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On the generation of spatially-periodic longitudinal vortices in shear flows.

A. D. D. Craik
School of Mathematical and Computational Sciences,
University of St Andrews, St Andrews, Fife KY16 9SS, U.K.
and
Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan.

Abstract

'Longitudinal vortices' are vortical structures with axis aligned approximately along the direction of the mean flow. They are common both in turbulent shear flows and as part of the developing disturbances in unstable laminar shear flows. Here, we consider physical mechanisms for the creation of periodic arrays of longitudinal vortices in unstable, nearly-parallel, shear flows.

1. Mode-interaction scenarios

In linear stability theory, the Squire transformation normally ensures that the linearly most unstable disturbance in a parallel laminar shear flow is two-dimensional, with no dependence on the spanwise coordinate. Thus, in Cartesian coordinates (x, y, z), when a primary shear flow $\boldsymbol{u}_0 = [U(z), 0, 0]$ becomes unstable, as the Reynolds number R is increased beyond some critical value, two-dimensional Tollmien-Schlichting waves develop. These have the form

$$\boldsymbol{u}_1 = \mathrm{Re}\{\varepsilon A[\phi', 0, -i\alpha\phi]\exp[i(\alpha x - \omega t)]\},$$

where ε is a small parameter, $\phi = \phi(z)$, α is the wavenumber and ω the frequency of the wave. In linear theory, the complex amplitude function A is constant; but this varies slowly in time or space when nonlinear effects are admitted. One or other, or both, of α, ω may be complex, depending on the problem studied: their values are connected by an eigenvalue relationship $F(\alpha, \omega, R) = 0$, resulting from the fourth-order Orr-Sommerfeld equation for $\phi(z)$ and appropriate boundary conditions. Their imaginary parts are usually numerically small. Real α and a positive imaginary part for ω corresponds to temporal growth; while real ω and a negative imaginary part for α corresponds to spatial (downstream) growth. For a comprehensive account of linear hydrodynamic stability theory, see Drazin & Reid [1].

However, when an unstable two-dimensional Tollmien-Schlichting wave reaches finite amplitude, it promotes the growth of initially infinitesimal three-dimensional waves which would otherwise be linearly damped. This is known as *secondary* or *parametric instability.* In boundary layers and non-symmetric channel flows, the first such instability to arise is that of a symmetric pair of *subharmonic* modes

$$u^{+}_{2} = \mathrm{Re}\{\varepsilon B_{+}[u_{+}, v_{+}, w_{+}]\exp[i(\alpha_2 x + \beta y - \omega_2 t)]\},$$
$$u^{-}_{2} = \mathrm{Re}\{\varepsilon B_{-}[u_{-}, v_{-}, w_{-}]\exp[i(\alpha_2 x - \beta y - \omega_2 t)]\},$$

with linear frequency ω_2 and x-wavenumber α_2 having real parts close to *half* those of the two-dimensional wave. The value of β must be such that this subharmonic resonance condition is nearly satisfied. The three participating modes interact nonlinearly, and equations governing the evolution of $A(t)$, $B_{+}(t)$, $B_{-}(t)$ may be found. These contain quadratic interaction terms, characteristic of *three-wave resonance*: see, for example, Craik [2], Bayly, Orszag & Herbert [3], Kachanov [4]. At first, the amplitudes B_{+} and B_{-} grow exponentially if A is constant, but the interaction later becomes more complex; and, eventually, other modes also grow to prominence and transition to turbulence takes place.

Before then, longitudinal vortices are generated by the quadratic interaction of modes u^{+}_{2} and u^{-}_{2}: their periodic structure, coming from the difference of the above exponents, is like $\sin 2\beta y$ and $\cos 2\beta y$ and their amplitude is of order $O(\varepsilon^2|B|^2)$. This final part of the mechanism was first suggested by Benney & Lin [5], who examined the weak nonlinear interaction of two constant-amplitude oblique waves; and there have been several more recent studies. I shall designate this mechanism as "*Scenario A*".

