

MULTIFRACTAL ANALYSIS FOR ONE-DIMENSIONAL MAPS  
WITH A NEUTRAL FIXED POINT

(中立不動点をもつ力学系のマルチフラクタル解析)

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1. INTRODUCTION & RESULTS

Recent developments of computers have made us possible to see fractal sets casually. If you have any experience of drawing fractal pictures by computer, you must have noticed that some points are easy to draw while others are not. This means that an invariant set does not necessarily have uniform structure from the dynamical point of view. One of the motivations for the multifractal analysis is to uncover this hidden structure.

Given a measure  $\mu$  on a set  $X$ , a typical multifractal object is the Hausdorff dimension of points with the same scaling  $\alpha$ ;

$$f_\mu(\alpha) := \text{HD} \left\{ x \in X \mid \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\}.$$

For conformal hyperbolic systems with Gibbs measures  $\mu$ , the dimension spectrum  $\alpha \mapsto f_\mu(\alpha)$  is real analytic (Pesin-Weiss[2]). On the other hand, based on the analogy of statistical mechanics, it is expected that spectra of non-hyperbolic systems fail to be real analytic and that the point of phase transition has some characteristics. In this work, we present some results along this line.

Let  $f : [0, 1] \rightarrow [0, 1]$  be a piecewise monotone interval map with the following properties;

- (A1)  $0 < \exists c < 1$  s.t. each of  $f|_{(0, c)}, f|_{(c, 1)}$  is strictly monotone and extends to a  $C^1$  function  $f_1$  and  $f_2$  with  $f_1[0, c] = f_2[c, 1] = [0, 1]$ .
- (A2)  $f_1(0) = 0$  and  $f_1$  is  $C^2$  except at  $z = 0$ .
- (A3)  $Df_1(0) = 1$  and  $Df_1(z) > 1$  for  $z \in [0, c]$ ,
- (A4)  $|Df_2(z)| > 1$  on  $f_2^{-1}([c, 1])$  and  $|Df_2(z)| \geq 1$  otherwise.
- (A5)  $D^2 f_1$  has an asymptotic expansion around 0 of the form

$$D^2 f(z) = bz^{\tau-1} + o(z^{\tau-1}), \quad b > 0, \quad \tau > 0.$$

**Example 1.1** (Farey map).  $c = 1/2$ ,

$$f_1(z) = z/(1-z), \quad f_2(z) = (1-z)/z$$

Let  $u$  be the measure of maximal entropy for  $f$ .

**Definition 1.1.** For  $\alpha \in \mathbb{R}^+$  define

$$f_u(\alpha) = \text{HD} \left\{ z \in [0, 1]; \lim_{r \rightarrow 0} \frac{\log u(B(z, r))}{\log r} = \alpha \right\},$$

$$l(\alpha) = \text{HD} \left\{ z \in [0, 1]; \lim_{n \rightarrow \infty} \frac{1}{n} \log |Df^n(z)| = \alpha \right\}$$

where HD stands for Hausdorff dimension.  $f_u(\alpha)$  is called the dimension spectrum of  $u$  and  $l_u(\alpha)$  the Lyapunov spectrum for  $f$ .

**Main Theorem .**  $\exists T(q)$  real analytic on  $\mathbb{R}^+$  s.t.

- (a)  $D^2T(q) > 0, T(0) = 1$  for  $q > 0$  and  $\lim_{q \rightarrow 0} DT(q)$  exists,  
 (b) The domain of  $f_u(\alpha)$  is unbounded.  $(f_u(\alpha), T(q))$  forms a Legendre transform pair;

$$f_u(\alpha(q)) = T(q) + q\alpha(q)$$

where  $\alpha(q) = -DT(q) > 0$ ,

- (c) If  $\alpha(0+)$  is finite then  $f_u(\alpha) = 1$  for every  $\alpha \geq \alpha(0+)$ ,  
 (d)  $l(\alpha) = f_u(\log 2/\alpha)$ .

**Corollary .** The followings are equivalent;

- (a)  $\alpha(0) = \infty$ .  
 (b)  $f_u(\alpha)$  and  $l_u(\alpha)$  are real analytic.  
 (c) The absolutely continuous invariant measure is infinite.  
 (d) The Lyapunov exponent with respect to the absolutely continuous invariant measure is 0.

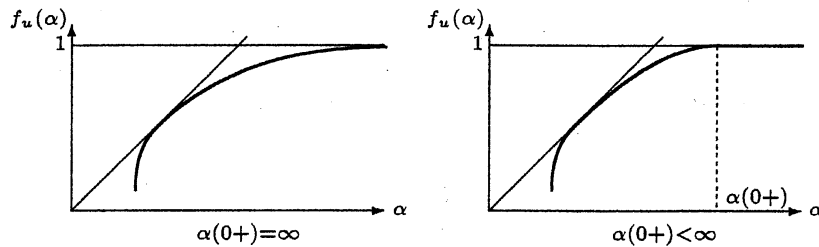


FIGURE 1. Typical spectral shapes

## 2. HOW TO PROVE

Put  $J = [c, 1], I_\infty = \{0\}$  and let  $I_k = f_1^{-k}(J)$  for all  $k \geq 0$ . Define the subintervals  $\{J_k\}$  of  $J$  by  $J_k = f_2^{-1}(I_{k-1})$ . The next lemma is simple but essential through this work.

