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# ON $(p,\omega)$-PRECISE FUNCTIONS WHOSE DERIVATIVES ARE SINGULAR INTEGRALS

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**1. Theorems.** As kernels of potentials we shall be concerned only with Riesz and Bessel kernels in $\mathbb{R}^d, d \geq 2$. We write $k_a(x) = |x|^{a-d}$ for $0 < a < d$ and $U_a^f = k_a * f$ for a function $f$ in case the potential is well-defined.

Let $\alpha$ be $(\alpha_1, ..., \alpha_d)$ with integers $\alpha_i \geq 0$, and set $|\alpha| = \alpha_1 + \cdots + \alpha_d$. We shall call $\alpha$ an index of order $|\alpha|$. For a function $f$ we write $D^\alpha f = D_x^\alpha f$ for $\partial^{|\alpha|} f / \partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}$ when this has a meaning. In case $\alpha = (0, ..., 0)$ we write $\alpha = 0$ and let $D^0 f$ mean $f$.

Let $d \geq 2$, $1 < p < \infty$, $\Gamma$ be a family of locally rectifiable curves in $\mathbb{R}^d$ and $\omega$ a weight. We take the definition of extremal length $\lambda_p(\Gamma; \omega)$ for granted. A function $f$ is called $(p, \omega)$-precise in an open set $G$ if the extremal length of the family of locally rectifiable curves in $G$ along each of which $f$ is not absolutely continuous is infinite and $\int_G |\operatorname{grad} f|^p \omega dx$ is finite.

We announce

**Theorem 1.** *Let $d \geq 2$, $1 < p < \infty$, $\omega$ be a weight in $\mathbb{R}^d$ satisfying Muckenhoupt's $A_p$ condition, $f \in L^{p,\omega}(\mathbb{R}^d)$ and $\alpha$ be an index of order $|\alpha| \geq 0$. Then writing $K^{i,\alpha}$ for $\partial D^\alpha k_{1+|\alpha|} / \partial x_i$, we see that*

$$\lim_{r \to 0} \int_{|x-y|>r} K^{i,\alpha}(x-y) f(y) dy$$

*exists in $L^{p,\omega}(\mathbb{R}^d)$. We denote this limit by $T^{i,\alpha} f$. If, in addition, $\int_{\mathbb{R}^d} (1+|x|)^{1-d} |f(x)| dx < \infty$, then $D^\alpha k_{1+|\alpha|} * f$ is $(p, \omega)$-precise, the relation*

$$\frac{\partial D^\alpha k_{1+|\alpha|} * f}{\partial x_i} = T^{i,\alpha} f$$

*holds a.e. in $\mathbb{R}^d$ for each $i$, and*

$$\| \operatorname{grad}(D^\alpha k_{1+|\alpha|} * f) \|_{p,\omega} \leq \text{const.}\, \|f\|_{p,\omega}$$

*is valid.*

Next for $a > 0$ we consider the Bessel kernel

$$G_a(r) = \frac{1}{(4\pi)^{a/2}\Gamma(a/2)} \int_0^\infty e^{-\pi|x|^2/t} e^{-t/(4\pi)} t^{(a-d)/2} \frac{dt}{t}.$$

**Theorem 2.** *Let $d, p, \omega, f, \alpha$ be the same as in Theorem 1. Writing $K_{i,\alpha}$ for $\partial D^\alpha G_{1+|\alpha|} / \partial x_i$, we see that*

$$\lim_{r\to 0} \int_{|x-y|>r} K_{i,\alpha}(x-y) f(y) dy$$

*exists in $L^{p,\omega}(\mathbb{R}^d)$. Denote this limit by $T_{i,\alpha} f$. Then $D^\alpha G_{1+|\alpha|} * f$ is $(p,\omega)$-precise, the relation*

$$\frac{\partial D^\alpha G_{1+|\alpha|} * f}{\partial x_i} = T_{i,\alpha} f + a_i f$$

