to another root.

Let $D_5 = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \{x, y, z\}\}$ be an antimatroid on a three-element set $\{x, y, z\}$. It is easy to check that D_5 is not in $\mathfrak{NS}_D$. Since the class $\mathfrak{NS}_D$ is closed under taking minors, containing no minor isomorphic to D_5 is a trivial necessary condition for an antimatroid to belong to $\mathfrak{NS}_D$. We shall show below that this is also sufficient.

In the following lemmas and arguments, we suppose that $\mathbb{F}$ is an antimatroid on a finite set V, and does not contain D_5 as a minor.

Lemma 2 For a path set X of $\mathbb{F}$, there exist a unique feasible ordering of the elements, say $x_1 \cdots x_k$, and $\{x_1, \ldots, x_i\}$ is a path set of $\mathbb{F}$ for each $i = 1, \ldots, k$.

(Proof) Otherwise, $\mathbb{F}$ would contain D_5 as a minor. □

From the path sets of $\mathbb{F}$, we shall construct a rooted directed graph, denoted by $G[\mathbb{F}]$, so that the unique ordering of each path set of $\mathbb{F}$ becomes a directed path in $G[\mathbb{F}]$. More precisely, the vertex set of $G[\mathbb{F}]$ is $V \cup \{r\}$, and for each path set A of $\mathbb{F}$ with its unique feasible ordering $a_1 a_2 \cdots a_n$, we add an edge (r, a_1) and edges (a_i, a_{i+1}) $(i = 1, \ldots, n-1)$ to $G[\mathbb{F}]$. By definitin, a path in $G[\mathbb{F}]$ which arises from a path set of $\mathbb{F}$ is elementary and free of short-cuts.

We first state two observations as lemmas below.

Lemma 3 $G[\mathbb{F}]$ is nonredundant.

The following is a crucial property of the antimatroids containing no minor isomorphic to D_5, and it is a key lemma for Theorem 1.

Lemma 4 Let a, b, $x_1, \ldots, x_n$ $(n \geq 2)$ be distinct elements of V. And suppose that $A_i = \{a, x_1, \ldots, x_i\}$ $(0 \leq i \leq n-1)$ and $B_i = \{b, x_1, \ldots, x_i\}$ $(0 \leq i \leq n)$ are feasible sets of $\mathbb{F}$, and B_n is a path set of $\mathbb{F}$. Then $A_n = \{a, x_1, \ldots, x_n\}$ is a feasible set of $\mathbb{F}$.

(Proof) By $(A3)$, we have $C_j = \{a, b, x_1, \ldots, x_j\}$ is a feasible set of $\mathbb{F}$ for $j = 1, \ldots, n$. Since $[A_{n-2}, C_n] \supseteq \{C_n, C_{n-1}, C_{n-2}, A_{n-1}, A_{n-2}\}$ and at the same time $[A_{n-2}, C_n]$ must not be isomorphic to D_5, either $A_{n-1} \cup x_n \in \mathbb{F}$ or $C_{n-2} \cup x_n \in \mathbb{F}$ holds. In the first case of $A_{n-1} \cup x_n \in \mathbb{F}$, we have $A_n = A_{n-1} \cup (A_{n-1} \cup x_n)$, which completes the proof. In the latter case of $C_{n-2} \cup x_n \in \mathbb{F}$, we have either $A_{n-2} \cup x_n \in \mathbb{F}$ or $C_{n-3} \cup x_n \in \mathbb{F}$ by similar argument. If $A_{n-2} \cup x_n \in \mathbb{F}$, then $A_n = A_{n-1} \cup (A_{n-2 \cup x_n})$ follows and the proof is completed. And in case of $C_{n-3} \cup x_n \in \mathbb{F}$, we can repeat the above argument until we have either $A_0 \cup x_n = \{a, x_n\} \in \mathbb{F}$ or $C_0 \cup x_n = \{a, b, x_n\} \in \mathbb{F}$. If $\{a, x_n\} \in \mathbb{F}$, then $A_n = A_{n-1} \cup \{a, x_n\}$ readily follows. And if not, $\{a, b, x_n\} \in \mathbb{F}$ holds and this implies $B_0 \cup x_n = \{b, x_n\} \in \mathbb{F}$ since otherwise $[\emptyset, \{a, b, x_n\}]$ would be isomorphic to D_5. Then $B_n - \{x_{n-1}\} = B_{n-2} \cup x_n = B_{n-2} \cup \{\} \in \mathbb{F}$ By assumption, $B_n - \{x_n\} = B_{n-1} \in \mathbb{F}$. But this contradicts the assumption that B_n is a path set. Hence the proof is completed. □

