数理解析研究所講究録
1120 巻 1999 年 59-67

有向グラフ上の確率的局所多数決ゲーム

A Probabilistic Local Majority Polling Game on Weighted Directed Graphs

中田 寿夫 [1] Toshio NAKATA
今林 裕 [2] Hiroshi IMAHAYASHI
山下 雅史 [2] Masafumi YAMASHITA

[1] 福岡教育大学
〒 811-4192 宗像市赤間町
nakata@crown.fukuoka-edu.ac.jp

[2] 九州大学システム情報科学研究科
〒 812-8581 福岡市東区箱崎
{imahayas,mak}@csce.kyushu-u.ac.jp

摘要: In this paper, we investigate a probabilistic local majority polling game on weighted directed graphs, keeping an application to the distributed agreement problem in mind. We formulate the game as a Markov chain, where an absorbing state corresponds to a system configuration that an agreement is achieved, and characterize on which graphs the game will eventually reach an absorbing state with probability 1. We then calculate, given a pair of an initial and an absorbing states, the absorbing probability that the game will reach the absorbing state, starting with the initial state.

キーワード: distributed agreement algorithm, majority polling game, Markov chain, monopoly

1 Introduction

Motivated by the importance of coordination problems among agents, Tennenholtz and his colleagues extensively discussed the problem of agreeing on what they called social laws [4, 10, 11]. Shoham and Tennenholtz, in particular, proposed and compared several algorithms for the agents to agree on a standard from some existing proposals [10]. Under the assumption that every agent knows who supports which proposal, the agreement can be made just by, for example, taking the one that the most agents support. The agent system however can be distributed too wide to admit this assumption, and this is why they suggested those heuristic algorithms based on partial information on the distribution of the agents' opinions.

For simplicity, suppose that there are two proposals, 0 and 1, and that the proposal that a majority of the agents support is to be selected as the standard. Given, for each agent, a group of neighboring agents whose opinions are available to it, a simple and natural heuristic to approximate the agreement is to take the majority of the opinions available to it, i.e., the opinions of its neighbors and itself. This is called the deterministic local majority polling system.

Peleg and his colleagues recently investigated this system and determined how many agents supporting 0 are necessary and sufficient for all agents to result in 0 [1, 7, 8]. They naturally model the system by a finite connected undirected graph $G = (V, E)$, where V and E respectively represent the set of agents and the (symmetric) neighborhood relation. A subset M of V is called a *monopoly*, if for any $v \in V$, members in M form a majority of the vertices adjacent to v (including v itself). Linial et al. discussed the problem as a packing and a covering problems on graphs

[7]. They showed that $|M|$ is $\Omega(\sqrt{n})$ and gave a graph with M of size $O(\sqrt{n})$, where $n = |V|$. Bermond and Peleg studied some of its modifications, r-monopoly and self-ignoring monopoly [1].

Peleg also discussed a repetitive version of the deterministic local majority polling system informally defined as follows [9]: each of the vertices u has a local state $\xi(u) \in \{0, 1\}$ and synchronously updates its local state to the one that a majority of its neighbors (including itself) are in. The game proceeds as a repetition of this process.

Since the number of possible configurations is finite, this dynamical system will eventually be periodic or stationary, and there is a pair of a graph and an initial configuration such that the system will never be stationary; the game may not end up with all vertices being in the same local state. The (repetitive) deterministic local majority polling game is hence not powerful enough for the purpose of making all agents agree on an adequate standard, let alone a one-shot deterministic local majority polling system.

We therefore study a probabilistic local majority polling game, the idea of which was suggested by Peleg on an undirected graph [8, 8.1.5 Other variants].[1] In this paper, the game is defined on a directed finite graph $G = (V, E)$ with an edge weight function $\mu : E \to \mathbf{R}^+$, where $\mathbf{R}^+$ is the set of positive real numbers. A directed edge $e = (u, v) \in E$ from a vertex u to a vertex v represents that v can read the local state $\xi(u)$ of u. There can exist a self-loop edge (v, v) in E. Note that v cannot make use of the local state $\xi(v)$ of itself, unless $(v, v) \in E$. The weight $\mu(u, v)$ assigned to an edge $(u, v) \in E$ intuitively denotes the degree of importance of the local state $\xi(u)$ for v.

