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INVERSE LIMIT STABILITY FOR SEMIFLOWS

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Abstract. On semiflows on Banach manifolds, Quandt stated that some special semiflows were inverse limit stable[15, 16]. However, the concepts and proofs used to show inverse limit stability of semiflows are rough and ambiguous. We consider semiflows on finite dimensional compact manifolds or on finite dimensional compact non-singular branched manifolds. In this paper we announce that more special semiflows are inverse limit stable. That is, Anosov semiflows are inverse limit stable.

1. Introduction

For diffeomorphisms (resp. flows) of *closed manifolds*, the theory of structural stability has been established. For a geometric study of evolution equations, Quandt [15, 16] *outlined* a corresponding theory for endomorphisms (resp. semiflows) of *Banach manifolds* (or Banach space). To extend the stability theory from diffeomorphisms (resp. flows) to endomorphisms (resp. semiflows) structural stability was generalized to inverse limit stability[10, 13]. Inverse limit stability does not guarantee preservation of the topological dynamics. However inverse limit stability gives the one-to-one correspondence between all global solutions for any two semiflows near original one. On semiflows on Banach manifolds, Quandt[16] stated that some special semiflows were inverse limit stable. However, the concept and the proof used to show inverse limit stability of semiflows are rough and ambiguous. In order to give rigorous proof we consider an appropriate setting in this paper. So we consider semiflows on *finite dimensional* smooth manifolds without boundary or on finite dimensional compact smooth non-singular branched manifolds without boundary. We call finite dimensional *compact* (resp. *noncompact*) smooth manifold without boundary *closed* (resp. *open*) manifold. Non-singular branched manifolds were introduced by R. F. Williams[20]. By a *non-singular branched n-dimensional manifold* of class C^k is meant a metrizable space K together with:

(i) a collection $\{U_i\}$ of closed subsets of K;
(ii) for each U_i a finite collection $\{D_{ij}\}$ of closed subsets of U_i;
(iii) for each i, a map $\pi_i : U_i \to D_i^n$, D_i^n a closed n-disk of class C^k in R^n;

subject to the following axioms:

(a) $\cup_j D_{ij} = U_i$ and $\cup_i Int U_i = K$;
(b) $\pi_i | D_{ij}$ is a homeomorphism onto D_i^n ;
(c) there is a cocycle of diffeomorphisms $\{\alpha_{i'i}\}$ of class C^k such that $\pi_{i'} = \alpha_{i'i} \circ \pi_i$ when defined. The domain of $\alpha_{i'i}$ is $\pi_i(U_i \cap U_{i'})$.

To show the existence of Anosov semiflows which are not flows, we construct a finite dimensional compact smooth non-singular branched manifold without boundary. In this paper we announce:

Theorem. *Let M be either a closed manifold or a finite dimensional compact smooth non-singular branched manifold without boundary. Then Anosov semiflows on M are C^1 inverse limit stable in* $\mathrm{Sem}_s^1(M)$.

2. Preliminaries

Let M be a finite dimensional smooth connected boundaryless (non-singular branched) manifold with a Riemannian metric. Let d be a metric on M induced by its Riemannian metric. A C^r *semiflow* $F(t)$ on M, $r \geq 1$, is a one parameter family of C^r maps $\{F(t) : M \to M | t \geq 0\}$ such that
(1) $F(s+t) = F(s)F(t)$ for all $s, t \geq 0$,
(2) $F(0)$ is the identity map of M,
(3) $F(\cdot)(x) : [0, \infty) \to M$ is continuous for every $x \in M$.

We shall sometimes use the notation $F(t,x) = F(t)(x)$ for $(t,x) \in [0, +\infty) \times M$. $\mathrm{Sem}^r(M)$ denotes the class of all C^r semiflows on M, $r \geq 0$, with the metric d_r, where $d_r(F(t), G(t)) = \sup\{ d(F(t)(x), G(t)(x)),\ |D^iF(t)(x) - D^iG(t)(x)| \ |$ $t \in [0,1],\ x \in M$ and $i \in \{1, \cdots, r\}\}$ for $F(t), G(t) \in \mathrm{Sem}^r(M)$.

