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# Reconnection and the road to turbulence

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**Abstract**

It will be noted that several initial conditions besides anti-parallel vortices give varying degrees of evidence for a singularity of the incompressible Euler equations. The primary test is whether the peak vorticity approximately obeys $\|\omega\| \sim C_\omega/(T_c - t)$. Another test is the integral enstrophy production growing as $\Omega_{pr} = \int dV \omega_i e_{ij} \omega_j \sim 1/(T_c - t)$. This is related to the requirement for two separate length scales being necessary to describe the collapse to a point. The only flow with evidence for the blowup of $\Omega_{pr}$ besides initially anti-parallel vortices is initially orthogonal vortices where $\|\omega\|$ is becoming singular at the point where arms from each initial vortex have become anti-parallel. $C_\omega \approx 19$ for both initially anti-parallel and orthogonal vortices.

## 1 Introduction

To form a singularity from smooth initial conditions without external forcing implies that the velocity that would be needed to bring two vortex lines together must develop from the nonlinear terms and must be self-induced by the vorticity near the lines themselves through the law of Biot-Savart.

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi} \int -\mathbf{y} \times \omega(\mathbf{x}+\mathbf{y}) \frac{d^3y}{|\mathbf{x}-\mathbf{y}|^3} \qquad (1)$$

A simplified picture of such a singularity is given in Fig. 1. If initially there is no vorticity between two regions of vorticity separated by a distance $\delta$, due to Kelvin's theorem vorticity cannot appear in this gap in the Euler equations unless the regions of vorticity touch, that is as $\delta \to 0$. A condition for a finite

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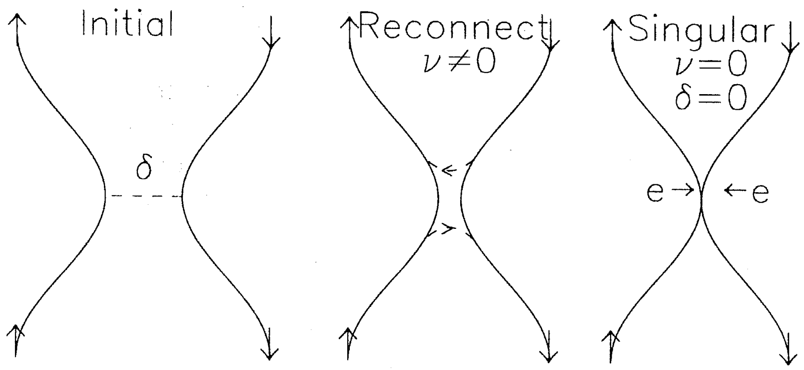


Figure 1: Diagram of the interaction of anti-parallel vortices. From an initial condition of anti-parallel vortices separated at their closest approach by $\delta$, if $\nu \neq 0$ there is reconnection that forms new vortices indicated by the dashed curves. However, if $\nu = 0$, a singularity can form when $\delta = 0$ if the vortices are pushed together by the self-induced strain indicated by $e$.

time singularity can be placed upon the the strain $e$ perpendicular to the vortex lines that pushes them together using an equation for $\delta$.

$$d\delta/dt = -e\delta \quad , \tag{2}$$

which integrated out becomes

$$\delta(t) = \exp(-\int_0^t e ds) \quad .$$

and implies that $\delta(t) \to 0$ if

$$\int_0^t e ds \to \infty \tag{3}$$

This can happen in a finite time of $t = T_c$ only if the strain $e$ goes to infinity with a form implied by this equation.

The original, precise definition of this condition [2] was written in terms of the $L_\infty$ norm of vorticity $||\omega||_\infty$ or the peak vorticity $\omega_p$ as the condition that if there is to be a singularity at a time $T_c$ then

$$\int_0^{T_c} ||\omega||_\infty ds \to \infty \tag{4}$$

is required. In other words, not only must $||\omega||_\infty \to \infty$, but it must do so in a time integrable manner. It has since been shown [22] that this condition applies to any derivative of the velocity field, including the strain pushing vortex lines together.

This condition (4) has several implications. First, there cannot be any singularities in higher derivatives of the velocity without there being a singularity in the vorticity. There is no need to look for singularities in higher derivatives, which makes the search for a singularity tractible with numerical methods. The second property of (4) that is important for simulations is that if the growth of $||\omega||_\infty$ is assumed to be algebraic

$$||\omega||_\infty \sim 1/(T_c - t)^\gamma \tag{5}$$

then (4) implies that $\gamma \geq 1$ if there is to be a singularity.

Based upon dimensional analysis, the expectation is that $\gamma \equiv 1$, that is its lower bound. This expectation was used in the analysis of the evolution of pair of anti-parallel vortex lines using the Euler equations [14] and in the analogous analysis of the growth of current in ideal magnetohydrodynamics [19].

