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半線形楕円型方程式の球対称解と構造

Radial solutions and the structure of semilinear elliptic equations

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1 Introduction

This is a joint work with Hiroshi Morishita (Hyogo University) and Eiji Yanagida (University of Tokyo).

It often happens that the radial solutions play the fundamental role to investigate the structure of solutions in semilinear elliptic equations. If we restrict that solutions are radially symmetric, problems are reduced to the analysis of the ordinary differential equations. However, it is not easy to investigate the existence, the non-existence, and the uniqueness of solutions even if we combine known technique of ordinary differential equations and the functional analysis.

In 1979, Gidas-Ni-Nirenberg [GNN1] showed a very important result concerning the dependence of the symmetricity of positive solutions of semilinear elliptic equations on the symmetricity of equations and the symmetricity of regions under the general conditions on the nonlinear term.

For instance, let us consider the Dirichlet problem

$$\Delta u+f(u,|x|)=0 \text{ in } B, \quad u>0 \text{ in } B, \quad u=0 \text{ on } \partial B,$$

where $B:=\{x:|x|<1\}\subset \mathbf{R}^{n}$ is a unit ball, and $f(u,r)$ is a smooth function with respect to u and $r=|x|$. Gidas-Ni-Nirenberg [GNN1] showed that if $f_{r}=\partial f/\partial r\leq 0$, then positive solutions of the above equation must be radially symmetric. Here the condition that the solutions are positive is essential. In fact, a solution which changes its sign is not necessarily radially symmeric. Moreover, there is an example of an equation whose positive solution does not have radial symmetricity provided that $f_{r}>0$ for some r.

Their proof of the radial symmmetricity based on so-called the moving plane method. This method was first developed by Alexandroff ([A]) to prove a theorem: The topological n-sphere embedded in $\mathbf{R}^{n+1}$ with constant mean curvature must be a standard n-sphere. This method was later applied to an over-determined system by Serrin [S], and developed to [GNN1] .

Gidas-Ni-Nirenberg [GNN2] extended the above result to the case the domain is $\mathbf{R}^n$, in which they showed that a solution with sufficiently fast decay must be necessarily radially symmetric. Later, the improvement and the simplification of the proof of [GNN1] and [GNN2] has been done. The assumption that the decay of solutions are sufficiently fast was almost removed by Li-Ni [LN2] and Li [L]. Now, various refinement and extension of the moving method and applications to a-priori estimates are being done (see, e.g., Berestycki-Caffarelli-Nirenberg [BCN], Chen-Lin [CL], Naito-Suzuki [NS]).

Consequently there are a lot of cases that to investigate the structure of positive solutions on a ball or entire space is reduced to invetigate the structure of positive radial solutions.

There are a lot of results about the structure of positive radial solutions including all regular solutions or some class of singular solutions. These problems are reduced to investigate the structure of solutions of one parameter (see, e.g., Ding-Ni [DN], Li-Ni [LN], Ni-Yotsutani [NY], Yanagida [Y] and Yanagida-Yotsutani [YY1, YY2, YY3, YY4]).

It has been investigated that the problems are studied one by one according to the problem, even if nonlinear terms and the boundary conditions are slightly different. However, it is becoming to be clear that a class of equations which are apparently different has a similar structure. Moreover, it is recently discovered that the boundary value problems which satisfy radial solutions are reduced to a kind of canonical form after suitable change of variables. By virtue of this fact, we can understand known results systematically, make clear unknown structure of various equations, and precisely investigate the structure.

It seems that there are no results about the structure of all positive radial solutions including both regular and singular solutions. This problem is reduced to investigate the structure of solutions of two parameters. We need new devices to treat the problem of two parameters.

As the first step to study the problem, we investigate the structure of all positive radial solutions including both regular and singular solutions to Matukuma's equation

$$\Delta u + \frac{1}{1+|x|^2} u^p = 0 \quad in \;\; \boldsymbol{R}^3 \setminus \{0\}.$$

2 Main results

Since we are interested in all positive radial solutions in $\boldsymbol{R}^3 \setminus \{0\}$, we investigate the structure of all solutions of

$$\begin{aligned} &u_{rr} + \frac{2}{r}u_r + \frac{1}{1+r^2}u_+^p = 0, \quad r \in (0,\infty), \\ &u_r(1) = \mu, \;\; u(1) = \nu \geq 0, \end{aligned} \tag{2.1}$$

where $\mu \geq 0$ and ν are dummy parameters, and $u_+ = max\{u, 0\}$. We note that the equation (2.1) has the unique solution, and denote it by $u = u(r; \mu, \nu)$.

