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Some topics of geodesics on tori

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1 Introduction

In [A1] and [A2] we constructed explicit conformal embeddings from once punctured tori to closed tori corresponding to real lattices. Our principal aim of the construction of these mappings was to know explicit relations between the Teichmüller spaces of the once punctured torus and the closed torus. In this note we consider these conformal embeddings from another point of view and state some related topics with respect to geodesics on a once punctured torus.

Our aim of this note is to show the following three points:

1. a number theoretical background of the conformal embedding from a once punctured torus to a closed torus (§3),

2. McShane's identity for the lengths of simple closed geodesics on a once punctured torus (§4, §5),

3. an alternative proof of McShane's identity by using binary trees (§6).

Relations among the cusp and closed geodesics on a once punctured torus are very important in 1 and 2. The relation $X^2+Y^2+Z^2=XYZ$ plays an important role in 1 and 3. This is also used in order to represent the Teichmüller space of the once punctured torus (§2).

2 A model of $\mathcal{T}_{1,1}$

Let M be a once punctured torus with a complete finite-area hyperbolic structure and let G be the fundamental group of M. G is a free group of two generators a_0, b_0 such that the commutator $[a_0, b_0]$ is peripheral. Throughout this note we use the convention that an element A in $\mathrm{PSL}(2,\mathbb{R})$ represents the Möbius transformation induced by A, *i.e.*,

$$\text{if } A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R}) \text{ then } A(z)=\frac{az+b}{cz+d}.$$

The group of isometries of the upper half-plane $\mathbb{H}$ can be identified with $\mathrm{PSL}(2,\mathbb{R})$. Then any complete finite-area hyperbolic structure on M is given by $\mathbb{H}/\rho(G)$, where $\rho : G \to \mathrm{PSL}(2,\mathbb{R})$ is a discrete faithful representation such that $\rho([a_0, b_0])$ is parabolic.

Let $\mathcal{C}$ be the set of isotopy classes of simple closed curves on M, where *simple* curves are curves which contain no self intersections. We can associate to each $\gamma \in \mathcal{C}$ the hyperbolic length $|\gamma|$ of the unique closed geodesic in the isotopy class. Moreover, each homotopy class of closed curves on M corresponds to an equivalent class in G, where $g, h \in G$ are equivalent if g is conjugate to h or h^{-1}. If the homotopy class γ is represented by a group element $g \in G$, then we can verify $\mathrm{tr}\,(\rho(g)) = 2\cosh \frac{1}{2}|\gamma|$, where tr denotes the trace of a matrix.

A Fuchsian group $\Gamma = \langle A, B\rangle$ for $A, B \in \mathrm{PSL}(2,\mathbb{R})$ is called *a Fricke group* if A, B are hyperbolic and $\mathrm{tr}\,[B^{-1}, A^{-1}] = -2$, where $\Gamma = \langle A, B\rangle$ is the free group generated by A and B. We can verify that the image of the discrete faithful representation ρ which determines a complete finite-area hyperbolic structure on a once punctured torus is a Fricke group. Conversely, we consider a once punctured torus which is uniformized by a Fricke group Γ and take a normalized form for the presentation of Γ (see [A1], [A2]). By using the quantities $X = \mathrm{tr}\, A, Y = \mathrm{tr}\, B$ and $Z = \mathrm{tr}\, AB$, the above description of the Fricke group is characterized by $X^2 + Y^2 + Z^2 = XYZ$ and $X, Y, Z > 2$. Moreover, we obtain the following theorem (see [W]).

Theorem 1 (Fricke, Keen) *The Teichmüller space $\mathcal{T}_{1,1}$ of the once punctured torus is the sublocus of $X^2 + Y^2 + Z^2 = XYZ$ with $X, Y, Z > 2$.*

In this note we denote a point in the Teichmüller space $\mathcal{T}_{1,1}$ of the once punctured torus by a triplet (X, Y, Z).

Example 1 Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$. Then $\Gamma = \langle A, B\rangle$ is a representation of $(3, 3, 3)$ and is a subgroup of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ of index 6.