An alternative mechanism, which I shall call "*Scenario B*", involves *cubic*, rather than quadratic, wave interactions, and so usually requires a larger two-dimensional mode to trigger the secondary instability. The characteristic form of the secondary modes is then

$$u^{+}_{2} = \mathrm{Re}\{\varepsilon B_{+}[u_{+}, v_{+}, w_{+}]\exp[i(\alpha_2 x + \beta y - \omega_2 t)]\},$$
$$u^{-}_{2} = \mathrm{Re}\{\varepsilon B_{-}[u_{-}, v_{-}, w_{-}]\exp[i(\alpha_2 x - \beta y - \omega_2 t)]\},$$

where α_2 and ω_2 are close to α and ω respectively, rather than their half as in *Scenario A*: but exact equality cannot be attained (except for $\beta = 0$) and the preferred β-value is less certain. The generation of longitudinal vortices by the quadratic interaction of these modes u^{+}_{2} and u^{-}_{2} proceeds as before, giving a spanwise periodicity in $\sin 2\beta y$ and $\cos 2\beta y$. But, additionally, quadratic interaction of u_1 with u^{+}_{2} and with u^{-}_{2} now drives longitudinal-vortex components with spanwise periodicity $\sin\beta y$ and $\cos\beta y$. The latter has an additional slow periodic dependence on x and t, in $\exp\{i[(\alpha_2 - \alpha)x - (\omega_2 - \omega)t]\}$, and amplitude of order $O(\varepsilon^2|A||B|)$.

Both *Scenario A* and *Scenario B* have been realised in experiments and in theoretical and numerical studies for the Blasius boundary layer: see [2], [3], [4]. But, for plane Poiseuille flow, the flow symmetry prohibits significant quadratic interactions and *Scenario A* is then unlikely to occur.

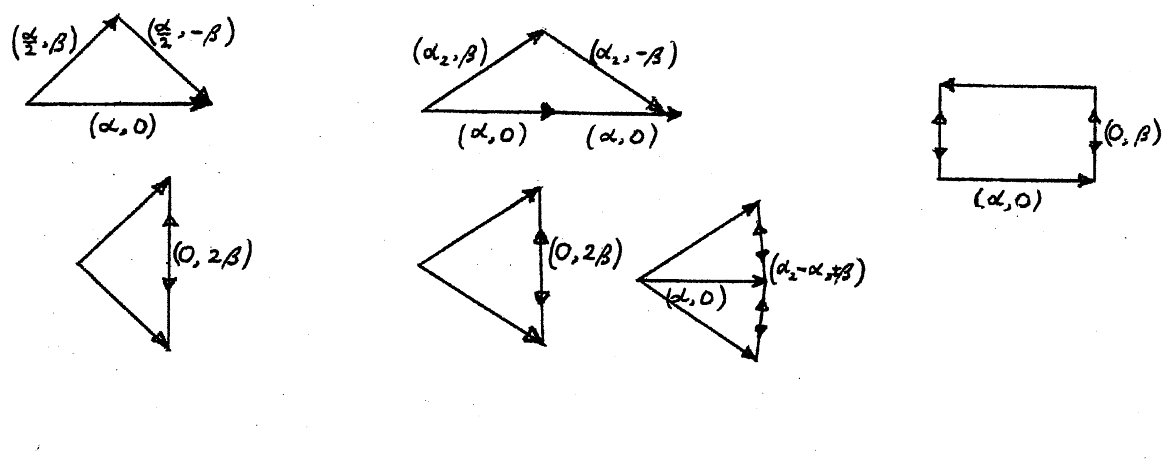
There is a third possible scenario, "*Scenario C*", in which longitudinal vortices are *directly* driven unstable by the two-dimensional wave. This requires cubic-order interactions, and may be envisaged as follows. In addition to the two-dimensional mode u_1, let there be an infinitesimal longitudinal-vortex mode of form