*holds with some constant $a_i$ a.e. in $\mathbb{R}^d$ for each $i$, and*

$$\| \operatorname{grad}(D^\alpha G_{1+|\alpha|} * f) \|_{p,\omega} \leq \text{const.} \|f\|_{p,\omega}$$

*is valid.*

**2. Proof.** We shall prove only Theorem 1 in this paper. We begin with

**Lemma 1.** *Let $d, p, \omega$ and $f$ be as in Theorem 1. Let $\Phi_n$ be an integrable function in $\mathbb{R}^d$ such that $|\Phi_n(x)| \leq c_1 n^d$ for $|x| < 1/n$ and $|\Phi_n(x)| \leq c_2 n^{-1} |x|^{-d-1}$ for $|x| > 1/n$, where $c_1$ and $c_2$ are constants. Set $a_n = \int_{\mathbb{R}^d} \Phi_n(x) dx$ and $h_n = f * \Phi_n$. Then $\|h_n - a_n f\|_{p,\omega} \to 0$ as $n \to \infty$.*

*Proof.* By a famous Muckenhoupt theorem [Mu, p.222, Theorem 9] $\|Mf\|_{p,\omega} \leq$ const. $\|f\|_{p,\omega}$, where $Mf$ is the Hardy-Littlewood maximal function. It follows that $Mf(x) < \infty$ a.e. in $\mathbb{R}^d$. Let us see that $f$ is locally integrable in $\mathbb{R}^d$. In fact, for any compact set $K$ in $\mathbb{R}^d$ we have

$$\int_K |f| dx \leq \left( \int_K |f|^p \omega dx \right)^{1/p} \left( \int_K \omega^{1/(1-p)} dx \right)^{1/p'} \leq \|f\|_{p,\omega} \left( \int_K \omega^{1/(1-p)} dx \right)^{1/p'} < \infty$$

because $\omega \in A_p$ shows that the last integral is finite.

Now set

$$k(r) = \int_{|y|<r} |f(x-y) - f(x)| dy.$$

Then $r^{-d}k(r) \le c_3(Mf(x)+|f(x)|) < \infty$ for a.e. $x \in \mathbb{R}^d$. Let $x$ be such a point. Moreover, since $f$ is locally integrable, $r^{-d}k(r) \to 0$ as $r \to 0$ also for a.e. $x$ by a well-known result in integration theory; see, for instance, [R, p.168, Theorem 8.8]. We suppose $x$ has such a property too. Given $\varepsilon > 0$ choose $r_0 > 0$ so that $r^{-d}k(r) < \varepsilon$ if $0 < r < r_0$. In order to show $\lim_{n\to\infty}(h_n - a_n f) = 0$, it is sufficient to consider $n$ such that $1/n < r_0$. We write

$$\begin{aligned} h_n(x) - a_n f(x) &= \left( \int_{|y|<1/n} + \int_{1/n<|y|<r_0} + \int_{|y|>r_0} \right) (f(x-y) - f(x))\Phi_n(y)dy \\ &= I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

Since $|\Phi_n(y)| \le c_1/n^{-d}$ if $|y| < 1/n$,

$$|I_1(x)| \le \frac{c_1}{n^{-d}} \int_{|y|<1/n} |f(x-y) - f(x)|dy \to 0$$

as $n \to \infty$ at our $x$. We note that

$$|I_2(x)| \le \frac{c_2}{n} \int_{1/n}^{r_0} \frac{1}{r^{d+1}} dk(r) \le \frac{c_2}{n} \left( \frac{k(r_0)}{r_0^{d+1}} + \varepsilon \int_{1/n}^{r_0} \frac{1}{r^2} dr \right) \le \frac{c_4}{n}(1+\varepsilon n) \le 2c_4\varepsilon$$

if $n$ is large. To avoid a repetition of similar computations we shall give a preliminary evaluation before evaluating $I_3(x)$. If $1/n \le \alpha < \infty$, then