The once punctured torus $(3, 3, 3)$ plays an important role when we give a geometrical interpretation of a classical result of the number theory (see §3).

3 Markoff forms and geodesics

We begin by recalling the classical theory. We define a quadratic form with integral coefficients and irrational roots ξ, η as follows:

$$Q(x, y) = ax^2 + bxy + cy^2 = a(x - \xi y)(x - \eta y), \quad d = b^2 - 4ac > 0,$$

and define

$$m(Q) = \inf_{(x,y)\neq(0,0)} \frac{|Q(x,y)|}{\sqrt{d}}.$$

If (m_1, m_2, m_3) is given by the positive solutions of $m_1^2 + m_2^2 + m_3^2 = 3m_1m_2m_3$, we call (m_1, m_2, m_3) a *Markoff triple.* Note that if we set $m_1 = X/3$, $m_2 = Y/3$ and $m_3 = Z/3$, the relation $X^2 + Y^2 + Z^2 = XYZ$ in §2 is obtained. Suppose that m_i, $i = 1, 2, 3$ are integers, it is easily verified that $m_i \in \tilde{M} = \{1, 2, 5, 13, 29, \cdots\}$. A number included in $\tilde{M}$ is called a *Markoff number.*

We can construct a quadratic form corresponding to a Markoff triple, which is called a *Markoff form.* The Markoff form corresponding to (m_1, m_2, m_3) with $m_1 \geq m_2 \geq m_3$ is defined by $Q_{m_1}(x, y) = ax^2 + bxy + cy^2$ such that

$$a = m_1, \quad b = 3m_1 - 2u, \quad c = \frac{1 + u^2}{m_1} - 3u,$$

$$\text{where } u = \text{the least positive integer of } \pm\frac{km_1 - m_2}{m_3} (\text{mod } 3m_1), \quad k = 0, 1, 2, \cdots.$$

We show the first five cases:

$$\text{if } (1,1,1) \text{ then } Q_1(x,y) = x^2 - xy - y^2 \text{ and } m(Q_1) = \frac{1}{\sqrt{5}},$$

$$\text{if } (2,1,1) \text{ then } Q_2(x,y) = 2x^2 - 4xy - 2y^2 \text{ and } m(Q_2) = \frac{1}{2\sqrt{2}},$$

$$\text{if } (5,2,1) \text{ then } Q_5(x,y) = 5x^2 - 11xy - 5y^2 \text{ and } m(Q_5) = \frac{5}{\sqrt{221}},$$

$$\text{if } (13,5,1) \text{ then } Q_{13}(x,y) = 13x^2 - 29xy - 13y^2 \text{ and } m(Q_{13}) = \frac{13}{\sqrt{1517}},$$

$$\text{if } (29,5,2) \text{ then } Q_{29}(x,y) = 29x^2 - 63xy - 31y^2 \text{ and } m(Q_{29}) = \frac{29}{\sqrt{7565}}.$$

We recall a relation between Markoff form and Diophantine approximation. Consider the problem of finding the following $\nu(\theta)$ for a given irrational number θ. Let p and q be positive integers,

$$\nu(\theta) = \inf\left\{c \,\middle|\, \left|\theta - \frac{p}{q}\right| < \frac{c}{q^2} \text{ for infinitely many } q\right\}.$$

A. Markoff proved that there is a discrete set of values ν_i decreasing to $1/3$ such that if $\nu(\theta) > 1/3$ then $\nu(\theta) = \nu_i$ for some i, where $\nu_i = 1/\sqrt{9 - \frac{4}{i^2}}$ and i is a Markoff number. For the first five cases we can confirm $m(Q_i) = \nu_i$ for $i = 1, 2, 5, 13, 29$.

Let ξ, η be irrational roots of a Markoff form Q. The form Q corresponds to the geodesic on $\mathbb{H}$ whose endpoints are ξ and η. We can assume that $0 < -\eta < 1 < \xi$ without loss of generality, then ξ and $-\eta$ are given by continued fractions as follows:

$$\xi = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}} = \{a_0, a_1, a_2, \cdots\} \text{ and } -\eta = \cfrac{1}{a_{-1} + \frac{1}{a_{-2} + \cdots}} = \{0, a_{-1}, a_{-2}, \cdots\},$$

where a_i, $i \in \mathbb{Z}$ are positive integers.

