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The Pompeiu and related problems and boundary behavior

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain with appropriately smooth boundary $\partial\Omega$, and ν be an exterior unit normal vector $\partial\Omega$ of the boundary. We consider the overdetermined boundary value problem

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = c_D = \text{a real constant} \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} = -c_N = \text{a real constant} \quad \text{on} \quad \partial\Omega. \tag{1.3}$$

Here λ, c_D, c_N are real constants.

The problem is to ask the shape of Ω when the above problem is solvable. In cases when $c_D = c_N = 0$ and $\lambda = c_N = 0$, the constant function $u \equiv c_D$ solves the problem for any Ω. Therefore to rule out in those cases as exceptional situations, we assume

$$c_D^2 + c_N^2 \neq 0, \tag{1.4}$$

and

$$\lambda \neq 0 \quad \text{when} \quad c_N = 0. \tag{1.5}$$

Rayleigh conjectured in [5] that when $c_D \neq 0$, $c_N = 0$, and Ω is a simply connected bounded plane domain, the only disk has a solution. His conjecture, however, has not solved yet. We have known that this is equivalent to the *the Pompeiu problem* [3, 4], which comes from differential geometry (see [6, Problem 80]). There are many partial answers (for examples, see references cited in [7]).

If we do not assume the simple connectivity or boundedness of Ω, many shapes are possible to solve the problem. For examples, the half plane, a strip domain, a cylinder, and domains written by their product, all of which have the constant mean curvature. Thus we would like to show that the solvability implies the constancy of mean curvature. This is true, if u behaves "gently" in some sense near the boundary.

Here we assume that $\partial\Omega$ is in the class of C^1, and that there exists a real-valued solution u in the class of $C^2(\bar{\Omega})$. We, however, impose no topological nor geometrical assumptions on $\partial\Omega$.

Kinderlehrer-Nirenberg [1] shows that the above assumptions imply that $\partial\Omega$ is real-analytic. This also imples the analiticity of solutions (eg. [2]), and we may assume

(1.6) $\partial\Omega$ is real analytic, and u is real analytic in $\bar{\Omega}$.

Furthermore we can define the mean curvature h. We take the signature positive for convex domains. What we want to see is conditions on the behavior of u near boundary which implies the constancy of h. To see this, we must study the expressions of higher normal derivatives of solutions of the Poisson equation in terms of boundary data and geometry of boundary. We shall give the recurrence formula of expressions and the sketch of its proof in § 2 (Theorem 2.1). In § 3 we apply it to (1.1) – (1.3) to get necessary and sufficient conditions of behavior of u near boundary which implies the constancy of mean curvature (Theorem 3.1). We shall also gain a condition for the constancy of the mean curvatures of higher order (Theorem 3.2).

2 The expression of higher normal derivatives

We denote the principal curvatures of $\partial\Omega$ by κ_i $(i = 1, 2, \ldots, n-1)$, which signature is taken positive when Ω is convex. Put $h = \dfrac{1}{n-1}\sum_{i=1}^{n-1}\kappa_i$, the mean curvature. Let (g_{ij}) and (h_{ij}) be the first and second fundamental forms of $\partial\Omega$ respectively induced by the immersion $\partial\Omega \hookrightarrow \mathbb{R}^n$. The inverse matrix and the determinant of (g_{ij}) are denoted by (g^{ij}) and g respectively. We use notation h_i^j and h_{ij} to mean $h_i^j = h_{ik}g^{kj}$, $h^{ij} = h_{k\ell}g^{ik}g^{j\ell}$. For $p \in \mathbb{N}$ we use notation $h^{(p)}{}_j^i$ to mean

$$h^{(p)}{}_i^j = h_i^{i_1}h_{i_1}^{i_2}\cdots h_{i_{p-1}}^j, \quad H^{(p)} = h^{(p)}{}_i^i, \quad h_{ij}^{(p)} = h^{(p)}{}_i^k g_{kj}, \quad h^{(p)ij} = h^{(p)}{}_k^i g^{kj}.$$