The probabilistic game proceeds in the same way as the deterministic one, except for the state update algorithm. The new local state $\xi(v)$ is determined through a stochastic procedure as follows: For any $v \in V$ and $b \in \{0, 1\}$, let

$$s_\xi(v, b) = \sum_{u \in V_b} \mu(u, v),$$

where $V_b = \{u \in V : \xi(u) = b\}$. Here we assume $\mu(u, v) = 0$ for any $(u, v) \notin E$. Then we select a bit b with probability $s_\xi(v, b)/(s_\xi(v, 0) + s_\xi(v, 1))$ as the new local state $\xi(v)$.

A directed graph with a weight function is a substantial extension over an undirected graph as a model of multi-agent systems; the former can for example treat asymmetric neighborhood relation and each agent's influence upon the decision process. In this paper, we further prepare a framework by which we can partly discuss the effect of the degree of synchrony on the game. In the study of the repetitive deterministic local majority polling game, all vertices v simultaneously update the local states $\xi(v)$, which assumption is unrealistic for widely distributed multi-agent systems to make. In this paper, k randomly selected vertices are assumed to simultaneously update their local states, which we will call the k-polling. By varying the value of k from 1 to $n = |V|$, we will observe the effect of the degree of synchrony upon the game.

In order to investigate the probabilistic local majority polling game, we first formalize the system as a Markov chain with absorbing states, and show that the game will eventually reach an absorbing state with probability 1, if G is strongly connected and, in addition, has a self-loop when $k = n$. Since the absorbing states correspond to configurations such that all vertices have the same local state, it guarantees that the modeled distributed system will achieve the agreement with probability 1. We next calculate the probability that all vertices will achieve the agreement with local state being 0, given an initial configuration. As a result, we will show that the probability is computable by solving a set of simultaneous linear equations with $2^n - 2$ variables, but obtaining an explicit form of it seems to be difficult. We

[1] It should be noted that independently Hassin and Peleg also studied a probabilistic local majority polling game [5].

hence present two classes of graphs and give an explicit form of the probability for them. Finally, we demonstrate that regular graphs have desirable property from the view of the distributed agreement application, by using the martingale theory.

Following an introduction of the probabilistic local polling game in Section 2, Section 3 investigates the game as a Markov chain. Section 4 discusses regular graphs by using the martingale theory. Some concluding remarks will be made in Section 5.

2 The Model and Notations

Consider a finite connected directed graph $G = (V, E)$ with an edge weight function $\mu : E \to \mathbf{R}^+$, where $\mathbf{R}^+$ is the set of positive real numbers. Throughout the paper, n is reserved to denote the order $|V|$ of G. We associate a boolean value $\xi(v) \in \{0,1\}$ with each vertex $v \in V$, and call it the local state of v. For any $v \in V$, let $\Gamma(v) = \{u \in V : (u,v) \in E\}$. Then the set of vertices whose local states v can read is the substantial meaning of $\Gamma(v)$ in the paper. We hence naturally assume $\Gamma(v) \neq \emptyset$ for any $v \in V$. A *configuration* (or a global state) $\xi = (\xi(v_1), \cdots, \xi(v_n)) \in \Xi = \{0,1\}^V$ is the vector of all local states. For any $\xi \in \Xi$, $v \in V$ and $b \in \{0,1\}$, let

$$s_\xi(v,b) = \sum_{u \in \Gamma(v), \xi(u)=b} \mu(u,v),$$

and

$$s(v) = s_\xi(v,0) + s_\xi(v,1), \tag{1}$$

i.e., $s_\xi(v,b)$ is the weighted sum of the number of vertices whose local states are available to v and are b, assuming configuration ξ. Note that $s(v)(= \sum_{u\in\Gamma(v)} \mu(u,v))$ does not depend on ξ and is positive.

Given a configuration $\xi = (\xi(v_1), \cdots, \xi(v_n))$, we consider the following transformation of ξ. First we uniformly randomly select k distinct vertices $u_1, \ldots, u_k$ from V. Next for each $1 \le i \le k$, we generate a random bit b_i, where the probability that $b_i = b$ is given by

$$q_\xi(u_i, b) = s_\xi(u_i, b)/s(u_i). \tag{2}$$

Then ξ is transformed into the configuration that is constructed from ξ by setting, for each $1 \le i \le k$, $\xi(u_i) = b_i$. Note that each of the updates of k states is independent and simultaneous. Now b_i is the (new) local state of u_i. We call this stochastic transformation procedure *k-polling*. The *probabilistic local majority k-polling game* on G with μ, which we will study in this paper, is a transition of configurations defined by a repetitive applications of k-polling.