Theorem 1 Let $\mathbb{F}$ be an antimatroid containing no minor isomorphic to D_5. Let $G[\mathbb{F}]$ be the rooted directed graph defined from the family of all the path sets of $\mathbb{F}$, and $\mathbb{F}(G[\mathbb{F}])$ denote the node-search antimatroid of the graph $G[\mathbb{F}]$. Then

$$\mathbb{F}(G[\mathbb{F}]) = \mathbb{F}$$

(Proof of Theorem 1)
Take a feasible set $A \in \mathbb{F}$ such that $A \neq \emptyset$. Since any element in a lattice is a union of join-irreducible elements and a join-irreducible element of the lattice of $\mathbb{F}$ is equal to a path set, there exist path sets $A_1, \dots, A_m$ such that $A = A_1 \cup \dots \cup A_n$. Since each path set A_i corresponds to a rooted path in $G[\mathbb{F}]$, A is a feasible set of a node-search antimatroid of $G[\mathbb{F}]$, i.e. $A \in \mathbb{F}(G[\mathbb{F}])$. Hence we have $\mathbb{F} \subseteq \mathbb{F}(G[\mathbb{F}])$.

Conversely, we shall show $\mathbb{F}(G[\mathbb{F}]) \subseteq \mathbb{F}$. Any feasible set of $\mathbb{F}(G[\mathbb{F}])$ is a join of vertex sets of paths of $G[\mathbb{F}]$ without short-cuts. Hence, it is sufficient to show that $A = \{a_1, \dots, a_n\}$ is a feasible set of $\mathbb{F}$ for any short-cut-free path $P = ra_1 \cdots a_n$ in $G[\mathbb{F}]$,

Suppose that P is a minimal path for which the assrtion fails to hold. Hence we have $A_i = \{a_1, \dots a_i\} \in \mathbb{F}$ $(i = 1, \dots, n-1)$ and $A(= A_n) = \{a_1, \dots a_n\} \notin \mathbb{F}$. By definition, there exists a path $Q = rb_1 \cdots b_m$ in $G[\mathbb{F}]$ such that the final edge of Q is equal to (a_{n-1}, a_n), that is, $a_{n-1} = b_{m-1}$ and $a_n = b_m$. By Lemma 2, $B_i = \{b_1, \dots, b_i\}$ is a path set for $i = 1, \dots, m$. By assumption, there exist $s \geq 2$ such that $a_{n-s} \neq b_{m-s}$ and $a_{n-j} = b_{m-j}$ for $j = 0, 1, \dots, s-1$. Since P and Q are shot-cut-free paths, we have $n - s \geq 1$ and $m - s \geq 1$. And a_{n-s} is not on Q, and b_{m-s} is not on P.

Let $X = A_{n-s-1} \cap B_{m-s-1}$ and $\mathbb{F}' = \mathbb{F}/X$. Then $A'_i = A_{n-s+i} \setminus A_{n-s-1}$ and $B'_i = B_{n-s+i} \setminus B_{n-s-1}$ for $(i = 0, 1, \dots, s)$ are feasible sets of $\mathbb{F}'$, and B'_n is a path set of $\mathbb{F}'$. Hence by Lemma 4, we have $A'_s \in \mathbb{F}'$, which implies $A = A'_s \cup X \in \mathbb{F}$. This completes the proof. □

From Theorem 1, we readily have

Corollary 1 A neccessary and sufficient condition for an antimatroid to be a node-search antimatroid of a rooted directed graph is that it has no minor isomorphic to D_5.

3 Node-search Antimatroids of Undirected Graphs

A node-search antimatroid of an undirected graph is a special case of those of directed graphs. In fact, if we are given an undirected graph, replacing each undirected edge with a pair of directed edges with reverse orientations gives a directed graph whose node-search antimatroid is the same with that of the undiredted graph. Hence $\mathfrak{NS}_{UND} \subseteq \mathfrak{NS}_D$, and D_5 is also a forbidden minor for the class $\mathfrak{NS}_{UND}$.