For convenience we put

$$h^{(0)}{}_i^j = \delta_i^j, \quad h_{ij}^{(0)} = g_{ij}, \quad h^{(0)ij} = g^{ij}.$$

Then it is easy to see

$$h^{(p)}{}_k^i h^{(q)}{}_j^k = h^{(p+q)}{}_j^i$$

for p, $q = 0, 1, 2, \ldots$. Put

$$\Delta_p v = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}\left(\sqrt{g}h^{(p)ij}\frac{\partial v}{\partial x_j}\right), \quad \langle\nabla_p v, \nabla_p w\rangle = h^{(p)ij}\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j}.$$

By use of the principal curvatures we find

$$H^{(p)} = \sum_{i=1}^{n-1} \kappa_i^p,$$

and

$$\langle \nabla_p v, \nabla_p w \rangle = \sum_{i=1}^{n-1} \kappa_i^p \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i}$$

at the center of the system. Note neither Δ_p is necessarily elliptic nor $\langle \nabla_p v, \nabla_p w \rangle$ is positive definite, except $p = 0$.

The following recurrence formula holds not only for the solution to (1.1) – (1.3) but also for solutions of the Poisson equation. We need not assume the constancy of boundary data.

Theorem 2.1 *Put $\Delta u = f$. Then it holds that*

$$\frac{\partial^k u}{\partial \nu^k} = \sum_{\ell=0}^{k-2} (k-2)! \left[\frac{(-1)^{\ell+1}}{(k-\ell-2)!} H^{(\ell+1)} \frac{\partial^{k-\ell-1} u}{\partial \nu^{k-\ell-1}} \right.$$

$$+ (-1)^{k-1} \left\{ \frac{(-1)^\ell (k-\ell-1)}{\ell!} \Delta_{k-\ell-2} \frac{\partial^\ell u}{\partial \nu^\ell} \right.$$

$$\left.\left. - \sum_{m=0}^{\ell} \frac{(-1)^m (k-\ell-2)}{(\ell-m+1)m!} \left\langle \nabla_{k-\ell-3} H^{(\ell-m+1)}, \nabla_{k-\ell-3} \frac{\partial^m u}{\partial \nu^m} \right\rangle \right\} \right] + \frac{\partial^{k-2} f}{\partial \nu^{k-2}}$$

on $\partial\Omega$ for $2 \leq k \leq k^*$. We interpret $(k-\ell-2)\langle \nabla_{k-\ell-3} \cdot, \nabla_{k-\ell-3} \cdot \rangle = 0$ for $\ell = k-2$. k^* is determined by the regularity of f, the Dirichlet and Neumann data, and $\partial\Omega$.

Sketch of Proof. At first we assume u and $\partial\Omega$ are real analytic. Then power series appearing in ρ in the sequel converges for sufficiently small ρ. The radii of convergence are dominated by the radius of curvature. Its proof is quite standard, so we omit it. Hence we can calculate in formal way.

Let Ω_ρ be the interior parallel set of Ω with distance ρ. If $\rho > 0$ is sufficiently small, then $\partial\Omega_\rho$ has the same regularity as that of $\partial\Omega$. We denote the exterior unit normal vector of $\partial\Omega_\rho$ by ν_ρ. The first fundamental form of $\partial\Omega_\rho$ is denoted by $(g_{ij}(\rho))$. We will denote other quantities on $\partial\Omega_\rho$ in the same manner. Let p be a point on $\partial\Omega_\rho$. We may assume that the direction of x_n-axis is that of ν_ρ at p, and $x' = (x_1, \ldots, x_{n-1})$ is a local coordinate system around p. $\partial\Omega_\rho$ has a local representation