We start with calculating the probability that ξ is transformed into η for any ξ and η in Ξ. For any $W \subseteq V$ and $j = 1, 2, \cdots, n$, we denote by $Sub_j(W)$ the set of all j-(sub)sets X of W, i.e., $Sub_j(W) = \{X \subseteq W : |X| = j\}$. For any configuration ξ and set $W \subseteq V$, let $\xi^W \in \Xi$ be the configuration such that the vertices exactly in W are in different local states from ξ; formally,

$$\xi^W(v) = \begin{cases} \xi(v) & \text{if } v \notin W, \\ 1 - \xi(v) & \text{if } v \in W. \end{cases}$$

Let $p_k(\xi, \eta)$ be the transition probability from ξ to η, i.e., the probability that ξ is transformed into η by an application of k-polling.

Proposition 1 *Let $\eta = \xi^W$ for some $W \subseteq V$. Then*

$$\begin{aligned} p_k(\xi,\eta) &= p_k(\xi,\xi^W) \\ &= \frac{1}{\binom{n}{k}} \sum_{U: W \subseteq U \in Sub_k(V)} \\ &\quad \prod_{u \in W} q_\xi(u, 1-\xi(u)) \\ &\quad \prod_{v \in U \setminus W} q_\xi(v, \xi(v)). \end{aligned} \tag{3}$$

Proof. Observe first that $p_k(\xi,\eta) = 0$, if $|W| > k$, since no $U \in Sub_k(V)$ is a superset of W.

Suppose hence that $|W| \le k$. By the definition of k-polling, a k-set $U \in Sub_k(V)$ is randomly selected from V with probability $1/\binom{n}{k}$. Provided that a particular $U \in Sub_k(V)$ is selected, the probability that ξ is transformed into ξ^W is

$$\prod_{u \in W} q_\xi(u, 1 - \xi(u)) \prod_{v \in U \setminus W} q_\xi(v, \xi(v)),$$

if $W \subseteq U$, and 0, otherwise, since the updates of $\xi(u)$ are independently and simultaneously made. ∎

By definition, for any $1 \leq k \leq n$, the transition matrix $P_k = (p_k(\xi, \eta))_{\xi, \eta \in \Xi}$ is a stochastic matrix, i.e., $p_k(\xi, \eta) \geq 0$ and $\sum_{\eta \in \Xi} p_k(\xi, \eta) = 1$, for $\xi, \eta \in \Xi$.

3 The Probabilistic Game as a Markov Chain

Now we introduce a finite Markov chain on Ξ with the transition probability $P_k = (p_k(\xi, \eta))_{\xi, \eta \in \Xi}$. Following the terminology of the theory of Markov chain, we will use term a 'state' (of a Markov chain) and a 'configuration' (of a graph) interchangeably. The elements of $P_k^t (= (P_k)^t)$ are denoted by $p_k^{(t)}(\xi, \eta)$ for $t \in \mathbf{N}$, where $\mathbf{N}$ is the set of non-negative integers. Let $\mathbf{0} = (0, \cdots, 0), \mathbf{1} = (1, \cdots, 1) \in \Xi$ and $\hat{\Xi} = \Xi \setminus \{\mathbf{0}, \mathbf{1}\}$. By definition, $\mathbf{0}$ and $\mathbf{1}$ are absorbing states, i.e., $p_k(\mathbf{0}, \mathbf{0}) = p_k(\mathbf{1}, \mathbf{1}) = 1$, and they are the only absorbing states, since graph G is assumed to be connected.

For each $k = 1, \cdots, n$, we say that $\xi \in \Xi$ is *reachable* to η, if there exists a $t \geq 0$, which may depend on ξ and η, such that $p_k^{(t)}(\xi, \eta) > 0$ holds. We first characterize the class of graphs such that every configuration in $\hat{\Xi}$ is reachable both to $\mathbf{0}$ and $\mathbf{1}$. The probabilistic local k-polling game on such a graph will eventually reach an absorbing state with probability 1, regardless of the initial configuration and k.