$$\begin{aligned} \int_\alpha^\infty |(f(x-y) - f(x))\Phi_n(y)|dy &\le \frac{c_2}{n} \int_\alpha^\infty \frac{1}{r^{d+1}} dk(r) \\ &= \frac{c_2}{n} \left( \frac{k(r)}{r^{d+1}} \bigg|_\alpha^\infty + (d+1) \int_\alpha^\infty \frac{k(r)}{r^{d+2}} dr \right) \\ &\le \frac{c_2 c_3}{n} \lim_{r\to\infty} \frac{Mf(x) + |f(x)|}{r} + (d+1)c_2c_3 \frac{Mf(x) + |f(x)|}{n} \int_\alpha^\infty \frac{1}{r^2} dr \\ &= (d+1)c_2c_3 \frac{Mf(x) + |f(x)|}{\alpha n}. \end{aligned} \tag{1}$$

By this evaluation we see that $|I_3(x)| \le (d+1)c_2c_3(Mf(x) + |f(x)|)/(r_0 n) \to 0$ as $n \to \infty$. Accordingly $\limsup_{n\to\infty} |h_n(x) - a_n f(x)| \le \text{const.}\,\varepsilon$ so that $\lim_{n\to\infty}(h_n(x) - a_n f(x)) = 0$.

Next we shall show that $|h_n - a_n f|$ is dominated by a function, which is independent of $n$ and belongs to $L^{p,\omega}(\mathbb{R}^d)$. We write

$$h_n(x) - a_n f(x) = \left( \int_{|y|<1/n} + \int_{1/n<|y|} \right) (f(x-y) - f(x))\Phi_n(y)dy = I_1(x) + I_2'(x).$$

We observe that $|I_1(x)| \leq c_3(Mf(x) + |f(x)|)$, and that $|I_2'(x)|$ is dominated by $(d+1)c_2c_3(Mf(x) + |f(x)|)$ by (1). We recall that $Mf + |f|$ belongs to $L^{p,\omega}(\mathbb{R}^d)$ in virtue of the Muckenhoupt theorem. Consequently we can apply Lebesgue's dominated convergence theorem and obtain

$$\lim_{n\to\infty} \|h_n - a_n f\|_{p,\omega} = \|\lim_{n\to\infty}(h_n - a_n f)\|_{p,\omega} = 0.$$

Thus the lemma is completely proved.

We shall give three more lemmas. Their proofs require a number of properties of $(p,\omega)$-precise functions and so they are omitted. We only refer to the corresponding results in [O].

We shall call a set $E$in $\mathbb{R}^d$ $(p,\omega)$-exc. if $\lambda_p(\Gamma;\omega) = \infty$ for the family $\Gamma$ of curves terminating at the points of $E$. This can be characterized as a kind of set of capacity zero. We shall say that a property holds $(p,\omega)$-a.e. if the exceptional set is a $(p,\omega)$-exc. set.

**Lemma 2.** [O, Theorems 4.4.4 and 4.4.5]. *Let $\omega \in A_p$. Let $f_1, f_2, \ldots$ be $(p,\omega)$-precise in $\mathbb{R}^d$ and assume*

$$\lim_{n,m\to\infty} \|\operatorname{grad} f_n - \operatorname{grad} f_m\|_{p,\omega} = 0.$$

*Then there exist a $(p,\omega)$-precise function $f$ in $\mathbb{R}^d$, a subsequence $\{n_j\}$ and a sequence $\{c_j\}$ of constans such that $\|\operatorname{grad} f_n - \operatorname{grad} f\|_{p,\omega} \to 0$ and $f_{n_j} - c_j \to f$ $(p,\omega)$-a.e. in $\mathbb{R}^d$.*

**Lemma 3.** [O, Corollary to Theorem 4.4.6]. *If a sequence $\{g_n\}$ of $(p,\omega)$-precise functions converges pointwise to a function $g$ $(p,\omega)$-a.e. in $\mathbb{R}^d$ and $\{\operatorname{grad} g_n\}$ is a Cauchy sequence in $L^{p,\omega}(\mathbb{R}^d)$, then $g$ is $(p,\omega)$-precise and $\|\operatorname{grad} g_n - \operatorname{grad} g\|_{p,\omega} \to 0$ as $n \to \infty$.*