**Lemma 2.1.** For arbitrarily small  $\epsilon > 0$  there exist constants  $C_1, C_2$  so that

$$C_1 k^{(1-\epsilon)(1+\tau)/\tau} \leq Df^k(z) \leq C_2 k^{(1+\epsilon)(1+\tau)/\tau}$$

for all  $k \in \mathbb{N}$  and all  $z \in I_k$ .

It can be shown that the induced map  $F$  on  $[c, 1]$  is an expanding Markov map with a countable partition and, using the lemma above, that  $F$  has bounded distortion. Thus we can apply to  $F$  the techniques of the thermodynamic formalism.

Define parameterized weight functions  $\phi_{s,q}$  by

$$\phi_{s,q}(z) = q \log \psi - s \log |DF(z)|$$

where  $\psi(z) = 2^{-k}$ ,  $z \in \text{int} J_k$ .

**Theorem 2.2** (Walters). *For each  $(s, q) \in \mathbb{R} \times \mathbb{R}^+ \cup D \times \{0\}$  there exists an  $F$ -invariant ergodic probability measure  $\nu_{s,q}$  on  $J$  so that for each  $n > 0$  there is a constant  $\Delta(n) \geq 1$  satisfying  $\lim_{n \rightarrow \infty} \log \Delta(n) / \log \sup_{K \in \mathcal{P}^n} |K| \rightarrow 0$  and*

$$\nu_{s,q}(F^n A) \stackrel{\Delta(n)}{\sim} \int_A \exp(nP(s, q) - \phi_{s,q}^{(n)}) d\nu_{s,q}$$

whenever  $F^n : A \rightarrow F^n A$  is invertible.

**Definition 2.1.** We call the invariant measure above the equilibrium state for  $\phi_{s,q}$  and the number  $P(s, q) = \log \lambda(s, q)$  the pressure for  $\phi_{s,q}$ .

The next gives the definition of  $T(q)$  (a generalization of Bowen's formula).

**Lemma 2.3.** *For each  $q \in \mathbb{R}^+ \cup \{0\}$ , the function  $s \mapsto P(s, q)$  is convex, strictly decreasing (hence continuous) and has a unique zero  $T(q)$ .*

Consider the one-parameter family of mappings  $\phi_{T(q),q}$  for  $q \in \mathbb{R}^+ \cup \{0\}$ . Let  $\nu_q$  be the equilibrium state for  $\phi_{T(q),q}$ . Since  $\nu_q$  is ergodic for each  $q > 0$ , there exists a positive constant  $\alpha = \alpha(q)$  such that for almost every  $z \in J$

$$\sum_{l=0}^{k-1} -\log \psi(F^l z) / \sum_{l=0}^{k-1} \log |DF(F^l z)| \rightarrow \alpha(q) = - \int \log \psi d\nu_q / \int \log |DF| d\nu_q.$$

Then define the set

$$K(\delta) = \left\{ z \in J \setminus J_{\text{inv}}; \lim_{k \rightarrow \infty} \frac{\sum_{l=0}^{k-1} -\log \psi(F^l z)}{\sum_{l=0}^{k-1} \log |DF(F^l z)|} = \delta \right\}$$

and denote the Hausdorff dimension of  $X$  (set or measure) by  $\text{HD}(X)$ .

**Proposition 2.4.** *For  $q \in \mathbb{R}^+$*

$$\text{HD}(\nu_q) = \text{HD}(K(\alpha(q))) = T(q) + q\alpha(q).$$

The next lemma relates the original map  $f$  with the induced  $F$  in the multifractal formalism.

**Lemma 2.5.** *For  $z \in J \setminus J_{\text{inv}}$  each of the following three implies the other two:*

- (a)  $\lim_{r \rightarrow 0} \frac{\log u(B(z, r))}{\log r} = \alpha.$
- (b)  $\lim_{n \rightarrow \infty} \frac{-\sum_{l=0}^{n-1} \log \psi(F^l z)}{\sum_{l=0}^{n-1} \log |DF(F^l z)|} = \alpha.$
- (c)  $\lim_{n \rightarrow \infty} \frac{1}{n} \log |Df^n(z)| = \frac{\log 2}{\alpha}.$

So far we have shown how to prove the main theorem except the analyticity of  $T(q)$ , which is closely connected to Corollary.

**Proposition 2.6.** (a)  $T(q)$  is real analytic for  $q > 0$  and satisfies

$$DT(q) = \frac{\int \log \psi(z) d\nu_q(z)}{\int \log |DF(z)| d\nu_q(z)} = -\alpha(q).$$

(b) If  $\Psi := \log \psi - DT(q) \log |DF|$  then

$$D^2T(q) = \left( \int \log |DF(z)| d\nu_q(z) \right)^{-1} \sigma^2(\Psi)$$

where

$$\sigma^2(\Psi) = \frac{d^2}{ds^2} P(T(q) + sDT(q), q + s) \Big|_{s=0}.$$

The formula in (a) leads to the proof of Corollary.

### 3. HOW TO FILL IN THE DETAILS

We refer to [5] for any reader who is interested in the details. See [1] also for the general background of multifractal analysis. The paper [4] contains number theoretical applications and shares some results with this work, but its proof seems to be missing some point.

#### REFERENCES

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