Theorem 1 *Suppose that a given graph G is not strongly connected. Then there exists a configuration $\xi \in \hat{\Xi}$ such that for any $1 \leq k \leq n$, it is not reachable to $\mathbf{0}$, i.e., for any $1 \leq k \leq n$ and $t \in \mathbf{N}$,*

$$p_k^{(t)}(\xi, \mathbf{0}) = 0. \tag{4}$$

Proof. Let $V_1, V_2, \cdots, V_\ell$ be the partition of V corresponding to the strongly connected components of G, i.e., the induced subgraph of G induced by V_j is a strongly connected component of G for any $1 \leq j \leq \ell$. Since G is not strongly connected, $\ell \geq 2$. Then there is a j such that there is no directed path from a vertex not in V_j to a vertex in V_j. For a configuration ξ such that $\xi(v) = 1$ if and only if $v \in V_j$, Eq. (4) holds, since every vertex in $\Gamma(v)$ is in 1 for any $v \in V_j$. ∎

Theorem 2 *Suppose that a given graph G is strongly connected. Then every configuration $\xi \in \hat{\Xi}$ is reachable both to $\mathbf{0}$ and $\mathbf{1}$, provided that G contains a self-loop when $k = n$.*

Remark 1 If a graph G is strongly connected but has no self-loops, then the n-polling may not always lead the system to an absorbing state.

Proof of Theorem 2.

Provided that $k \neq n$, we show $p_k^{(m_0)}(\xi, \mathbf{0}) > 0$ for some $m_0 \in \mathbf{N}$. The fact that ξ is reachable to $\mathbf{1}$ is provable by a similar argument.

Consider any configuration $\xi_0 \in \hat{\Xi}$. There is then a vertex $u_0 \in V$ having local state $\xi_0(u_0) = 0$ by definition. Since the given graph G is strongly connected, there is a directed path from u_0 to v for any $v \in V$. Let $V_j \subseteq V$ be the set of vertices v such that the length of a shortest path from u_0 to v is j, i.e., the distance of v from u_0 is j. Sets $V_0, V_1, \cdots, V_\ell$ clearly form a partition of V, where $V_0 = \{u_0\}$ and ℓ is the distance of a farthest vertex from u_0.

Let Ξ_j be the set of configurations $\xi \in \Xi$ such that $\xi(v) = 0$ for any $v \in \bigcup_{i=0}^{j} V_i$. By definition, $\xi_0 \in \Xi_0$ and

$$\Xi_0 \supset \Xi_1 \supset \cdots \supset \Xi_\ell = \{\mathbf{0}\}.$$

For any $0 \leq j \leq \ell - 1$, we show that any configuration in Ξ_j is reachable to a configuration in Ξ_{j+1}. This, together with $\xi_0 \in \Xi_0$, implies that ξ_0 is reachable to $\mathbf{0}$.

Consider any configuration $\xi \in \Xi_j \setminus \Xi_{j+1}$. Let $u_1 \in V_{j+1}$ be a vertex such that $\xi(u_1) = 1$, which witnesses the fact $\xi \notin \Xi_{j+1}$. Observe that $q_\xi(v, 0) > 0$ for any $v \in \bigcup_{i=1}^{j+1} V_i$, since letting $v \in V_i$, there is an $x \in \Gamma(v) \cap V_{i-1}$ and $\xi(x) = 0$ by definition.

Although $q_\xi(u_0, 0)$ can be 0, since $k < n$, there is a k-set $U \in Sub_k(V)$ that contains u_1 but not u_0. Letting $U_0 \subseteq U$ be the set of vertices v such that $q_\xi(v, 0) > 0$, we construct a configuration η from ξ by setting the local states of all $v \in U_0$ to 0 and those of all $v \in U \setminus U_0$ to 1. By definition, we observe the following facts, which altogether imply that $\xi \in \Xi_j \setminus \Xi_{j+1}$ is reachable to a certain configuration in Ξ_{j+1}:

- $\eta(u_1) = 0$,
- if $\xi(v) = 0$ and $v \in \cup_{i=0}^{j+1} V_i$ then $\eta(v) = 0$, and
- $p_k(\xi, \eta) > 0$.

We would like to go on the proof for $k = n$, provided that G has a self-loop. Let u_0 be a vertex having a self-loop. For any configuration ξ such that $\xi(u_0) = 0$, $p_n^{(t)}(\xi, \mathbf{0}) > 0$ for some $t \in \mathrm{N}$ since $q_\xi(u_0, 0) > 0$. (One can repeat a similar proof to that for case $k < n$, provided that $q_\xi(u_0, 0) > 0$.) Hence all what we need to do is to show that for any configuration $\xi (\neq \mathbf{1})$, there is a configuration η such that $\eta(u_0) = 0$ and $p_n^{(t)}(\xi, \eta) > 0$ for some $t \in \mathrm{N}$ hold.