$$x_n = \varphi(x').$$

And assume that Ω_ρ is located locally in $\{x_n < \varphi(x')\}$. By use of the principal coordinate system, the Laplace-Beltrami operator on $\partial\Omega_\rho$ is $\Delta_{g(\rho)} = \sum_{i=1}^{n-1} \nabla_{\partial x_i}\nabla_{\partial x_i}$ at p, and we have

$$\Delta_{g(\rho)}u_\rho = \Delta u - \frac{\partial^2 u}{\partial x_n^2} - \frac{\partial u}{\partial x_n}\sum_{i=0}^{n-1}\kappa_i(p) = f - \frac{\partial^2 u}{\partial \nu_\rho^2} - H(\rho)\frac{\partial u}{\partial \nu_\rho}.$$

Here $\kappa_i(p)$'s are the principal curvatures at $p \in \partial\Omega_\rho$. Consequently it holds that

$$\frac{\partial^2 u}{\partial \nu_\rho^2} = -H(\rho)\frac{\partial u}{\partial \nu_\rho} - \Delta_{g(\rho)}u_\rho + f$$

on $\partial\Omega_\rho$. Operating $(-1)^{k-2}\left.\dfrac{d^{k-2}}{d\rho^{k-2}}\right|_{\rho=0}$, we have

$$\text{(2.1)} \quad \frac{\partial^k u}{\partial \nu^k} = (-1)^{k-1}\frac{d^{k-2}}{d\rho^{k-2}}\left.\left(H(\rho)\frac{\partial u}{\partial \nu_\rho}\right)\right|_{\rho=0} + (-1)^{k-1}\frac{d^{k-2}}{d\rho^{k-2}}\left.\left(\Delta_{g(\rho)}u_\rho\right)\right|_{\rho=0} + \frac{\partial^{k-2} f}{\partial \nu^{k-2}}.$$

In a consequence of Riemannian geometry we have easily

$$\text{(2.2)} \qquad g_{ij}(\rho) = g_{ij} - 2\rho h_{ij} + \rho^2 h_{ij}^{(2)},$$

$$\text{(2.3)} \qquad g^{ij}(\rho) = \sum_{p=0}^{\infty}(p+1)\rho^p h^{(p)ij},$$

$$\text{(2.4)} \qquad g(\rho) = g\exp\left(-2\int_0^\rho H(\rho)\,d\rho\right),$$

$$\text{(2.5)} \qquad h_{ij}(\rho) = h_{ij} - \rho h_{ij}^{(2)},$$

$$\text{(2.6)} \qquad h_i^j(\rho) = \sum_{p=0}^{\infty}\rho^p h^{(p+1)}{}_i^j,$$

$$\text{(2.7)} \qquad h^{ij}(\rho) = \sum_{p=0}^{\infty}\frac{(p+1)(p+2)}{2}\rho^p h^{(p+1)ij},$$

and

$$\text{(2.8)} \qquad H(\rho) = \sum_{p=0}^{\infty}\rho^p H^{(p+1)}.$$

To obtain the assertion of theorem we may substitute (2.3) – (2.8) into (2.1). This completes the proof for analytic case.

In non-analytic case, we may replace all of above power series in ρ by the Taylor expansions of finite order with remainder terms. The calculations can be proceeded exactly the same manner up to $k = k^*$. □

3 Applications to the Pompeiu and related problems

We always assume (1.4) – (1.6). In a consequence of Theorem 2.1, if u solves (1.1) – (1.3), then we have

$$\frac{\partial^2 u}{\partial \nu^2} = -\lambda c_D + c_N H^{(1)}, \tag{3.1}$$

and

$$\frac{\partial^3 u}{\partial \nu^3} = \lambda c_D H^{(1)} - c_N \left(H^{(2)} + {H^{(1)}}^2 - \lambda \right). \tag{3.2}$$

Needless to say, we can obtain the expressions of more higher derivatives.