We can classify each solution of (2.1) according to its behavior as $r \to \infty$. We say that

(i) $u(r;\mu,\nu)$ is a *crossing solution* in$(1,\infty)$ if $u(r;\mu,\nu)$ has a zero in $(1,\infty)$,

(ii) $u(r;\mu,\nu)$ is a *slow-decay solution* at $r=\infty$ if $u(r;\mu,\nu)>0$ on $(1,\infty)$ and $\lim_{r\to\infty} ru(r;\mu,\nu)=\infty$,

(iii) $u(r;\mu,\nu)$ is a *rapid-decay solution* at $r=\infty$ if $u(r;\mu,\nu)>0$ on $(1,\infty)$ and $\lim_{r\to\infty} ru(r;\mu,\nu)$ exists and is finite and positive.

Similarly, we can classify each solution of (2.1) according to its behavior as $r \to 0$. We say that

(i) $u(r;\mu,\nu)$ is a *crossing solution* in $(0,1)$ if $u(r;\mu,\nu)$ has a zero in $(0,1)$,

(ii) $u(r;\mu,\nu)$ is a *singular solution* at $r=0$ if $u(r;\mu,\nu)>0$ on $(0,1)$ and $\lim_{r\to 0} u(r;\mu,\nu)=\infty$,

(iii) $u(r;\mu,\nu)$ is a *regular solution* at $r=0$ if $u(r;\mu,\nu)>0$ on $(0,1)$ and $\lim_{r\to 0} u(r;\mu,\nu)$ exists and is finite and positive.

Now we state main theorems.

Theorem 2.1 *Let $p>1$ be fixed. There exists a continuous function $R(\theta;p)\in C([0,3\pi/4])$ with $R(\theta;p)>0$ for $\theta\in[0,3\pi/4)$ and $R(3\pi/4;p)=0$ such that*

(i) If $(\mu,\nu)\in R_{out}$ then $u(r;\mu,\nu)$ is a crossing solution in $(1,\infty)$,

(ii) If $(\mu,\nu)\in R_{on}$ then $u(r;\mu,\nu)$ is a rapid-decay solution at $r=\infty$,

(iii) If $(\mu,\nu)\in R_{in}$ then $u(r;\mu,\nu)$ is a slow-decay solution at $r=\infty$,

where

$$
\begin{aligned}
R_{out} &:= \{(\rho\cos\theta,\rho\sin\theta):\rho>R(\theta;p), 0<\theta<3\pi/4\},\\
R_{on} &:= \{(R(\theta;p)\cos\theta, R(\theta;p)\sin\theta):0<\theta<3\pi/4\},\\
R_{in} &:= \{(\rho\cos\theta,\rho\sin\theta):0<\rho<R(\theta;p), 0<\theta<3\pi/4\}.
\end{aligned}
$$

Theorem 2.2 *Let $p>1$ be fixed. There exists a continuous function $L(\theta;p)\in C([\pi/2,\pi])$ with $L(\pi/2;p)=0$, $L(\theta;p)>0$ for $\theta\in(\pi/2,\pi)$, $L(\pi;p)>0$ $(0<p<5)$, $L(\pi;p)=0$ $(p\geq 5)$ such that*

(i) If $(\mu,\nu)\in L_{out}$ then $u(r;\mu,\nu)$ is a crossing solution in $(0,1)$,

(ii) If $(\mu,\nu)\in L_{on}$ then $u(r;\mu,\nu)$ is a regular solution at $r=0$,

(iii) If $(\mu,\nu)\in L_{in}$ then $u(r;\mu,\nu)$ is a singular solution at $r=0$,

where

$$\begin{array}{lcl} L_{out} & := & \{(\rho\cos\theta, \rho\sin\theta) : \rho > L(\theta;p), \pi/2 < \theta < \pi\}, \\ L_{on} & := & \{(L(\theta;p)\cos\theta, L(\theta;p)\sin\theta) : \pi/2 < \theta < \pi\}, \\ L_{in} & := & \{(\rho\cos\theta, \rho\sin\theta) : 0 < \rho < L(\theta;p), \pi/2 < \theta < \pi\}. \end{array}$$

Theorem 2.3 *The following relations hold:*

(i) If $1 < p < 5$, *then* $L_{on} \cap R_{on} = \{\text{one point}\}$.