Let $\xi (\neq \mathbf{1})$ and u_1 be any configuration and a vertex in V such that $\xi(u_1) = 0$, respectively. Since G is strongly connected, let $X : x_0(= u_1), x_1, \cdots, x_\ell(= u_0)$ be a shortest path from u_1 to u_0. With each $x_j (0 \leq j \leq \ell)$, we associate, in the following, a configuration ξ_j such that $\xi_j(x_j) = 0$ and $p_n(\xi_{j-1}, \xi_j) > 0$ hold. By taking $\xi = \xi_0$ and $\eta = \xi_\ell$, we have $p_n^{(\ell)}(\xi, \xi_\ell) > 0$.

First ξ is associated with $x_0(= u_1)$. For any $1 \leq j \leq \ell$, letting U_j be the set of vertices v such that $q_{\xi_{j-1}}(v, 0) > 0$, the configuration ξ_j associated with x_j, is constructed from ξ_{j-1} by setting the local states of all vertices in U_j to 0. Then clearly $p_n(\xi_{j-1}, \xi_j) > 0$ by definition. In order to observe $x_j \in U_j$ for any $1 \leq j \leq \ell$, we point out that $\xi(x_0) = 0$, and that for any $1 \leq j \leq \ell$, $q_{\xi_{j-1}}(x_j, 0) > 0$, since $x_{j-1} \in \Gamma(x_j)$. ∎

Next we discuss how to calculate the absorbing probability that, from a given initial configuration $\xi \in \hat{\Xi}$, the system reaches a given absorbing state in $\{\mathbf{0}, \mathbf{1}\}$. For any initial distribution π, there exists a (unique) stationary distribution $(\lim_{t\to\infty} \pi P_k^t)$ whose form is $(\overset{\mathbf{0}}{p_1}, 0, \cdots, 0, \overset{\mathbf{1}}{p_2})$ for some $p_1, p_2 > 0$ such that $p_1 + p_2 = 1$, i.e., the support of its stationary distribution is $\{\mathbf{0}, \mathbf{1}\} \subset \Xi$. The following theorem is well-known in the Markov chain theory (see e.g., Feller [3, P.403 Theorem 2]). Let $\mathrm{Absorb}_k(\xi, \mathbf{0})$ be the absorbing probability that, starting from a configuration ξ, the system is absorbed into $\mathbf{0}$.

Theorem 3 $\mathbf{x} = (\mathrm{Absorb}_k(\xi, \mathbf{0}))_{\xi \in \Xi}$ *is a solution of the following equation that satisfies the boundary conditions* $x_\mathbf{0} = 1$ *and* $x_\mathbf{1} = 0$:

$$P_k \mathbf{x} = \mathbf{x}, \tag{5}$$

where $P_k = (p_k(\xi, \eta))_{\xi, \eta \in \Xi}$ *and* $\mathbf{x} = (x_\eta)_{\eta \in \Xi}$.

Conversely there exists a unique $\mathbf{x}$ *that satisfies* Eq. (5).

The above theorem guarantees that $\mathrm{Absorb}_k(\xi, \mathbf{0})$ can be calculated by solving the set of simultaneous linear equations (5) in $O(h^3)$ time, where $h = 2^n - 2$ is the number of variables appeared [2]. However, obtaining an explicit form of $\mathrm{Absorb}_k(\xi, \mathbf{0})$ seems to be difficult in general. In the rest of this section, we give an explicit form of $\mathrm{Absorb}_k(\xi, \mathbf{0})$ for two classes of graphs.

Lemma 1 *Suppose that for any* $\xi \in \Xi$,

$$\sum_{\xi(v)=1} s_\xi(v, 0) = \sum_{\xi(v)=0} s_\xi(v, 1). \tag{6}$$

Then for any $\xi \in \Xi$ *and* $k = 1, \cdots, n$,

$$\mathrm{Absorb}_k(\xi, \mathbf{0}) = \frac{\sum_{\xi(v)=0, v \in V} s(v)}{\sum_{v \in V} s(v)}. \tag{7}$$