We denote the interior parallel set of Ω with distance ρ and its exterior unit normal vector by Ω_ρ and ν_ρ again. For sufficiently small $\rho > 0$, Π_ρ is the projection from $\partial\Omega$ to $\partial\Omega_\rho$ given by $\Pi_\rho(x) = x - \rho\nu(x) \in \partial\Omega_\rho$ for $x \in \partial\Omega$. By use of (3.1) with (1.3), we have

$$-\frac{1}{\rho}\left(c_N + \frac{\partial u}{\partial \nu_\rho} \right) = \frac{1}{\rho}\left(\frac{\partial u}{\partial \nu} - \frac{\partial u}{\partial \nu_\rho} \right) \to \frac{\partial^2 u}{\partial \nu_\rho^2} = -\lambda c_D + c_N H^{(1)} \quad \text{as} \quad \rho \downarrow 0.$$

Therefore it holds that for x, $y \in \partial\Omega$

$$\lim_{\rho\downarrow 0} \frac{1}{\rho}\left(\frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(x)) - \frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(y)) \right) = -c_N \left(H^{(1)}(x) - H^{(1)}(y) \right). \tag{3.3}$$

In a similar manner when $c_N = 0$, (3.1) and (3.2) give us

$$\lim_{\rho\downarrow 0} \frac{1}{\rho}\left(\frac{\partial^2 u}{\partial \nu_\rho^2}(\Pi_\rho(x)) - \frac{\partial^2 u}{\partial \nu_\rho^2}(\Pi_\rho(y)) \right) = -\lambda c_D \left(H^{(1)}(x) - H^{(1)}(y) \right).$$

Since $\dfrac{\partial u}{\partial \nu}$ is constant on $\partial\Omega$, we can apply L'Hospital's law to get

$$\lim_{\rho\downarrow 0} \frac{1}{\rho^2}\left(\frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(x)) - \frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(y)) \right) = \lim_{\rho\downarrow 0} \frac{1}{2\rho}\left(\frac{\partial^2 u}{\partial \nu_\rho^2}(\Pi_\rho(x)) - \frac{\partial^2 u}{\partial \nu_\rho^2}(\Pi_\rho(y)) \right).$$

Hence we obtain

$$\lim_{\rho\downarrow 0} \frac{1}{\rho^2}\left(\frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(x)) - \frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(y)) \right) = -\frac{\lambda c_D}{2} \left(H^{(1)}(x) - H^{(1)}(y) \right). \tag{3.4}$$

We remark that x and y need not belong to the same connected component of $\partial\Omega$ in the above relations (3.3) – (3.4). Consequently we get a sufficient condition for the constancy of mean curvature of $\partial\Omega$.

Theorem 3.1 *Suppose Ω admits a solution u of* (1.1) – (1.3). *Then the following conditions are equivalent to to the global constancy of mean curvature of $\partial\Omega$.*

1. *When* $c_N \neq 0$, $\lim_{\rho \downarrow 0} \frac{1}{\rho} \left(\frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(x)) - \frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(y)) \right) = 0$ *for any* $x, y \in \partial\Omega$.

2. *When* $c_N = 0$, $\lim_{\rho \downarrow 0} \frac{1}{\rho^2} \left(\frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(x)) - \frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(y)) \right) = 0$ *for any* $x, y \in \partial\Omega$.

In particular, if Ω is bounded and the solution u behaves as above, then Ω is a sphere. We can generalize (3.3) – (3.4) as follows.

Theorem 3.2 *Suppose* Ω *admits a solution* u *of* (1.1) – (1.3). *Let* k *be an integer satisfying*

$$k \geq \begin{cases} 1 & \textit{when } c_N \neq 0, \\ 2 & \textit{when } c_N = 0. \end{cases}$$

Then the following are equivalent to each other.

1. $\lim_{\rho \downarrow 0} \frac{1}{\rho^k} \left(\frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(x)) - \frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(y)) \right) = 0$ for any $x, y \in \partial\Omega$.