**Lemma 4.** [O, Theorem 4.2.5]. *Let $f, g$ be $(p,\omega)$-precise functions in $\mathbb{R}^d$ which are equal $(p,\omega)$-a.e. in $\mathbb{R}^d$. Then $\operatorname{grad} f = \operatorname{grad} g$ a.e. in $\mathbb{R}^d$.*

*Proof of Theorem 1.* The first assertion of the theorem is a consequence of Theorem III due to Coifman and Fefferman [CF, Studia Math., 1974]. Assume $\int_{\mathbb{R}^d}(1+|x|)^{1-d}|f(x)|dx < \infty$. This is a necessary and sufficient condition for $U_1^{|f|}$ to be finite a.e. in $\mathbb{R}^d$.

Set $\varphi_n = D^\alpha k_{1+|\alpha|,1/n}$ and $g_n = \varphi_n * f$, where $k_{a,b}(x) = (|x|^2+b^2)^{(a-d)/2}$ in general. We can observe easily

$$\left|\frac{\partial}{\partial x_i}\varphi_n(x-y)f(y)\right| \le \begin{cases} \text{const.}\, n^d|f(y)| & \text{if } |y| < 2(|x|+1), \\ \text{const.}\, \dfrac{|f(y)|}{|y|^d} & \text{if } |y| \ge 2(|x|+1). \end{cases}$$

We infer that $\partial g_n/\partial x_i = (\partial \varphi_n/\partial x_i) * f$ and it is continuous in $\mathbb{R}^d$. Therefore $g_n$ is absolutely continuous along all locally rectifiable curves in $\mathbb{R}^d$. Setting $K^{i,\alpha}_{1/n}(x) = K^{i,\alpha}\chi_{|\cdot|>1/n}(x)$, we know that $K^{i,\alpha}_{1/n} * f \to T^{i,\alpha}f$ in $L^{p,\omega}(\mathbb{R}^d)$ as $n \to \infty$. We can write

$$\frac{\partial g_n}{\partial x_i}(x) - K^{i,\alpha}_{1/n} * f(x) = \Phi_n * f,$$

where

$$\Phi_n(x) = \begin{cases} \dfrac{\partial}{\partial x_i}D^\alpha\left(|x|^2+\dfrac{1}{n^2}\right)^{(1+|\alpha|-d)/2} & \text{on } |x| < 1/n, \\ \dfrac{\partial}{\partial x_i}D^\alpha\left(|x|^2+\dfrac{1}{n^2}\right)^{(1+|\alpha|-d)/2} - \dfrac{\partial}{\partial x_i}D^\alpha|x|^{1+|\alpha|-d} & \text{on } |x| \ge 1/n. \end{cases}$$

By the mean value theorem we have $|\Phi_n(x)| \le \text{const.}\, 1/(n^2|x|^{d+1})$ on $|x| \ge 1/n$. Hence $\int_{\mathbb{R}^d}|\Phi_n(x)|dx < \infty$. As in [Mi, p.219, Lemma 4.1] we see that $\int_{\mathbb{R}^d}\Phi_n(x)dx$ vanishes.

In view of Lemma 1 and the equality $\int_{\mathbb{R}^d}\Phi_n(x)dx = 0$, we obtain

$$\lim_{n\to\infty}\left\|\frac{\partial g_n}{\partial x_i} - K^{i,\alpha}_{1/n} * f\right\|_{p,\omega} = \lim_{n\to\infty}\left\|\int_{\mathbb{R}^d} f(x-y)\Phi_n(y)dy\right\|_{p,\omega} = 0. \tag{1}$$