Now let us consider a rooted directed graph $G_4 = (V_4 \cup r,\ E)$ such that

$$V_4 = \{1, 2, 3, 4\}, \tag{3}$$
$$E = \{\ (r,1),\ (r,2),\ (1,3),\ (2,4),\ (3,4)\ \}, \tag{4}$$

and let $S_{10} \in \mathfrak{NS}_D$ be the node-search antimatroid of G_4, which is described as

$$S_{10} = \{\ \emptyset,\ \{1\},\ \{2\},\ \{1,2\},\ \{1,3\},\ \{2,4\}, \tag{5}$$
$$\{1,2,3\},\ \{1,3,4\},\ \{1,2,4\},\ \{1,2,3,4\}\ \}. \tag{6}$$

It is a routine to check that S_{10} cannot be realized as a node-search antimatroid of an undirected graph, and is minimal with respect to this property.

Hence S_{10} is another forbidden minor of $\mathfrak{NS}_{UND}$. And we can further show

Theorem 2 Let $\mathbb{F}$ be an antimatroid containing no minor isomorphic to D_5, and $G = G[\mathbb{F}]$ be an nonredundant directed graph defined from the path sets of $\mathbb{F}$. Let G^0 be an undirected graph which is defined from G by considering each directed edge as an undirected one, and $\mathbb{F}^0$ denote the node-search antimatroid of the undiredted graph G^0. The the following are equivalent.

(1) $\mathbb{F}^0 = \mathbb{F}$,

(2) $\mathbb{F}$ does not contain S_{10} as a minor,

(3) $G[\mathbb{F}]$ does not contain G_4 as an r-minor graph.

(Proof) (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

We shall show that (3) implies (1). $\mathbb{F} \subseteq \mathbb{F}^0$ is obvious from the definition. We shall show the opposite inclusion $\mathbb{F}^0 \subseteq \mathbb{F}$. Take any undirected path $P^0 = ra_1 \cdots a_n$ $(n \geq 1)$ in G^0 without an (undirected) short-cut. Then it is sufficient to show that $A = \{a_1, a_2, \ldots, a_n\}$ is a feasible set of $\mathbb{F}$. If (a_{i-1}, a_i) is an edge in $E(G)$ for each $i = 1, \ldots, n$, then $ra_1 \cdots a_n$ is a directed path of G and $A = \{a_1, a_2, \ldots, a_n\} \in \mathbb{F}$ is obvious. Otherwise, let k be the smallest index such that $(a_{k-1}, a_k) \notin E(G)$ $(1 \leq k \leq n)$, and P be the directed path $ra_1 \cdots r_k$ in G.. Since G is nonredundant, we have $k \geq 3$. And $\{a_{k-1}, a_k\}$ is an undirected edge in G^0. Hence there exists a path set $B = \{b_1, \ldots, b_m\}$ of $\mathbb{F}$ with its unique ordering $b_1 \cdots b_m$ and $(b_{m-1}, b_m) = (a_k, a_{k-1}) \in E(G)$ If the size m of B is two, then the edge (r, b_1) is a short-cut of P^0, a contradiction. Hence $m \geq 3$ holds.

Let $X = \{a_1, \ldots, a_{k-3}\} \cup \{b_1, \ldots, b_{m-3}\}$ and $Y = A \cup B$. And let $G[X, Y]$ denote the associated r-minor graph of G. Since the path $Q = rb_1b_2 \cdots b_m$ is free of short-cuts, we have $b_{m-2} \neq a_{k-2}$. Hence the set of nodes of $G[X, Y]$ consists of r, a_{k-2}, $a_{k-1}(= b_m)$, $b_{m-1}(= a_k)$ and b_{m-2}. By definition of r-minor, there is no in-edge to a_{k-2} nor to b_{m-2}. Since P and Q do not have short-cuts, the edges (r, b_{m-1}), (r, a_{k-1}) and (b_{m-2}, b_m) do not exist in $G[X, Y]$. And $G[X, Y]$ does not have an edge (a_{k-2}, a_k) since the undirected path P^0 has no short-cut.

Hence the edge set of $G[X, Y]$ consists of (r, a_{k-2}), (a_{k-2}, a_{k-1}), (r, b_{m-2}), (b_{m-2}, b_{m-1}) and (b_{m-1}, a_{k-1}), and $G[X, Y]$ is shown to be isomorphic to G_4, which is a contradiction. This completes the proof. □

We can rewrite the theorem as

Corollary 2 A necessary and sufficient condition for an antimatroid to be a node-serach antimatroid of an undirected graph is that it contains neither D_5 nor S_{10} as a minor.

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