While a relationship between strain and the growth of the vorticity is implied by the mathematics used to bound vorticity (4), the equation relating strain to vorticity directly was not used. This equation is that the strain in the direction of vorticity $\alpha$ at position $\mathbf{x}$ [9] is

$$\alpha(\mathbf{x}) = \frac{3}{4\pi} \int D(\hat{\mathbf{y}}, \hat{\omega}_x, \hat{\omega}_{x+y}) \omega_x \frac{d^3y}{|\mathbf{y}|^3} \tag{6}$$

and $D$ is a geometrical factor determined by finding the volume of the trapezoid formed by the three unit vectors given. This equation can be used to derive higher order conditions for the existence of a singularity of Euler that bring role of the curvature of vortex lines [10, 11] into play. These conditions are used to bound the BKM result, and so become bounds on a singularity themselves.

One condition written in terms of algebraic collapse is that a length

$$R \sim |\nabla(\hat{\omega}(x+y) - \hat{\omega}(x))|^{-1} \tag{7}$$

that could be physically interpreted as a radius of curvature, should obey

$$\int_0^{T_c} R^{-2}(t)ds \rightarrow \infty \tag{8}$$

if there is a singularity [10]. If there is an algebraic decrease like $R \sim (T_c - t)^p$, this implies $p \geq 1/2$ and $p \equiv 1/2$ is expected, just as $\gamma \equiv 1$ is expected for $||\omega||_\infty$ (5).

Another length scale that comes into the proof of this result in [10] is due to the necessity that Biot-Savart contributions to the growth of the peak vorticity cannot come within some distance $\rho$ of $||\omega||_\infty$. That is, a self-similar form consistent with $p = 1/2$ cannot go all the way to $y = 0$. The mathematics would suggest that $\rho/R \sim \int_0^t (1/R)ds$ or $\rho \sim (T_c - t)$ if $p = 1/2$ for $R$. Evidence for two length scales collapsing respectively as $R$ and $\rho$ has been presented before [18]. An important consequence of having two length scales collapsing in this manner is that the volume integral of the enstrophy production should go as the peak vorticity cubed times the volume of a sheet of extent $R$ and thickness $\rho$:

$$\Omega_{pr} = \int dV \omega_i e_{ij} \omega_j \sim O(\rho R^2 \omega_p^3) \sim 1/(T_c - t) \tag{9}$$

The three tests used to provide evidence for a singularity of incompressible Euler [14] are summarized in Figure 2.

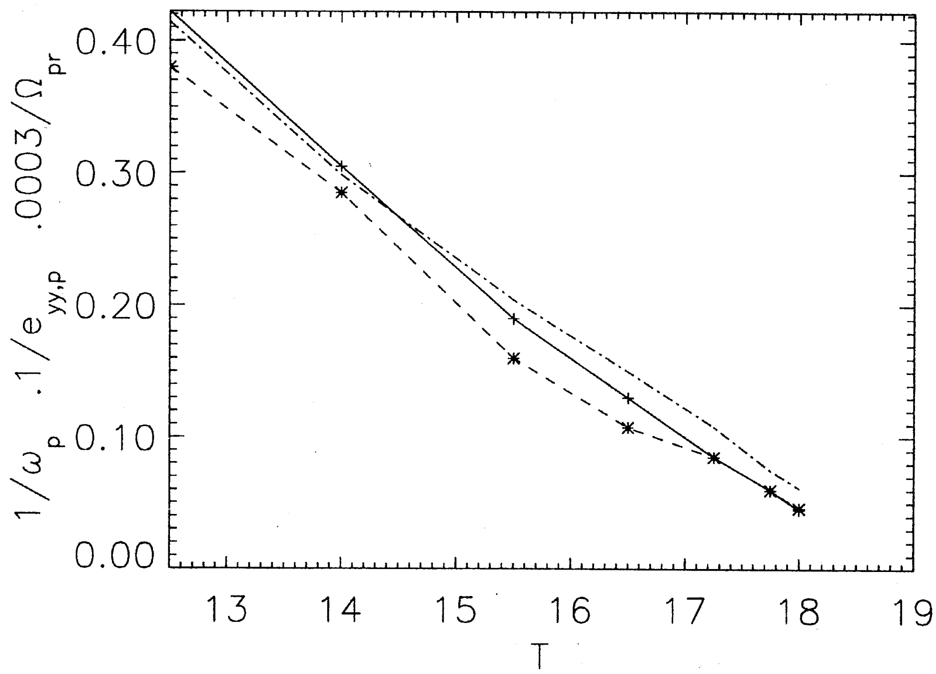


Figure 2: Dependence of $1/||\omega||_\infty$, $1/||e_{yy}||_\infty$ and $1/\int dV \omega_i e_{ij} \omega_j$ on time from the anti-parallel Euler calculation [14] showing convergence to a singular time of about $T = 18.7$.