(ii) If $p \geq 5$, *then* $L_{in} \cup L_{on} \subset R_{in}$.

3 Outline of the proof

It is difficult to treat (2.1) directly, since the problem includes two parameters μ and ν. Instead of (2.1), we introduce the following initial value problem satisfying the third boundary condition

$$u_{rr} + \frac{2}{r}u_r + K(r)u_+^p = 0 \ \ (r > 0), \ \ u_r(1) = \ell\, u(1), \ \ u(1) = \nu > 0, \qquad (3.1)$$

where $K(r) = (1+r^2)^{-1}$, ℓ is a fixed real number, and positive number ν is moved. The unique solution of (3.1) is denoted by $u(r;\ell\nu,\nu)$.

First we treat the simplest case.

Proposition 3.1 *Let* $p > 1$. *The following properties hold.*

(i) If $\ell \leq -1$, *then* $u(r;\ell\nu,\nu)$ *is a crossing solution in* $(1,\infty)$ *for all* $\nu > 0$.

(ii) If $\ell \geq 0$, *then* $u(r;\ell\nu,\nu)$ *is a crossing solution in* $(0,1)$ *for all* $\nu > 0$.

Let us consider (3.1) in $(1,\infty)$ by fixing the parameter $\ell > -1$.

Proposition 3.2 *Let* $p > 1$ *and* $\ell > -1$. *There exists the unique* $\nu^* = \nu^*(\ell;p)$ *such that* $\nu^*(\ell;p)$ *is continuous with respect to* $\ell \in (-1,\infty)$, $\nu^*(\ell;p) \to 0$ *as* $\ell \to -1$, $\nu^*(\ell;p) \to \nu^*(\infty;p)$ *as* $\ell \to \infty$ *for some* $\nu^*(\infty;p) > 0$, *and*

(i) $u(r;\ell\nu,\nu)$ *is a crossing solution in* $(1,\infty)$ *for* $\nu \in (\nu^*,\infty)$,

(ii) $u(r;\ell\nu^*,\nu^*)$ *is a rapid-decay solution,*

(iii) $u(r;\ell\nu,\nu)$ *is a slow-decay solution in* $(1,\infty)$ *for* $\nu \in (0,\nu^*)$.

Let us consider (3.1) in the interval $(0,1)$ by fixing the parameter $\ell < 0$.

Proposition 3.3 *Let $p > 1$ and $\ell < 0$. There exists the unique $\nu_* = \nu_*(\ell;p)$ such that $\nu_*(\ell;p)$ is continuous with respect to $\ell \in (-\infty, 0)$, $\nu_*(\ell;p) \to 0$ as $\ell \to 0$, $\nu_*(\ell;p) \to \nu_*(-\infty;p)$ as $\ell \to -\infty$ for some $\nu_*(-\infty;p) > 0$, and*

(i) $u(r;\ell\nu,\nu)$ is a crossing solution in $(0,1)$ for $\nu \in (\nu_*, \infty)$,

(ii) $u(r;\ell\nu_*,\nu_*)$ is a regular solution at $r = 0$,

(iii) $u(r;\ell\nu,\nu)$ is a singular solution at $r = 0$ for $\nu \in (0, \nu_*)$.

We can prove Theorem 2.1 by using Propositions 3.1 and 3.2. Similarly, we can prove Theorem 2.2 by using Propositions 3.1 and 3.3. For the proof of Theorem 2.3, we combine Theorems 2.1, 2.2 and the following fact (see, e.g., [NY],[LN] and [Y]) .

Proposition 3.4 *The following facts hold:*

(i) If $1 < p < 5$, then there exists the unique positive solution of (2.1), which is regular at $r = 0$ and rapid-decay at $r = \infty$.

(ii) If $p \geq 5$, then there exist no positive solutions of (2.1) which are regular at $r = 0$ and rapid-decay at $r = \infty$.