Proof. By Theorem 3, it is sufficient to show

$$\sum_{\eta\in\Xi} p_k(\xi,\eta)\left(\sum_{\eta(u)=0} s(u)\right) = \sum_{\xi(u)=0} s(u),$$

for any $\xi \in \hat{\Xi}$ and $k = 1,\cdots,n$. Letting $\eta = \xi^W$ for some $W \subseteq V$, we can rewrite $\sum_{\eta(u)=0} s(u)$ as $\sum_{\xi(u)=0} s(u) + \sum_{u\in W,\xi(u)=1} s(u) - \sum_{u\in W,\xi(u)=0} s(u)$. In the following, we show

$$\sum_{W\subseteq V} p_k(\xi,\xi^W)\left(\sum_{u\in W,\xi(u)=1} s(u) - \sum_{u\in W,\xi(u)=0} s(u)\right) = 0, \tag{8}$$

since

$$\begin{aligned}
&\sum_{\eta\in\Xi} p_k(\xi,\eta) \sum_{\xi(u)=0} s(u)\\
=& \sum_{\xi(u)=0} s(u) \sum_{\eta\in\Xi} p_k(\xi,\eta)\\
=& \sum_{\xi(u)=0} s(u).
\end{aligned}$$

By Proposition 1, Eq. (8) is equivalent to

$$\sum_{U\in Sub_k(V)} \sum_{W\subseteq U} \prod_{w\in W} q_\xi(w,1-\xi(w)) \prod_{v\in U\backslash W} q_\xi(v,\xi(v)) \left(\sum_{u\in W,\xi(u)=1} s(u) - \sum_{u\in W,\xi(u)=0} s(u)\right) = 0.$$

For any $U \in Sub_k(V)$, let

$$A(U) = \sum_{W\subseteq U} \prod_{w\in W} q_\xi(w,1-\xi(w)) \prod_{v\in U\backslash W} q_\xi(v,\xi(v)) \left(\sum_{u\in W,\xi(u)=1} s(u) - \sum_{u\in W,\xi(u)=0} s(u)\right).$$

Then we need to show

$$\sum_{U\in Sub_k(U)} A(U) = 0.$$

We now partially evaluate $A(U)$:

$$\begin{aligned}
A_1(U) =& \sum_{W\subseteq U} \prod_{w\in W} q_\xi(w,1-\xi(w)) \prod_{v\in U\backslash W} q_\xi(v,\xi(v)) \sum_{u\in W,\xi(u)=1} s(u)\\
=& \sum_{u\in U,\xi(u)=1} s(u) \sum_{W:u\in W\subseteq U} \prod_{w\in W} q_\xi(w,1-\xi(w)) \prod_{v\in U\backslash W} q_\xi(v,\xi(v))\\
=& \sum_{u\in U,\xi(u)=1} s(u) q_\xi(u,1-\xi(u)) \sum_{W:u\in W\subseteq U} \prod_{w\in W\backslash\{u\}} q_\xi(w,1-\xi(w)) \prod_{v\in U\backslash W} q_\xi(v,\xi(v))\\
=& \sum_{u\in U,\xi(u)=1} s(u) q_\xi(u,0) \prod_{v\in U\backslash\{u\}} \left\{q_\xi(v,1-\xi(v)) + q_\xi(v,\xi(v))\right\}\\
=& \sum_{u\in U,\xi(u)=1} s(u)\cdot\frac{s_\xi(u,0)}{s(u)}\cdot 1\\
=& \sum_{u\in U,\xi(u)=1} s_\xi(u,0).
\end{aligned}$$

In the same reduction,

$$\begin{aligned}
A_2(U) =& \sum_{W\subseteq U} \prod_{w\in W} q_\xi(w,1-\xi(w)) \prod_{v\in U\backslash W} q_\xi(v,\xi(v)) \sum_{u\in W,\xi(u)=0} s(u)\\
=& \sum_{u\in U,\xi(u)=0} s_\xi(u,1).
\end{aligned}$$

So we obtain

$$\begin{aligned}
A(U) =& \sum_{\xi(u)=1,u\in U} s_\xi(u,0)\\
&- \sum_{\xi(u)=0,u\in U} s_\xi(u,1).
\end{aligned}$$

Since

$$\sum_{U\in Sub_k(V)} A(U) = \binom{n-1}{k-1} \left(\sum_{\xi(v)=1} s_\xi(v,0) - \sum_{\xi(v)=0} s_\xi(v,1)\right),$$

the proof completes by assumption. ∎

We then give two classes of graphs satisfying the condition given by Eq. (6) in Lemma 1. A graph $G = (V, E)$ is said to be *semi-symmetric*, if for any vertex $v \in V$ its indegree coincides with its outdegree, i.e., $|\{u \in V : (u, v) \in E\}| = |\{w \in V : (v, w) \in E\}|$ for any $v \in V$. A graph $G = (V, E)$ is said to be *symmetric* if for any pair $u, v \in V$ of vertices, $(u, v) \in E$ if and only if $(v, u) \in E$. Note that a symmetric graph is semi-symmetric and that there can be a vertex that does not have a self-loop in a symmetric graph.