$$u_v = \mathrm{Re}\{\delta C[u, v, w]\exp(i\beta y - \sigma t)\}$$

with spanwise wavenumber β and small linear damping rate σ. Here, δ is an independent small parameter. Quadratic interaction of u_1 and u_v gives forced $O(\varepsilon\delta|A||C|)$ terms with exponentials of form $\exp[i(\alpha x + \beta y - \omega t) - \sigma t]$ and $\exp[i(\alpha x - \beta y - \omega t) - \sigma t]$. These, in turn, interact quadratically with u_1 to provide nonlinear terms of order $O(\varepsilon^2\delta|A|^2|C|)$ with exponentials of form $\exp(i\beta y - \sigma t)$ and $\exp(-i\beta y - \sigma t)$ which can act to reinforce u_v. Note that the mechanism is linear in δC, and will involve a possible $O(\varepsilon^2|A|^2)$ nonlinear amplification rate which must be large enough to overcome the linear damping rate σ.

It is likely that this *Scenario C* will operate, along with *Scenarios A* and *B*, in Blasius and other shear flows, but it has received comparatively little attention to date. It certainly operates effectively in *weak* shear flows in the presence of sufficiently large-amplitude two-dimensional waves. Depending on the nature of the dominant wave field, the phenomenon of *Langmuir circulations* in lakes and seas may be caused by *Scenario C* or by direct forcing by oblique-wave components, as in *Scenario A* and *B*: see Craik [6], Craik & Leibovich [7], Leibovich [8]. A photograph of Langmuir circulations is shown in [2, p.121]. Physical explanations for such growth of longitudinal vorticity are given in the next section.

All three scenarios are depicted schematically in Figure 1, which shows typical wavenumber interaction diagrams for the participating modes.



(a) (b) (c)

Figure 1: Wave-vector interaction diagrams for (a) Scenario A; (b) Scenario B, (c) Scenario C.

2. Physical mechanisms

The creation of longitudinal vortices is now considered from a physical point of view. In each of the above three scenarios, nonlinear interactions take place among the various participating Fourier modes.

(i) Eulerian view

Such weak interactions may be interpreted in terms of the so-called perturbation "*Reynolds stresses*" which appear in the Eulerian formulation of the equations of fluid motion. Denoting each velocity component in Cartesian tensor form as u_i (i = 1,2,3), the nonlinear convective terms of the momentum equation may be expressed as gradients of the Reynolds stresses τ_{ij}, defined as $\tau_{ij} \equiv u_i u_j$. (We take the constant density to be unity without loss of generality.) Thus, the momentum equation in the i-direction contains the terms $\tau_{ik,k}$ with summation over k = 1,2,3: see, for example, [9]. Clearly, longitudinal vortices are likely to result when Reynolds-stress terms of appropriate periodicity are present.

Equivalently, the Reynolds stress gradients may instead be viewed as a "*vortex force*" which has the form $\boldsymbol{u} \times \boldsymbol{\omega}$, the vector product of the velocity and vorticity. In the vorticity equation, the corresponding terms appear concisely as the curl of this "force", or torque, $\nabla \times (\boldsymbol{u} \times \boldsymbol{\omega})$. Longitudinal vorticity must result when the x-component of this torque has terms of appropriate spanwise-periodic form: cf [8].

However, a strictly Eulerian viewpoint somewhat conceals the dynamics and kinematics of vorticity; for, in non-viscous flows, the well-known Helmholtz vorticity theorems tell us that vortex lines *coincide with material fluid particles.* They therefore move with the Lagrangian velocity, or the velocity of fluid particles, rather than the Eulerian velocity, which is the velocity measured at fixed points in coordinate space. In time-dependent flows, and so in our "wavy" flows, these velocities are not the same.