We recall that $K^{i,\alpha}_{1/n} * f \to T^{i,\alpha}f$ in $L^{p,\omega}(\mathbb{R}^d)$ as $n \to \infty$. So naturally each $\|K^{i,\alpha}_{1/n} * f\|_{p,\omega}$ is finite. Hence (1) gives $\|\partial g_n/\partial x_i\|_{p,\omega} < \infty$ for each $n$. The absolute continuity along all locally rectifiable curves being known, it follows that $g_n$ is $(p,\omega)$-precise. From (1) and the fact $\lim_{n\to\infty}\|K^{i,\alpha}_{1/n} * f - T^{i,\alpha}f\|_{p,\omega} = 0$ we infer that $\{\partial g_n/\partial x_i\}$ form a Cauchy sequence in $L^{p,\omega}(\mathbb{R}^d)$. Using Lemma 2 we find a $(p,\omega)$-precise fuction $g_0$, a subsequence $\{n_j\}$ and a sequence $\{c_j\}$ of constants such that $\|\operatorname{grad}(g_n - g_0)\|_{p,\omega} \to 0$ and $g_{n_j} - c_j \to g_0$ $(p,\omega)$-a.e. The assumption $\int_{\mathbb{R}^d}(1+|x|)^{1-d}|f(x)|dx < \infty$ implies that

$$|g_{n_j}(x)| = |D^\alpha k_{1+|\alpha|,1/n} * f(x)| \le \text{const.}\, U_1^{|f|}(x) < \infty$$

for a.e. $x$. Hence we may assume that all $c_j$ are zero so that $g_{n_j} \to g_0$ $(p,\omega)$-a.e.

From (1) it follows that there exists a subsequence of $\{n_j\}$, which will be denoted again by $\{n_j\}$, such that

$$\lim_{j\to\infty}\left(\frac{\partial g_{n_j}}{\partial x_i} - K^{i,\alpha}_{1/n_j} * f\right) = 0 \tag{2}$$

a.e. in $\mathbb{R}^d$. The relations $\lim_{n\to\infty}\|\operatorname{grad}(g_n - g_0)\|_{p,\omega} = 0$ and $\lim_{n\to\infty}\|K^{i,\alpha}_{1/n} * f - T^{i,\alpha}f\|_{p,\omega} = 0$ show that we may assume that $\lim_{j\to\infty}\partial g_{n_j}/\partial x_i = \partial g_0/\partial x_i$ and $\lim_{j\to\infty} K^{i,\alpha}_{1/n_j} * f = T^{i,\alpha}f$ a.e. in $\mathbb{R}^d$ for each $i$. Taking into account (2) we obtain the equality

$$\frac{\partial g_0}{\partial x_i} = T^{i,\alpha}f \qquad \text{a.e. in } \mathbb{R}^d \text{ for each } i. \tag{3}$$

In the special case $\alpha = 0$ and $f \geq 0$ $g_n$ increases to $k_1 * f = U^f_1$ everywhere in $\mathbb{R}^d$ and $\{\operatorname{grad} g_n\}$ form a Cauchy sequence. Lemma 3 shows that $U^f_1$ is $(p,\omega)$-precise. In the general case $|D^\alpha k_{1+|\alpha|,1/n}| \leq \text{const.}\, k_1$ and $U^{|f|}_1$ is finite a.e. in $\mathbb{R}^d$. Hence applying Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} D^\alpha k_{1+|\alpha|,1/n} * f(x) = D^\alpha k_{1+|\alpha|} * f(x)$$

at every point $x$ with finite $U^{|f|}_1(x)$. Again by Lemma 3 we infer that $D^\alpha k_{1+|\alpha|} * f$ is $(p,\omega)$-precise.

Next, we recall that $g_{n_j} \to g_0$ as $j \to \infty$ $(p,\omega)$-a.e. and obtain $g_0 = D^\alpha k_{1+|\alpha|} * f$ $(p,\omega)$-a.e. in $\mathbb{R}^d$. Using Lemma 4 and (3) we derive

$$\frac{\partial D^\alpha k_{1+|\alpha|} * f}{\partial x_i} = \frac{\partial g_0}{\partial x_i} = T^{i,\alpha}f$$

a.e. in $\mathbb{R}^d$ for each $i$. Finally since $\|T^{i,\alpha}f\|_{p,\omega} \leq \text{const.}\,\|f\|_{p,\omega}$, $\|\operatorname{grad}(D^\alpha k_{1+|\alpha|}*f)\|_{p,\omega} \leq \text{const.}\,\|f\|_{p,\omega}$ as announced.

## References


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