Theorem 4 *The absorbing probability is given by Eq. (7), if*

(C1) *$G = (V, E)$ is symmetric and μ satisfies that $\mu(u, v) = \mu(v, u)$ for any $(u, v) \in E$, or*

(C2) *G is semi-symmetric and μ is a constant function.*

Proof. We show that the condition (6) holds for any $\xi \in \Xi$, if either (C1) or (C2) holds.

If (C1) holds, by definition, Condition (6) obviously holds for any $\xi \in \Xi$.

As for (C2), since G is semi-symmetric, E can be decomposed into several directed rings $E_1, E_2, \cdots, E_\ell$. Consider any configuration $\xi \in \Xi$. With respect to an arbitrary fixed ring E_i, $|\{(u, v) \in E_i : \xi(u) = 0, \xi(v) = 1\}| = |\{(u, v) \in E_i : \xi(u) = 1, \xi(v) = 0\}|$, which implies that condition (6) holds, since μ is constant. ∎

4 Regular Graphs and Martingales

In this section, using the martingale theory (see, e.g., [6]) we analyze the probabilistic local majority polling game satisfying Eq. 6.

4.1 Martingales

Let $\{X_t\}_{t=0}^{\infty}$ denote the Ξ-valued stochastic process defined by the transition probability given in Proposition 1 on the probability space $(\Xi^{\mathbf{N}}, \mathcal{F}, \mathbf{P}_\xi)$, where $\mathcal{F}$ is the σ-field, and $\mathbf{P}_\xi\{X_0 = \xi\} = 1$ for $\xi \in \Xi$. Namely X_t is the state of Ξ at time t and

$$p_k(\eta_1, \eta_2) = \mathbf{P}_\xi\{X_{t+1} = \eta_2 | X_t = \eta_1\}$$

for $t \in \mathbf{N}$. Let $\mathcal{F}_t \subset \mathcal{F}$ be the smallest σ-field such that all of $X_0, \cdots, X_t$ are measurable. Provided Eq. 6, by the proof of Lemma 1,

$$\sigma(\xi) = \sum_{\xi \in \Xi} p_k(\xi, \eta)\sigma(\eta) \quad \text{for } \xi \in \Xi, \tag{9}$$

where $\sigma(\xi) = \sum_{v \in V, \xi(v)=0} s(v)$. It means that σ is harmonic with respect to the transition matrix P_k. By Eq. (9), we have the following theorem ([6, P.87]):

Theorem 5 *If* Eq. (6) *holds then $(\sigma(X_t), \mathcal{F}_t)$ is a martingale, that is, for any $t \in \mathbf{N}$*

$$\mathbf{E}[\sigma(X_{t+1})|\mathcal{F}_t] = \sigma(X_t) \quad \mathbf{P}_\xi\text{-a.s.}$$

4.2 Regular graphs

Now we consider a regular graph with a constant edge weight function. Here by a *regular graph*, we mean a symmetric graph G such that

(i) every vertex $v \in V$ has a self-loop (v, v), and

(ii) all vertices $v \in V$ have the same indegree $|\Gamma(v)|$ (and hence have the same outdegree).

For a regular graph G and a constant function μ, by Theorem 4 and Eq. (7),

$$\text{Absorb}_k(\xi, \mathbf{0}) = \frac{|\{v \in V : \xi(v) = 0\}|}{|V|}, \tag{10}$$

which property seems to be desirable for a fair agreement. It is worth emphasizing that the probability depends only on the number of vertices having state 0, but not on their positions.

Theorem 6 *Suppose that G is regular (in the sense of this subsection), and μ is constant. Then* *for any $1 \leq k \leq n$, the mean of the number of vertices whose local states* 0 *(or* 0*) is invariant under k-polling, that is, for any $t \in \mathbf{N}$*

$$\begin{aligned}&\mathbf{E}[|\{v \in V : X_t = \eta, \eta(v) = 0\}|] \\ = \ & |\{v \in V : \xi(v) = 0\}|,\end{aligned}$$

where ξ is a given initial configuration.