An important implication concerning blowup of the enstrophy production $\Omega_{pr}$ in eq. (9) is that it implies that $\Omega = ||\omega||_2^2 \sim -\log(T_c - t)$, that is marginally singular growth of enstrophy.

## 2 What is needed next.

The anti-parallel initial condition is rather contrived and thus raises many questions if we are to be able to claim that this has anything to do with turbulence.

One is whether the evolution of other structures could satisfy the mathematical requirements. A related question is whether from relatively arbitrary initial conditions, does something resembling anti-parallel vortices always evolve? Then there is the question of what happens after viscosity plays a role and how does something resembling fully-developed turbulence, with the scaling laws that have been discussed, appear. At the last time of the Euler calculations the spectra are very steep, possibly approaching $E(k) \sim k^{-3}$, and the structure near the singular point is more like a sheet than a tube. There must be some significant changes if it is to go from this state to a fully turbulent state with a $E(k) \sim k^{-5/3}$ spectrum dominated by vortex tubes.

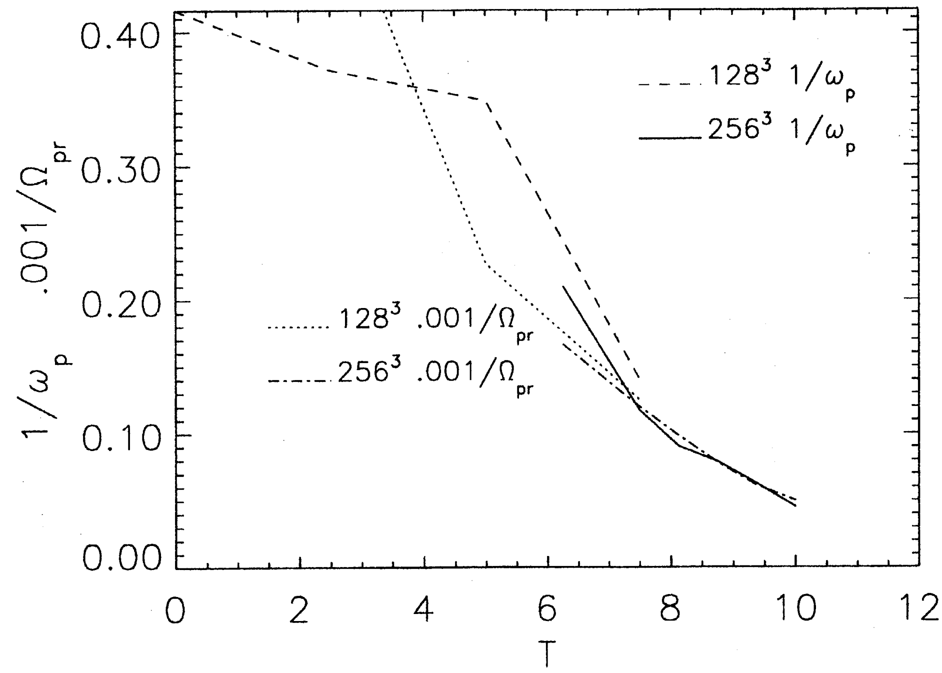


Figure 3: $1/||\omega||_\infty$ and $1/\omega_i e_{ij} \omega_j$ in Euler for orthogonal vortices.

Calculations that have looked at reconnection or singularities can be classified as either vortex filament calculations or simulations from smooth initial conditions of only a few Fourier modes. A possibly important parameter might be the helicity, a second quadratic invariant of the three-dimensional Euler equations,

$$H = \int dV \mathbf{u} \cdot \omega \quad . \tag{10}$$

Besides the anti-parallel vortex filament calculations, two additional vortex filament calculations with net initial helicity are linked vortex rings and othogonal tubes. Linked tubes have only been studied at low resolution [1] where surprising effects in helicity $H$ were found. The recent success in finding signs of singular behavior linked magnetic flux rings suggests that the hydrodynamic problem be revisited.

Figure 3 shows $1/||\omega||_\infty$ and $\int dV \omega_i e_{ij} \omega_j$ for a new orthogonal vortex tube simulation. This is the only evidence for a singularity of Euler besides the anti-parallel case that this author believes meets the BKM tests and supports the previous claim [4] that this type of initial condition has singular behavior. It was in fact anamolous trends in one of the early orthogonal vortex simulations by Melander and Zabusky presented at a meeting in Los Alamos in 1988 that led this author to conclude that there was a possibility of a singularity of Euler. The important property that might predispose this initial condition to singular behavior might be that it naturally localizes the interaction. Figure 4 shows isosurfaces of this flow. Near the point of $||\omega||_\infty$ at $t = 10$, the vortices appear to be anti-parallel, while away from this point there are sheets of vorticity being shed off that look similar to the winding of vortex lines in the outer parts in the singular anti-parallel case shown in [17]. The singular time is predicted to be about $T = 2$. At any given time $(T-t)||\omega||_\infty \approx 20$, which is consistent with the anti-parallel analysis in Figure 2. It is also consistent with the growth of $||\omega||_\infty$ for the case of a perturbed cylindrical shear calculation [12].