(ii) Mixed Eulerian-Lagrangian view for weak shear flows

When the mean flow is weak, as in the Langmuir circulation problem mentioned above, a particularly simple physical interpretation exists in terms of the kinematics of vortex lines. This derives from a mixed Eulerian and Lagrangian formulation, developed in great generality by Andrews & McIntyre [10,11] as their *Generalised Lagrangian Mean* (or "*GLM*") *equations*, and later applied to Langmuir circulations in [7] and [12].

For simplicity, consider inviscid fluid of infinite depth $0 < z < \infty$ with undisturbed free surface at $z = 0$. The free surface supports irrotational gravity waves of linearised form

$$\boldsymbol{u}_1 = [\partial\phi/\partial x,\ 0,\ \partial\phi/\partial z], \qquad \phi = \mathrm{Re}\{\varepsilon A e^{-\alpha z} \exp[i(\alpha x - \omega t)]\},$$

where ϕ is the velocity potential, the wavenumber α is real, and $\omega(\alpha)$ satisfies the linear dispersion relation $\omega^2 = \alpha g$ where g is gravitational acceleration. Suppose, also, that there is a *weak* $O(\varepsilon^2)$

Eulerian mean shear flow $\boldsymbol{u}_0 = \varepsilon^2[U(z), 0, 0]$. At $O(\varepsilon^2)$, there is a small second-harmonic wave contribution in $\exp[2i(\alpha x - \omega t)]$, but no wave-induced alteration to the mean flow. (However, when viscosity is taken into account, additional mean-flow terms do arise: see for example Craik [13]). It might be thought, therefore, that individual fluid particle velocities consist only of the $O(\varepsilon)$ circular orbital motion of the irrotational waves, plus the $O(\varepsilon^2)$ mean flow $\boldsymbol{u}_0(z)$ evaluated at the average depth of the particle. But this is not the case. In addition, there is an $O(\varepsilon^2)$ contribution from the "*Stokes drift*" $\boldsymbol{u}_S$ which is the difference between the mean Lagrangian velocity and the mean Eulerian velocity. The latter is

$$\boldsymbol{u}_S \equiv \varepsilon^2\{u^s{}_i\} \equiv \varepsilon^2\langle u^w{}_{i,k}\textstyle\int u^w{}_k dt\rangle,$$

where the angled brackets denote time-average over the period of the waves, and the linear wave velocity field is re-expressed in tensor form as $\boldsymbol{u}_1 \equiv \varepsilon\{u^w{}_i\}$. In deep water, this Stokes drift is

$$\boldsymbol{u}_S = \varepsilon^2|A|^2[(\alpha^3/\omega)\exp(-2\alpha z), 0, 0].$$

The above definition of $\boldsymbol{u}_S$ is applicable to *any* specified $O(\varepsilon)$ wave field $\varepsilon u^w{}_i$, with suitable averaging. A pair of obliquely-propagating gravity waves with respective potentials

$$\phi_+ = \mathrm{Re}\{\varepsilon A e^{-\gamma z}\exp[i(\alpha x + \beta y - \omega t)]\}, \qquad \phi_- = \mathrm{Re}\{\varepsilon A e^{-\gamma z}\exp[i(\alpha x - \beta y - \omega t)]\},$$

where $\gamma \equiv (\alpha^2 + \beta^2)^{1/2}$, has the wave velocity field $\boldsymbol{u}_1 \equiv \varepsilon\{u^w{}_i\} = [\partial\phi/\partial x, \partial\phi/\partial y, \partial\phi/\partial z]$ with $\phi = \phi_+ + \phi_-$. Its corresponding Stokes drift has both mean and spanwise-periodic components:

$$\boldsymbol{u}_S = \varepsilon^2|A|^2[u^s, 0, 0], \qquad u^s \equiv (\alpha/\omega)\exp(-2\alpha z)\{\alpha^2 + \beta^2 + \alpha^2\cos(2\beta y)\}.$$