Proof. Since G is regular, $s(v) = s$ is a constant, and

$$|\{v \in V : \xi(v) = 0\}| = \sigma(\xi)/s. \tag{11}$$

By Theorem 5,

$$\mathbf{E}[\sigma(X_t)/s] = \mathbf{E}[\sigma(X_0)/s].$$

By Eq. (11),

$$\begin{aligned}&\mathbf{E}[\sigma(X_t)/s] \\ = \ & \mathbf{E}[|\{v \in V : X_t = \eta, \eta(v) = 0\}|],\end{aligned}$$

and since $\mathbf{P}_\xi\{X_0 = \xi\} = 1$,

$$\mathbf{E}[\sigma(X_0)/s] = |\{v \in V : \xi(v) = 0\}|.$$

The proof completes. ∎

5 Concluding Remarks

This paper investigated a probabilistic local majority polling game on a weighted directed graph, by formulating it as a Markov chain. From the view of designing an agreement algorithm for all agents to agreeing on an opinion, the deterministic local majority polling game discussed so far is not powerful enough, which motivated our study. We characterized on which graphs the probabilistic game always finishes in an absorbing state, i.e., the agents achieve an agreement. We mainly investigated the probability that the game reaches an arbitrarily given absorbing state.

However, there remain many open problems, among which the the problem of calculating the absorbing time is perhaps the most important. Given a configuration ξ, calculating the mean time $\mathrm{T}_k(\xi, \mathbf{0})$ necessary for the system to reach absorbing state $\mathbf{0}$ is the problem. The following theorem holds using an argument similar to that of the ruin problem [3, P.348]:

Theorem 7 *$\mathbf{y} = (\mathrm{T}_k(\xi, \{\mathbf{0}, \mathbf{1}\}))_{\xi \in \Xi}$ is a solution of the following equation with the boundary conditions $y_\mathbf{0} = 0$ and $y_\mathbf{1} = 0$:*

$$P_k\mathbf{y} + \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} = \mathbf{y}, \tag{12}$$

where $\mathbf{y} = (y_\eta)_{\eta \in \Xi}$.

Conversely there exists a unique $\mathbf{y}$ *that satisfies* Eq. (12).

As in the case of Theorem 3, the time can be calculated by solving the set of simultaneous linear equations (12) in $O(h^3)$ time, where $h = 2^n - 2$ is the number of variables appeared [2]. It is however difficult to obtain an explicit form of the solution of the inhomogeneous difference equation (12), even if the graph is complete, and leave it as a future work.

Acknowledgments The authors would like to thank Mr. Yehuda Hassin at the Weizmann Institute of Science, Israel, for pointing out a bug regarding Theorem 4 in an earlier version of the paper.

参考文献

[1] J-C. Bermond and D. Peleg, "The power of small coalitions in graphs," *Proc. 2nd Structural Information & Communication Complexity,* Olympia, Greece, Carleton Univ. Press, 173-184, 1995.

[2] T. H. Cormen, C. E. Leiserson, and R. L. Rivest, "Introduction to Algorithms," Massachusetts: The MIT Press, 1990.

[3] W. Feller, "An Introduction to Probability Theory and its Applications Vol. I (3rd ed.), " New York: John Wiley & Sons, 1950.

[4] D. Fitoussi and M. Tennenholtz, "Minimal social laws," *AAAI'98,* 26–31, 1998.

[5] Y. Hassin and D. Peleg, "Distributed probabilistic polling and applications to proportionate agreement," *Proc. 26th International Colloquium on Automata, Languages, and Programming,* Prague, Czech Republic, July 1999 (to appear).

[6] J. G. Kemeny, J. L. Snell and A. W. Knapp, "Denumerable Markov chains". Princeton, N.J., D. Van Nostrand Co. Inc. 1966.

[7] N. Linial, D. Peleg, Y. Rabinovich and M. Saks, "Sphere packing and local majorities in graphs," *Proc. 2nd Israel Symposium on Theoretical Computer Science,* IEEE Computer Soc. Press, 141–149, 1993.

[8] D. Peleg, "Local majority voting, small coalitions and controlling monopolies in graphs: A review," Technical Report, Weizmann Institute, 1996.

[9] D. Peleg, "Size bounds for dynamic monopolies," *Discrete Applied Mathematics,* 86, 263–273, 1998.

[10] Y. Shoham and M. Tennenholtz, "Emergent conventions in multi-agent systems: initial experimental results and observations," *Proc. International Conference on Principles of Knowledge Representation and Reasoning,* 225–231, 1992.

[11] Y. Shoham and M. Tennenholtz, "On the systhesis of useful social laws for artificial agent societies," *AAAI-92,* 276–281, 1992.