Thus the orthogonal vortex case could be important for showing that the singular scaling of the anti-parallel case might be unique, that is the only possibility. However, what BKM analysis has been done with viscous, smooth initial conditions [15] finds $(T-t)||\omega||_\infty < 10$ not $\approx 20$ for the Taylor-Green case. It would therefore be useful to investigate these smooth initial conditions at sufficient resolution to clarify these trends, and also to determine what happens after a near singular event. To date, $\nu = 0$ simulations starting with smooth initial conditions only show sheet-like non-singular behavior [8]. At least for Taylor-Green the equivalent of at least $2048^3$ could be run, which should be sufficient to get out of the sheet-regime and into a regime that in viscous calculations [15] shows $||\omega||_\infty \sim 1/(T-t)$.

Higher resolution calculations from smooth initial conditions such as Taylor-Green [7], Kida flow [5], and from random initial conditions [13] are needed because for each of these cases has either when weakly linear $1/||\omega||_\infty$ behavior or another singular behavior. In all of these calculations, the spectrum at $T_c$ is always the order of $k^{-4}$, which is very steep in a manner similar to the anti-parallel Euler case [14] which found $k^{-3}$ as $t \to T_c$. To go from nearly singular behavior to turbulence the energy transfer mechanism after the near singularity must provide a means to go from this $k^{-3}$ spectrum to $k^{-5/3}$. For all of the cases mentioned this occurs at $t \approx 2T_c$ as well as a peak in the enstrophy and a flow dominated by vortex tubes. That is, all the properties of fully-developed turbulence.

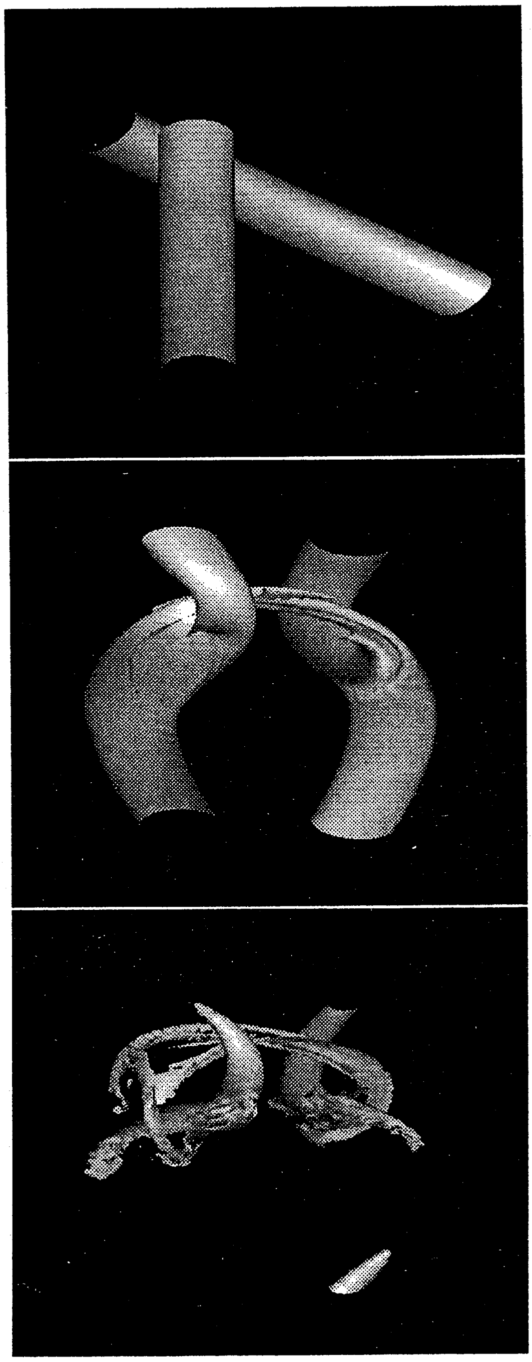

Figure 4: $1/||\omega||_\infty$ and $1/\omega_i e_{ij}\omega_j$ in Euler for orthogonal vortices. The three frames are $t = 0$, 6 and 10. Arms are pulled out of the original vortices, become anti-parallel, then vorticity within the arms develops singular behavior.

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