It turns out that the mean vorticity vector $\langle\omega\rangle \equiv \varepsilon^2\omega_0$, averaged over the wave period, satisfies the equation

$$\partial\omega_0/\partial\tau = (\omega_0 . \nabla)(\boldsymbol{u}_0 + \boldsymbol{u}_S) - (\boldsymbol{u}_0 + \boldsymbol{u}_S).\nabla\omega_0,$$

where τ is a slow timescale, defined by $t \equiv \varepsilon^2\tau$, and a weak viscous diffusion term may also be introduced if required (see [7]). It is obvious from this equation that the mean vortex lines are advected, stretched and tilted by the *Lagrangian* particle velocity $\boldsymbol{u}_0 + \boldsymbol{u}_S$ as stated above. The role of the Stokes drift in the kinematics of mean vorticity is thereby made apparent.

The development of longitudinal vortices is seen as a simple kinematic effect, in the case of the oblique wave pair just described. Consider a set of vortex lines associated with the mean Eulerian flow $\boldsymbol{u}_0$ initially with no longitudinal vortices present: these lines are directed along the y-axis. But they are carried by the average Lagrangian velocity, which has a spanwise-periodic component deriving from the Stokes drift, as stated above. Accordingly, they are soon distorted and a spanwise-periodic x-component of vorticity appears, as required. This is just the kinematic interpretation of the final stage of *Scenarios A* and *B* described above.

With a purely two-dimensional wave field, the process is a little more subtle; this corresponds to our *Scenario C*. First, envisage that there are very weak pre-existing longitudinal vortices. Since these distort the Eulerian velocity field $\boldsymbol{u}_0$, they are necessarily three-dimensional, with spanwise-periodic velocity components in all three directions. Accordingly, the mean vortex lines which initially lay along the y-direction have spanwise-periodic components in both the x-

and z- directions. But the z-vorticity is tilted by the z-dependent Stokes drift, so creating new x-vorticity and intensifying the original longitudinal vortices. This strengthens exponentially on the timescale τ. But, if there were no z-vorticity, there would be no intensification of the x-vorticity. This intensification takes place *entirely* because of the Stokes drift: the Eulerian velocity is ineffective because of cancellation of the stretching and tilting due to the two spanwise-periodic terms in $\boldsymbol{u}_0$. In other words, for the mechanism to operate, it needs *both* an Eulerian shear and a z-dependent Stokes drift.

A fuller description of this mechanism, and the supporting analysis, is given in [12]. There, it is shown that the inviscid vorticity equation yields

$$d^2w/dz^2 + 4\beta^2[(u_s'u_0'/\sigma^2) - 1]w = 0, \qquad (*)$$

where w is the vertical velocity component of the longitudinal vortices, and u_s', u_0' are respectively the z-derivatives of the Stokes drift and the primary Eulerian mean shear flow. The quantity σ is an eigenvalue, corresponding to the required exponential growth rate. With suitable boundary conditions applied at the free surface $z = 0$ and at $z = -\infty$ (or at a horizontal bottom $z = -h$), the most unstable eigenvalue may be found by standard methods.

There is a striking similarity between this eigenvalue problem, and that for inviscid Rayleigh-Bénard convection in horizontal layers of fluid heated from below. In the latter, buoyancy takes the place of $u_s'u_0'$. An even stronger connection exists between our problem and that for inviscid Taylor-Görtler instability of flow $u_0(z)$ over a concave wall. There, the corresponding equation is

$$d^2w/dz^2 + 4\beta^2[(2Ku_0u_0'/\sigma^2) - 1]w = 0,$$

where K denotes the wall curvature (see [12, p.50]). It is natural to seek some connection between $2Ku_0$ and the Stokes drift gradient u_s'. In fact, if we evaluate the average quantity $\langle 2Ku \rangle$ where K is the streamline curvature of our wavy flow and u is the total Eulerian x-velocity, it turns out that the leading-order contribution, at $O(\varepsilon^2)$, is precisely equal to u_s', the z-derivative of the Stokes drift! This allows yet another, rather unexpected, physical interpretation of the instability mechanism: it may be viewed as a sort of averaged Taylor-Görtler instability, driven by the mean "centrifugal force" within the wavy flow.