A *global solution* of a semiflow $F(t)$ is a function $v : \mathbf{R} \to M$ such that $F(t)(v(\tau)) = v(\tau + t)$ for all $t \geq 0$ and $\tau \in \mathbf{R}$. (3) of the above definition implies that a global solution is a continuous function of $\mathbf{R}$ to M. Let $C(\mathbf{R}, M)$ be the space of continuous functions from $\mathbf{R}$ to M. Let $\widetilde{S}(F)$ be the set of all global solutions of $F(t)$. For each $t \in \mathbf{R}$ we define a *shift* $\widetilde{F}(t)$ on $\widetilde{S}(F)$ by $[\widetilde{F}(t)v](\tau) = v(\tau + t)$ for $\tau \in \mathbf{R}$, $v \in \widetilde{S}(F)$. Let $\widetilde{p}_s$:$C(\mathbf{R}, M) \to M$ be a projection defined by $\widetilde{p}_s(v) = v(s)$ for $v \in C(\mathbf{R}, M)$. Let $A(F) = \bigcap_{t \geq 0} F(t)[M]$ for any semiflow $F(t)$. Note that $A(F) = \widetilde{p}_s(\widetilde{S}(F))$ for every $s \in \mathbf{R}$. We sometimes call $A(F)$ an *attractor* of $F(t)$. A subset J of M is *ω-invariant* for $F(t)$ if $F(t)J \subset J$ for all $t \geq 0$. Then $\widetilde{J}(F)$ or $\widetilde{J}$ denotes the set of all global solutions of $F(t)$ contained in J. We will say that a subset J of M is *invariant* for $F(t)$ if $F(t)J = J$ for all $t \geq 0$. We introduce a metric $\widetilde{d}$ on $\widetilde{S}(F)$ (more generally on $C(\mathbf{R}, M)$) by

$$\widetilde{d}(v,w) = \sup_{t \in \mathbf{R}} e^{-|t|} d(v(t), w(t)) \quad \text{for} \quad v, w \in C(\mathbf{R}, M).$$

Then $\widetilde{S}(F)$ is endowed with a topology induced by the metric $\widetilde{d}$. We shall say that a C^r semiflow $F(t)$ on M is *strong* if $F(t)$ satisfies the following:
(4) every global solution $v(t) \in \widetilde{S}(F)$ is differentiable for $t \in \mathbf{R}$.

Let $\mathrm{Sem}_s^r(M) = \{F(t) \in \mathrm{Sem}^r(M) \mid F(t) \text{ is strong}\}$ be the class of all C^r strong semiflows on M, $r \geq 0$, with the metric d_r. Let $\mathcal{F}^r(M)$ be the space of all C^r flows with the metric d_r. It is obvious that $\mathcal{F}^r(M) \subset \mathrm{Sem}_s^r(M) \subset \mathrm{Sem}^r(M)$.

A global solution v of $F(t)$ is called an *equilibrium solution* if there exists a point p such that $v(t) = p$ for all $t \in \mathbf{R}$, and a *periodic solution* if there exists a constant $T > 0$ such that $v(t+T) = v(t)$ for all $t \in \mathbf{R}$ and $v(t) \neq v(0)$ for $0 < t < T$. The time $T > 0$ is called the *period* of the solution. Thus we distinguish equilibrium solutions from periodic solutions. Let $\mathrm{Per}(F)$ be the set of all points on periodic solutions of $F(t)$. Let $\mathrm{E}(F) = \{v(0) \mid v \text{ is an equilibrium solution.}\}$ be the set of all fixed points for $F(t)$. For a global solution v of $F(t)$ define $\mathcal{O}(v) = \{v(t) | t \in \mathbf{R}\}$.

Let $F(t), G(t) \in \mathrm{Sem}^r(M)$. We say that $G(t)$ is *inverse limit semiconjugate* to $F(t)$ if there exists a continuous surjection $H : \widetilde{S}(F) \to \widetilde{S}(G)$ which takes the global solutions of $F(t)$ onto the global solutions of $G(t)$ and preserves the orientation in time; i.e. for each $v \in \widetilde{S}(F)$, there exists a nondecreasing automorphism $\beta_v : \mathbf{R} \to \mathbf{R}$ such that

$$[H \circ \widetilde{F}(t)](v) = [\widetilde{G}(\beta_v(t)) \circ H](v) \qquad \text{for all} \quad t \in \mathbf{R}.$$

Furthermore, if H is injective then $F(t)$ and $G(t)$ are *inverse limit conjugate.* A C^r semiflow $F(t)$ is C^r *inverse limit stable* if there exists a neighborhood $\mathcal{U}$ of $F(t)$ in $\mathrm{Sem}^r(M)$ such that for each $G(t) \in \mathcal{U}$, $F(t)$ and $G(t)$ are inverse limit conjugate. Similarly $F(t)$ is C^r *inverse limit stable in* $\mathrm{Sem}^r_s(M)$ if there exists a neighborhood $\mathcal{U}$ of $F(t)$ in $\mathrm{Sem}^r_s(M)$ such that for each $G(t) \in \mathcal{U}$, $F(t)$ and $G(t)$ are inverse limit conjugate. Equilibrium solutions and periodic solutions are preserved by inverse limit conjugacy.