Theorem 2 (Cohn) *(a) Markoff forms are characterized by the geodesics which remain below the horizontal line* $\mathrm{Im}\, z = 3/2$. *The projections of these geodesics to the once punctured torus* $(3,3,3)$ *are closed.*

(b) The sequence $\{-\eta, \xi\} = \{\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots\}$ *is a periodic sequence consisting of integer pairs* $(1,1)$ *and* $(2,2)$.

A geodesic on $\mathbb{H}$ is transferred to the extended plane $\mathbb{C}$ by using the relation

$$1 - J(z) = \wp'(u)^2 = 4\wp(u)^3 + 1 \quad \text{for} \quad z \in \mathbb{H}, u \in \mathbb{C}. \tag{3.1}$$

The relation (3.1) is obtained by considering an abelianization of actions of A and B introduced in Example 1 (see [C], [A1]). Then it follows from Theorem 2 (b) that a closed geodesic on the once punctured torus $(3,3,3)$ corresponds to a direction in $\mathbb{C}$. Indeed, the integer pairs $(1,1)$ and $(2,2)$ means the following transformations, respectively,

$$z \mapsto 1 + \frac{1}{1+\frac{1}{z}} = \frac{2z+1}{z+1} = B^{-1}(z) \text{ and } z \mapsto 2 + \frac{1}{2+\frac{1}{z}} = \frac{5z+2}{2z+1} = B^{-1}A^{-1}B^{-1}(z).$$

The assertion of Theorem 2 (a) was generalized to the case of Fricke groups by A. L. Schmidt [S]. Generalizing the relation (3.1), we constructed conformal embeddings from closed tori corresponding to real lattices to once punctured tori (see [A1], [A2]).

4 Simple geodesics on a hyperbolic surface

In the previous section we recalled that Markoff forms are characterized by the closed geodesics on the once punctured torus $(3,3,3)$ which is not near to the cusp. Here we consider simple geodesics on a general hyperbolic surface.

Let M be a hyperbolic surface, *i.e.*, M can be identified with the quotient space $\mathbb{H}/G$ of the upper half-plane $\mathbb{H}$ under a Fuchsian group G. Geodesics on $\mathbb{H}$ are projected to geodesics on M.

Definition 1 A geodesic on M is said to be *complete* if it is not strictly contained in any other geodesic, *i.e.*, it is either closed and smooth, or open and of infinite length in both directions.

Theorem 3 (Jørgensen) *Let M be a hyperbolic surface with cusps. Then there exist neighborhoods of cusps which do not have intersection with any simple geodesic on M except for ones having both ends going up cusps.*

It follows immediately from Theorem 3 that the hyperbolic surface M with cusps is not covered by its simple geodesics. Then we have the problem: what is the density of the set of points on simple geodesics? J. S. Birman and C. Series gave an answer for this problem in [BS]. For $k \geq 0$ we define $G_k = \{\gamma \mid \gamma$ is a complete geodesic on M which have at most k transversal self-intersections$\}$.

Theorem 4 (Birman-Series) *For each $k \geq 0$, $S_k = \{p \in M \mid p \in \gamma \in G_k\}$ is nowhere dense and Hausdorff dimension one.*

Note that $S_0 \subset S_1 \subset S_2 \subset \cdots$ and that $\cup_{k=1}^{\infty} S_k$ is a dense subset of M. Let $\mathbb{T}$ be a closed torus which can be identified with the quotient space of the extended plane $\mathbb{C}$ under a lattice group. We can define the set S_k for $\mathbb{T}$ in the same way as for M. Then $S_0 = \mathbb{T}$ which presents a striking contrast to Theorem 4. For simplicity, we consider the case $\mathbb{T} = \mathbb{C}/\langle 1, i\rangle$. A geodesic on $\mathbb{T}$ is the projection of a line on $\mathbb{C}$. Let l be a line on $\mathbb{C}$. The projection of l on $\mathbb{T}$ is always complete and simple, and is closed if and only if the slope of l is rational. Thus we have $S_0 = \mathbb{T}$.