4 Reduction to a canonical form

We explain the idea of the proof of Propositions 3.1, 3.2 and 3.3. We transform the equation (3.1) to

$$v_{tt} + k(t)v_+^p = 0 \ \ (-1 < t < 1), \ \ v_t(0) = m\, v(0), \ \ v(0) = \nu > 0, \tag{4.1}$$

where

$$\begin{aligned} v(t) &:= (1+t)u(r), \quad r := (1+t)/(1-t), \\ k(t) &:= 4(1+t)^{1-p}(1-t)^{-4}K((1+t)/(1-t)) \\ &= 4(1+t)^{1-p}(1-t)^{-2}/\{(1+t)^2 + (1-t)^2\}, \\ m &:= 2\ell + 1. \end{aligned}$$

We denote the unique solution of (4.1) by $v = v(t;\nu)$. We may investigate the behavior of solutions of (4.1).

It is easily seen that the following lemma holds, which is equivalent to Proposition 3.1.

Lemma 4.1 *Let $p > 1$. The following properties hold.*

(i) If $m \leq -1$, then $v(t;\nu)$ has a zero in $(0,1)$ for all $\nu > 0$.

(ii) If $m \geq 1$, then $v(t;\nu)$ has a zero in $(-1,0)$ for all $\nu > 0$.

We classify each solution of (4.1) in $(0,1)$ according to its behavior as $t \to 1$. We say that

(i) $v(t)$ is a crossing solution if $v(t)$ has a zero in $(0,1)$,

(ii) $v(t)$ is a singular solution if $v(t) > 0$ on $(0,1)$ and $\lim_{t\to 1}\{v(t)/(1-t)\} = \infty$

(iii) $v(t)$ is a regular solution if $v(t) > 0$ on $(0,1)$ and $\lim_{t\to 1}\{v(t)/(1-t)\}$ exists and is finite and positive.

It is easy to see that, if $v(t) > 0$ on $(0,1)$, then $v(t)/(1-t)$ is non-decreasing in t. This implies that any solution of (4.1) in $(0,1)$ is classified into one of the above three types.

It follows from Lemma 4.1 that we may investigate the behavior of solutions for $m \geq -1$.

Let $G(t)$ and $H(t)$ be functions defined by

$$G(t) := \frac{(m+1)^p}{p+1}\int_0^t \left\{(ms+1)^{-6}(s(1-s))^{\frac{p+3}{2}}k(s)\right\}_s \left(\frac{s}{1-s}\right)^{\frac{p+1}{2}} ds,$$

$$H(t) := \frac{(m+1)^p}{p+1}\int_1^t \left\{(ms+1)^{-6}(s(1-s))^{\frac{p+3}{2}}k(s)\right\}_s \left(\frac{1-s}{s}\right)^{\frac{p+1}{2}} ds.$$

The integrals in the definitions of $G(t)$ and $H(t)$ are well-defined. We note that the integral should be understood by using the integration by parts.

Finally we define

$$t_G := inf\{t \in (0,1); G(t) < 0\},$$
$$t_H := sup\{t \in (0,1); H(t) < 0\}.$$

Here we put $t_G = 1$ if $G(t) \geq 0$ on $(0,1)$, and $t_H = 0$ if $H(t) \geq 0$ on $(0,1)$. Under the condition $t_H \leq t_G$, we can completely classify the structure of solutions.

Theorem 4.1 *Let $m > -1$ be fixed. If $t_G = 1$, then $v(t;\nu)$ is a crossing solution for every $\nu > 0$.*

Theorem 4.2 *Let $m > -1$ be fixed. If $0 \leq t_H \leq t_G < 1$, then there exists a unique positive number ν^* such that*

(i) $v(t;\nu)$ is a crossing solution for every $\nu \in (\nu^*, \infty)$,

(ii) $v(t;\nu^*)$ is a regular solution, and

(iii) $v(t;\nu)$ is a singular solution for every $\nu \in (0, \nu^*)$.

The above theorems are obtained by applying the methods developed in [YY1,YY2,YY3].

We can prove Proposition 3.2 as an application of Theorems 4.1 and 4.2. The proof of Proposition 3.3 is similar.

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