Definition. Let $F(t)$ be a C^r semiflow on M, $r \geq 1$. We will say that an ω-invariant set J is a *hyperbolic* set for $F(t)$ if every global solution $v \in \widetilde{J}(F)$ is differentiable for t and there exist a Riemannian norm $|\cdot|$ on M and constants $c, \mu > 0$ such that for every $v \in \widetilde{J}(F)$ there exists a continuous splitting of $\bigcup_{t=-\infty}^{\infty} T_{v(t)}M$ into a direct sum $E^s \oplus E^o \oplus E^u$, where

$$\text{(1)} \qquad E^i = \bigcup_{t=-\infty}^{\infty} E^i_{v(t)} \quad \text{for} \quad i = s, o, u, \qquad E^o_{v(t)} = \mathbf{R} \cdot \frac{d}{dt} v(t) \quad \text{for each} \quad t \in \mathbf{R}$$

$$(T_{v(t)}F(\tau))[E^i_{v(t)}] = E^i_{v(t+\tau)} \quad \text{for} \quad i = s, o, u, \quad \text{and every} \quad \tau \geq 0,$$

$$\text{(2)} \qquad |(T_{v(t)}F(\tau))(w)| \leq ce^{-\mu\tau}|w| \quad \text{for} \quad w \in E^s_{v(t)}, \quad t \in \mathbf{R} \text{ and every } \tau \geq 0,$$

$$|(T_{v(t)}F(\tau))(w)| \geq c^{-1}e^{\mu\tau}|w| \quad \text{for} \quad w \in E^u_{v(t)}, \quad t \in \mathbf{R} \text{ and every } \tau \geq 0.$$

If Λ is a hyperbolic set for $F(t)$ such that $\dim E^s_{v(0)} = j$ for all $v \in \widetilde{\Lambda}(F)$, then we call j the *stable index* of Λ for $F(t)$. For convenience we will say that $v \in \widetilde{\Lambda}(F)$ is *hyperbolic* for $F(t)$ if Λ is hyperbolic for $F(t)$.

For $v \in \widetilde{S}(F)$ we can define a tangent space $T_vC(\mathbf{R}, M)$ of $C(\mathbf{R}, M)$ at v. Let $\Gamma(v^*TM)$ be the space of continuous sections of the pullback bundle of TM by v. $\Gamma(v^*TM)$ is endowed with a norm $||\xi|| = \sup_{t\in\mathbf{R}} |\xi(t)|$, where $|\cdot|$ is a Riemannian norm on M. Then we can consider that $T_vC(\mathbf{R}, M) = C(\mathbf{R}, v^*TM) = \Gamma(v^*TM)$ is a Banach space. Thus we can take an open neighborhood $U^*(v) = \Gamma_\varepsilon(v^*TM) = \{\xi \in \Gamma(v^*TM) | \; ||\xi|| < \varepsilon\} \subset C(\mathbf{R}, v^*TM)$ of zero section of v^*TM such that $U^*(v)$ can be identified with a neighborhood $U(v)$ of v in $C(\mathbf{R}, M)$ by a homeomorphism $G : U^*(v) \to U(v)$ defined by $G(\xi)(t) = \exp_{v(t)} \xi(t)$ for all $t \in \mathbf{R}$. Let $C^r(\mathbf{R}, M)$ be the space of C^r functions from $\mathbf{R}$ to M, $r \geq 1$. If it is further assumed that every $v \in \widetilde{S}(F)$ is C^r, then we can argue the similarity to the above and obtain that $T_vC^r(\mathbf{R}, M) = C^r(\mathbf{R}, v^*TM) = \Gamma^r(v^*TM)$.

Definition. We say that a C^1 semiflow $F(t)$ is an *Anosov semiflow* if $F(t)$ satisfies the following:

(i) $A(F)$ is a compact hyperbolic set for $F(t)$ with stable index $j = \text{constant} > 0$;

(ii) $A(F)$ contains no equilibrium solutions.

Intensionally we don't refer to manifolds in the definition. Because we consider general case in consideration of extension to infinite dimensional case. We shall describe the motivation of the above definition. First of all remember the definition of Anosov flows. If the entire manifold M possesses a hyperbolic structure then the flow is called an *Anosov flow.* In the case of semiflows, it is appropriate that only the attractor possesses a hyperbolic structure (i.e. condition (i)). In the case of Anosov flows on a compact manifold M, hyperbolicity of M implies that M has constant stable index. In finite dimensional case of semiflows in this paper, we easily have that $A(F)$ is equal to M and has constant stable index. However, in infinite dimensional case or special finite dimensional case (i.e. open manifolds) we cannot guarantee that $A(F) = M$. So constant stable index of $A(F)$ is not guaranteed by only hyperbolicity of $A(F)$. If $A(F) \subsetneqq M$ then there exists the possibility of decomposition of $A(F)$ into disjoint hyperbolic sets which have different stable index. Therefore, from intuition of original Anosov flows or diffeomorphisms it is natural to require constant stable index of $A(F)$ and no equilibrium solutions.

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