Let M be a hyperbolic surface with cusps and without boundary.

Definition 2 We call a portion of the surface isometric to $\{z \mid \mathrm{Im}\ z \geq 1\}/\langle P\rangle$, $P = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$ a *cusp region.*

It follows from Theorem 3 that there exists a cusp region so that if an end of a simple geodesic enters this region then it will never leave, that is, the end goes up the cusp. We call this region a *small cusp region.*

Definition 3 A geodesic has an *end up the cusp* if that end is contained in a small cusp region on M. A geodesic is a *cusp geodesic* if it has both its ends up the cusp.

Let $\mathcal{H} \subset M$ be a small cusp region. Define $E = \{x \in \mathcal{H} \mid x$ is on a complete simple geodesic$\}$. Then we can obtain that E is homeomorphic to $(\partial\mathcal{H} \cap E) \times \mathbb{R}^+$ and that $\partial\mathcal{H} \cap E =$ Cantor set $\cup$ {countably many isolated points}, where $\partial\mathcal{H}$ means the boundary of the small cusp region $\mathcal{H}$ and the definition of isolated point is as follows: let Y be a topological space and $X \subset Y$ then $x \in X$ is *isolated* if there is an open set U, $x \in U$ such that $U \cap X = \{x\}$. Using same notations, we introduce another definition: x is a *boundary point* if x is the limit of points in X and there is a connected open set U contained in the complement $Y \backslash X$ such that x is in the closure of U. We call a maximal component of $\partial\mathcal{H}\backslash E$ a *gap*.

If $x \in \partial\mathcal{H} \cap E$, the simple geodesic γ_x on which x lies has an end up the cusp. There is another end of γ_x, we call this *the other end.*

Theorem 5 (McShane) *Let $M, \mathcal{H}, E$ be as above and $x \in \partial\mathcal{H} \cap E$.*

(a) The point x is isolated if and only if the other end also goes up a cusp.

(b) The point x is a boundary point if and only if the other end spirals to a closed simple geodesic.

(c) Each gap is bounded by two points $y, z \in \partial\mathcal{H} \cap E$ which satisfy the following conditions: The point y lies on a cusp geodesic. The point z lies either on a geodesic with an end spiraling to a closed simple geodesic or on a geodesic with its other end up some other cusp. If the surface has only one cusp, only the first case occurs.

By using Theorem 4 and 5 we can prove the following striking identity for the lengths of simple closed geodesics on a once punctured torus.

Theorem 6 (McShane) *Let M be a once punctured torus then*

$$\sum_{\gamma} \frac{1}{1 + e^{|\gamma|}} = \frac{1}{2},$$

where the sum is over all simple closed geodesics γ on M.

5 Outline of a proof of Theorem 6

First we consider a general surface: let M be a hyperbolic surface with cusps. Take a cusp of M and $\mathcal{H}$ denotes its small cusp region. We define E in the same way as in §4. Let $x \in \partial\mathcal{H} \cap E$ be on a cusp geodesic γ. From Theorem 5, x is an isolated point and