2. $\frac{\partial^i u}{\partial \nu^i}$ is globally constant for $1 \leq i \leq k+1$.

3. $H^{(j)}$ is globally constant for

$$1 \leq j \leq \begin{cases} k & \text{when } c_N \neq 0, \\ k-1 & \text{when } c_N = 0. \end{cases}$$

We prove the above by induction on k. The assertion for $k = 1$ when $c_N \neq 0$, and for $k = 2$ when $c_N = 0$ holds, because it is Theorem 3.1. Therefore the proof is completed if we show the next fact which is generalization of (3.3) – (3.4).

Lemma 3.1 *Assume the assertion of Theorem 3.2 holds for some* $k = \ell - 1$. *If*

$$\lim_{\rho \downarrow 0} \frac{1}{\rho^{\ell-1}} \left(\frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(x)) - \frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(y)) \right) = 0,$$

then

$$\lim_{\rho \downarrow 0} \frac{1}{\rho^\ell} \left(\frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(x)) - \frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(y)) \right) = \frac{(-1)^\ell}{\ell!} \left(\frac{\partial^{\ell+1} u}{\partial \nu^{\ell+1}}(x) - \frac{\partial^{\ell+1} u}{\partial \nu^{\ell+1}}(y) \right)$$

$$= \begin{cases} -\frac{c_N}{\ell} \left(H^{(\ell)}(x) - H^{(\ell)}(y) \right) & \textit{when } c_N \neq 0, \\ \frac{\lambda c_D}{\ell} \left(H^{(\ell-1)}(x) - H^{(\ell-1)}(y) \right) & \textit{when } c_N = 0. \end{cases}$$

Proof. By the assumption $\dfrac{\partial^i u}{\partial \nu^i}$ is globally constant for $1 \le i \le \ell$. Therefore we can apply L'Hospital's law ℓ times to obtain

$$\lim_{\rho \downarrow 0} \frac{1}{\rho^\ell}\left(\frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(x)) - \frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(y))\right) = \frac{(-1)^\ell}{\ell!}\left(\frac{\partial^{\ell+1}u}{\partial \nu^{\ell+1}}(x) - \frac{\partial^{\ell+1}u}{\partial \nu^{\ell+1}}(y)\right).$$

Since $H^{(j)}$ is also globally constant for $1 \le j \le \ell - 1$ when $c_N \neq 0$, for $1 \le j \le \ell - 2$ when $c_N = 0$, we have

$$\frac{\partial^{\ell+1}u}{\partial \nu^{\ell+1}} = \begin{cases} -(-1)^\ell(\ell-1)!c_N H^{(\ell)} + \text{const.} & \text{when } c_N \neq 0, \\ (-1)^\ell(\ell-1)!\lambda c_D H^{(\ell-1)} + \text{const.} & \text{when } c_N \neq 0 \end{cases}$$

by careful use of Theorem 2.1. □

Since $H^{(j)}$ can be written by the elementary symmetric polynomials of principal curvatures (sometimes called the *mean curvatures of higher order*), the constancy of $H^{(j)}$ for $1 \le j \le n-1$ implies that of $H^{(k)}$ for all $k \in \mathbb{N}$. Therefore we obtain a following saturation property.

Corollary 3.1 *Put*

$$n^* = \begin{cases} n-1 & \text{when } c_N \neq 0, \\ n & \text{when } c_N = 0. \end{cases}$$

Then

$$\lim_{\rho \downarrow 0} \frac{1}{\rho^{n^*}}\left(\frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(x)) - \frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(y))\right) = 0$$

is equivalent to

$$\lim_{\rho \downarrow 0} \frac{1}{\rho^k}\left(\frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(x)) - \frac{\partial u}{\partial \nu_\rho}(\Pi_\rho(y))\right) = 0 \quad \text{for all } k \in \mathbb{N}.$$

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