(ii) Mixed Eulerian-Lagrangian view for stronger shear flows

The above interpretation in terms of Stokes drift is inadequate for $O(1)$ shear flows and $O(\varepsilon)$ waves, such as Tollmien-Schlichting waves in a Blasius boundary layer; but it remains sufficient for $O(\varepsilon)$ waves in $O(\varepsilon)$ mean flows after some rescaling (see [12]). Extension to $O(1)$ shear flows was also considered in [12], using the GLM equations. It is inappropriate to repeat the details here. It is enough to remark that, in place of Stokes drift, Andrews & McIntyre's concept of "*pseudomomentum*" appears: this is an $O(\varepsilon^2)$ entity, which is known explicitly in terms of the specified $O(\varepsilon)$ wave field. For weak mean flows, this is identical with the Stokes drift, but the two quantities differ for strong mean shear. The longitudinal-vortex instability in presence of two-dimensional waves in strong shear is found to be governed by a more complicated eigenvalue problem than that described above. An equation somewhat resembling (*) is obtained; but u_s' is replaced by the corresponding z-gradient of pseudomomentum, and there is an additional term on the right-hand side which describes the influence of spanwise-periodic modifications of the pseudomomentum caused by the presence of the longitudinal vortices. To evaluate this term it is necessary first to solve another, inhomogeneous, equation which governs the modification of the waves, at order $O(\varepsilon\delta)$, due to the presence of the longitudinal vortices.

To illustrate this process, Craik [12] obtained analytic solutions in cases where the vortex spacing is small compared with the wavelength. For simplicity, he examined non-viscous uniform shear flows between two sinusoidal "wavy walls" with differing amplitudes. He established that the expected instability exists *whenever the z-dependent wave amplitude decreases in the direction of increasing speed of the primary shear flow.* Also, instability persists when the amplitude *increases* in this direction, provided the waves are sufficiently long compared with the channel width.

No further developments of this line of enquiry took place for more than a decade, until W.C.R. Phillips and co-workers [14], [15], [16] undertook several extensions of the analysis of [12]. Employing the GLM formulation allied with numerical computations, and incorporating weak viscous effects, they have significantly increased our knowledge of wave-induced longitudinal-vortex instabilities. Since space does not permit an account of their recent work, the reader is referred to their papers cited here, and others forthcoming.

It is worth adding a comment regarding the relative merits of a purely Eulerian approach and the GLM approach just discussed. It is certainly the case that, following intensive study over several decades, the Eulerian approach has reached a high degree of sophistication: incorporated into numerical codes, it is now capable of great precision. in contrast, the GLM formulation has received scant attention, and it suffers from the great disadvantage of unfamiliarity: the various concepts, such as pseudomomentum, pseudoenergy and so on at first seem strange, and an apparently unnecessary complication.

However, once one is familiar with these concepts, the GLM approach offers a useful alternative method of analysis that is sometimes more direct that the purely Eulerian one. For instance, the present author's first calculations of Langmuir circulations were performed with a Eulerian formulation which, though (eventually!) yielding correct results, completely *obscured* the simple physical understanding of these results in terms of the Stokes drift. And, when the analysis was repeated using the GLM formulation, the same results were obtained after far less algebraic complexity. However, the GLM method has some drawbacks. The most important of these concerns "critical layers" where the wave velocity and Eulerian mean flow velocity almost coincide: for, there, the usual GLM averaging procedures do not apply. When critical layers are absent or unimportant, and viscous effects are quite weak, the GLM method has much to commend it.

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