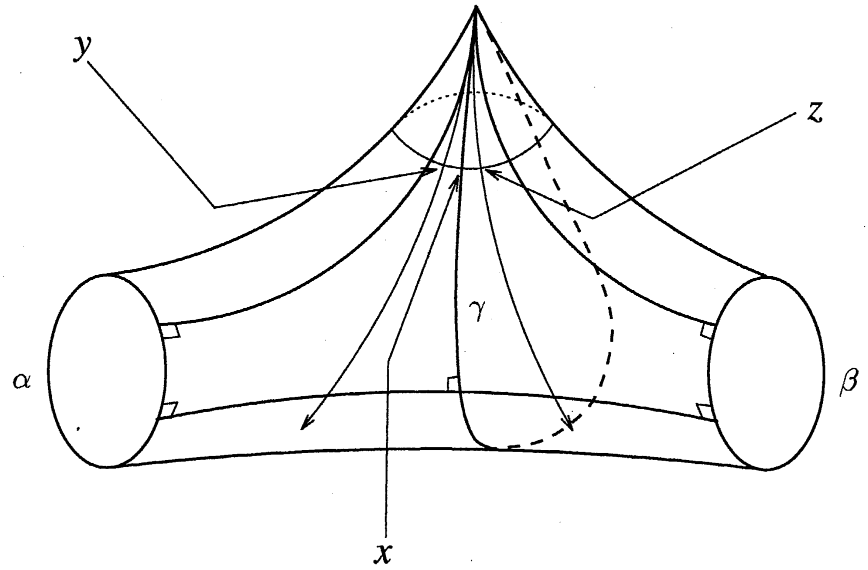


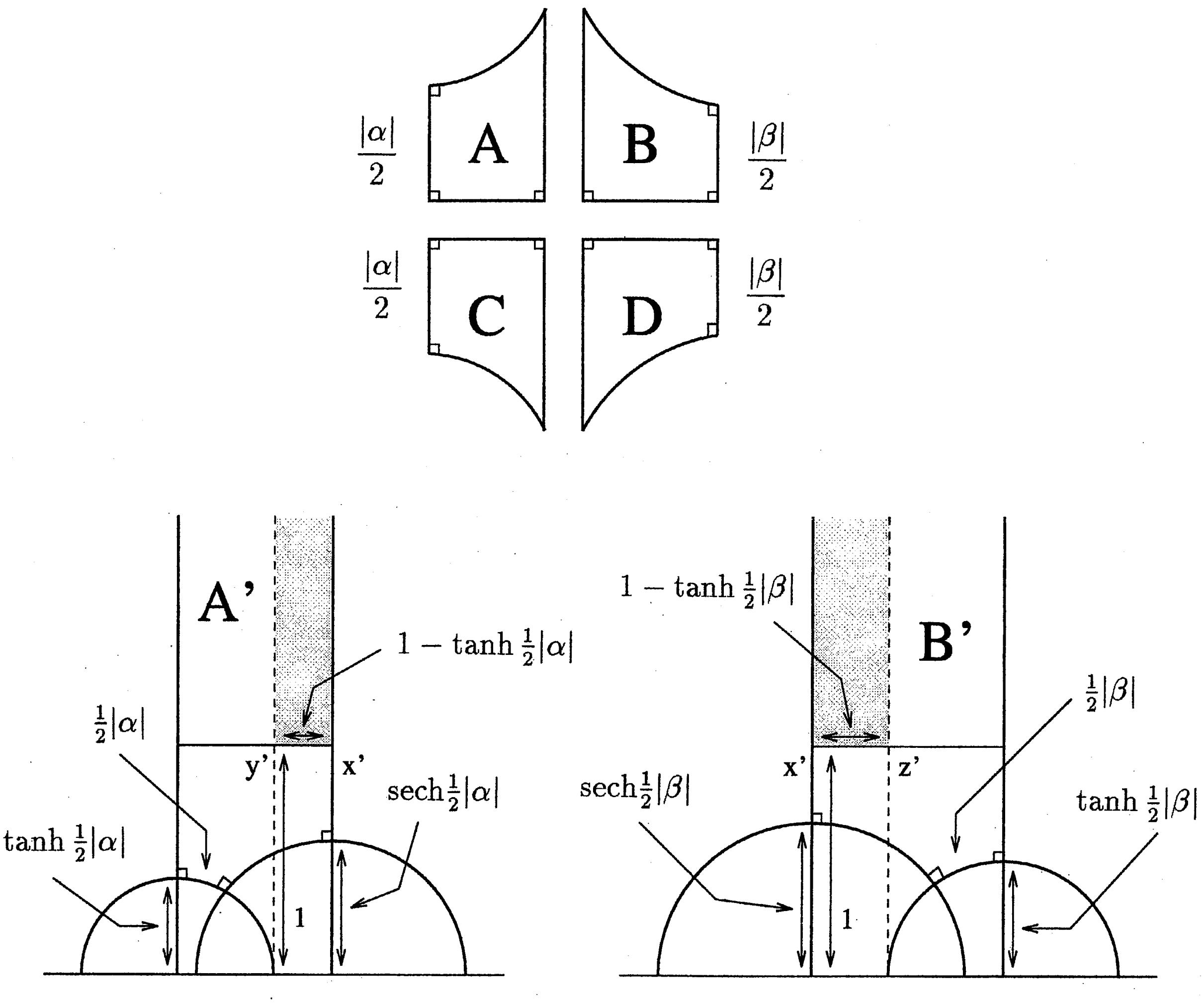
Fig. 5.1

is in the closure of two gaps. Suppose that one gap is bounded by x, y and the other is bounded by x, z. The cusp geodesic γ is contained in a unique embedded pair of pants (see Fig. 5.1). α (resp. β) is a closed simple geodesic in Theorem 5 (c) to which the other end of the geodesic through y (resp. z) spirals. We need not consider the second case of Theorem 5 (c), because our final aim is to prove the assertion for a once punctured torus.

The pants can be decomposed into four quadrilaterals A, B, C and D. Moreover, we obtain quadrilaterals A' and B' in $\mathbb{H}$ which are congruent to A and B, respectively (see Fig. 5.2). The broken lines in Fig. 5.2 represent geodesics ends of whose projections on M spiral to the simple closed geodesics α and β. The points x', y', z' are congruent to x, y, z which bound two gaps. Then we call the shaded parts in Fig. 5.2 *gap regions*. Now we can calculate the ratio of the area of the gap region to the area of the small cusp region:

$$\begin{aligned}&\frac{2(1-\tanh\frac{1}{2}|\alpha|)\operatorname{sech}\frac{1}{2}|\beta| + 2(1-\tanh\frac{1}{2}|\beta|)\operatorname{sech}\frac{1}{2}|\alpha|}{2\operatorname{sech}\frac{1}{2}|\alpha| + 2\operatorname{sech}\frac{1}{2}|\beta|}\\ =\ &\frac{(1-\tanh\frac{1}{2}|\alpha|)\cosh\frac{1}{2}|\alpha| + (1-\tanh\frac{1}{2}|\beta|)\cosh\frac{1}{2}|\beta|}{\cosh\frac{1}{2}|\alpha| + \cosh\frac{1}{2}|\beta|} = \frac{2}{1+e^{\frac{1}{2}(|\alpha|+|\beta|)}}.\end{aligned}$$

On the one hand, since these arguments can be applied to all cusp geodesics in the



A
B
C
D
|α|/2
|β|/2
A'
B'
1 − tanh ½|α|
1 − tanh ½|β|
½|α|
½|β|
tanh ½|α|
tanh ½|β|
sech ½|α|
sech ½|β|
y'
x'
z'
1

Fig. 5.2

small cusp region $\mathcal{H}$, we obtain

$$\sum_{\alpha,\beta} \frac{2}{1+e^{(|\alpha|+|\beta|)}} \times (\text{the area of small cusp region}) = (\text{the total area of gap region}).$$

On the other hand, Theorem 4 means that (the area of small cusp region) is equal to (the total area of gap region), so the above equation is changed into

$$\sum_{\alpha,\beta} \frac{1}{1+e^{\frac{1}{2}(|\alpha|+|\beta|)}} = \frac{1}{2}.$$

If M is a once punctured torus, the simple geodesics α and β coincide. Therefore we obtain the identity:

$$\sum_{\alpha} \frac{1}{1+e^{|\alpha|}} = \frac{1}{2}.$$

6 Another approach to Theorem 6

There exists an alternative proof of Theorem 6 by using binary trees (see [B]). In this section we rewrite Theorem 6 by using binary trees and state a part of the proof in order to show the advantage of using them.

Let Σ be a binary tree, *i.e.*, a simplicial tree all of whose vertices have degree 3. Suppose that Σ is embedded in the plane. We call the closure of a connected component of the complement of Σ a *complementary region.* Let Ω be the set of complementary regions.

We recall that M is a once punctured torus with a complete finite-area hyperbolic structure and $\mathcal{C}$ denotes the set of isotopy classes of simple closed curves on M. There is a natural bijective correspondences between Ω and $\mathcal{C}$. Let $\{a, b\}$ be a free basis for the fundamental group $G = \pi_1(M)$. We consider simple closed curves in $\mathcal{C}$ whose homotopy classes are represented by the group elements a, b, ab and $a^{-1}b$. There exists a discrete faithful representation ρ with $\rho(a) = A$, $\rho(b) = B$, $\rho(ab) = AB$ and $\rho(a^{-1}b) = A^{-1}B$ where $\Gamma = \langle A, B\rangle$ is a Fricke group. We set $X = \text{tr } A$, $Y = \text{tr } B$, $Z = \text{tr } AB$ and $W = \text{tr } A^{-1}B$. Then X, Y, Z, W are elements of Ω and are visualized as in Fig. 6.1.

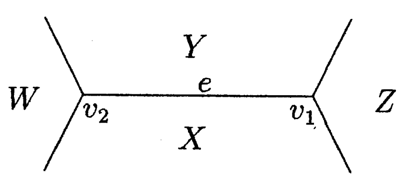


Fig. 6.1

Since $\langle A, B\rangle$ is a Fricke group, we get the relation $X^2+Y^2+Z^2 = XYZ$. We call this the *vertex relation* because v_1 is the intersection of $X, Y, Z \in \Omega$. Moreover, from the formula $\mathrm{tr}\, A\, \mathrm{tr}\, B = \mathrm{tr}\, AB + \mathrm{tr}\, A^{-1}B$, we obtain the relation $XY = Z + W$, which is called the *edge relation* because e is the edge connecting v_1 and v_2. A binary tree each of whose vertex (resp. edge) satisfies the vertex (resp. edge) relation is called a *Markoff binary tree.*

Note that in Fig. 6.1 the relation $W^2 + X^2 + Y^2 = WXY$ is obtained from $X^2 + Y^2 + Z^2 = XYZ$ and $XY = Z + W$. Generally, if there exists a single vertex satisfying the vertex relation, by using the edge relation inductively, we can construct the Markoff binary tree including the vertex.

Example 2 A part of the Markoff binary tree constructed from $(3,3,3)$ is as in Fig. 6.2.

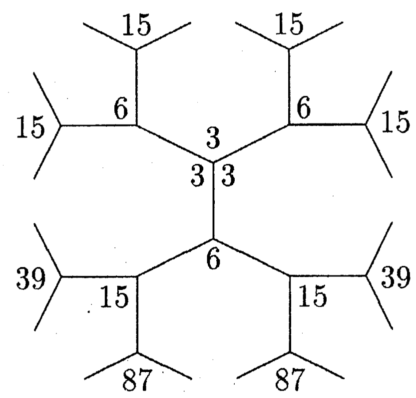


Fig. 6.2

Let Σ be the markoff binary tree constructed from $(X,Y,Z) \in \mathcal{T}_{1,1}$ and let Ω be the set of complementary regions associated with Σ. If we take $\Omega \ni \omega = 2\cosh \frac{1}{2}|\gamma_\omega|$, where γ_ω is the simple closed geodesic on the once punctured torus corresponding to ω, then

$$h(\omega) = \frac{1}{1+e^{|\gamma_\omega|}} \quad \text{where } h(x) = \frac{1}{2}\left(1 - \sqrt{1-\frac{4}{x^2}}\right).$$

By using this notation, the identity in Theorem 6 can be represented as follows:

$$\sum_{\omega\in\Omega} h(\omega) = \frac{1}{2}.$$

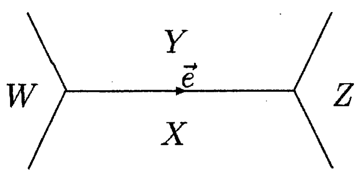


Fig. 6.3

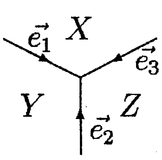


Fig. 6.4

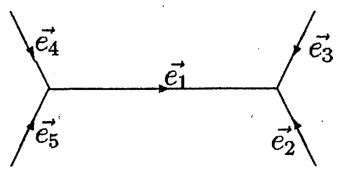


Fig. 6.5

Here we only show $\sum_{\omega\in\Omega} h(\omega) \leq \frac{1}{2}$. We think of a directed edge. Define $\psi(\vec{e}) = \frac{Z}{XY}$ in the case of Fig. 6.3. Immediately we get $\psi(\vec{e}) + \psi(-\vec{e}) = \frac{Z}{XY} + \frac{W}{XY} = 1$, which means the edge relation. In Fig. 6.4 it is obtained that $\psi(\vec{e_1}) + \psi(\vec{e_2}) + \psi(\vec{e_3}) = \frac{Z}{XY} + \frac{X}{YZ} + \frac{Y}{XZ} = 1$, which means the vertex relation. We consider the case of Fig. 6.5, from $\psi(\vec{e_1}) + \psi(\vec{e_2}) + \psi(\vec{e_3}) = 1$ and $\psi(\vec{e_4}) + \psi(\vec{e_5}) + \psi(-\vec{e_1}) = 1$, we have $\psi(\vec{e_2}) + \psi(\vec{e_3}) + \psi(\vec{e_4}) + \psi(\vec{e_5}) = 1$. Let C be the set of all directed edges which touch and are directed to a subtree (see Fig. 6.6). By using the above calculations inductively, we obtain $\sum_{\vec{e_i}\in C} \psi(\vec{e_i}) = 1$.

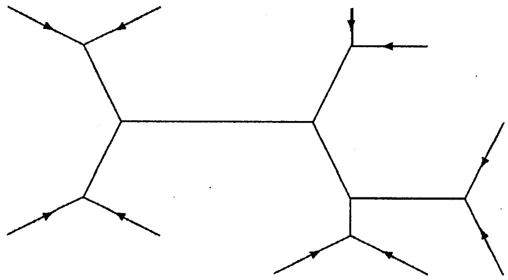


Fig. 6.6. In this figure C is the set of edges with direction.

We go back to the case of Fig. 6.3. From direct calculations we can verify the following inequality:

$$h(X) + h(Y) \leq h(X+Y) \leq \psi(\vec{e}) \quad \text{where} \quad h(X+Y) = h\left(\sqrt{\frac{1}{X^2} + \frac{1}{Y^2}}\right).$$

Consider the case of Fig. 6.4, there exist the following three inequalities:

$$h(X) + h(Y) \leq \psi(\vec{e_1}), \quad h(Y) + h(Z) \leq \psi(\vec{e_2}) \quad \text{and} \quad h(Z) + h(X) \leq \psi(\vec{e_3}).$$

Summing each side of inequalities, respectively, we obtain $h(X) + h(Y) + h(Z) \leq \frac{1}{2}$. By using this argument inductively, $\sum_{\omega\in\Omega} h(\omega) \leq \frac{1}{2}$ is proved.

References

[A1] R. Abe, *On correspondences between once punctured tori and closed tori: Fricke groups and real lattices*, to appear in Tokyo J. of Math.

[A2] R. Abe, *Abelianization of Fricke groups and rhombic lattices*, Preprint, 1998.

[BS] J. S. Birman and C. Series, *Geodesics with bounded intersection number on surfaces are sparsely distributed*, Topology 24 (1985), 217-225.

[B] B. H. Bowditch, *A proof of McShane's identity via Markoff triples*, Bull. London Math. Soc. 28 (1996), 73-78.

[C] H. Cohn, *Representation of Markoff's binary quadratic forms by geodesics on a perforated torus*, Acta. Arith. 18 (1971), 125-136.

[J] T. Jørgensen, *Simple geodesics on Riemann surfaces*, Proc. A. M. S. 86 (1982), 120-122.

[M] G. McShane, *Simple geodesics and a series constant over Teichmüller space*, Invent. Math. 132 (1998), 607-632.

[S] A. L. Schmidt, *Minimum of quadratic forms with respect to Fuchsian groups. I*, J. reine angrew. Math. 286/287 (1976), 341-468.

[W] S. Wolpert, *On the Käler form of the moduli space of once punctured tori*, Comm. Math. Helv. 58 (1983), 246-256.

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