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MODULI OF SEXTICS AND ITS GEOMETRY

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1. INTRODUCTION

Let $\mathcal{M}$ be the moduli space of sextics with 6 cusps and 3 nodes. A sextic C is called *of (2,3)-torus type* if its defining polynomial f has the expression $f(x,y)=f_2(x,y)^3+f_3(x,y)^2$ for some polynomials f_2, f_3 of degree 2, 3 respectively. Hereafter we simply say *of torus type* in the sense *of (2,3)-torus type*. We denote by $\mathcal{M}_{torus}$ the component of $\mathcal{M}$ which consists of curves of torus type and by $\mathcal{M}_{gen}$ the curves of non-torus type. We denote the dual curve of C by C^*. In our previous paper [O2], we have shown that *the dual curve operation $C\mapsto C^*$ gives an involution on $\mathcal{M}$ and it preserves the type of the curve in $\mathcal{M}$*, i.e., *$C^*\in\mathcal{M}_{torus}$ if and only if $C\in\mathcal{M}_{torus}$*. Let $\mathcal{N}_3$ be the moduli space of sextics with 3 (3,4)-cusps as in [O2]. For brevity, we denote $\mathcal{N}_3$ by $\mathcal{N}$. We have shown that $\mathcal{N}$ is in the closure of $\overline{\mathcal{M}}$ and the dual curve C^* of a generic $C\in\mathcal{N}$ is a sextic with 6 cusps and three nodes i.e., $C^*\in\mathcal{M}$ ([O2]). Let $G:=\mathrm{PGL}(3,\mathbf{C})$. The quotient moduli spaces are by definition the quotient spaces of the moduli spaces by the action of G.

In §2, we will study the quotient moduli space $\mathcal{M}/G$ and we will show that there exists an involution $\bar{\iota}$ on $\mathcal{M}/G$ such that $\bar{\iota}$ is different from the dual curve operation and $\bar{\iota}$ preserves the types of the sextics (Theorem 2.3).

In §3, we study the quotient moduli space $\mathcal{N}/G$. We will show that $\mathcal{N}/G$ is one dimensional and consists of two components $\mathcal{N}_{torus}/G$ and $\mathcal{N}_{gen}/G$ consisting of sextics of torus type and non-torus type respectively. Using their normal forms, we show that $\mathcal{N}_{torus}/G$ contains a unique sextic which is self dual (Theorem 3.9).

2. INVOLUTION ON THE QUOTIENT MODULI $\mathcal{M}/G$

Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be the moduli space of sextics with three nodes and 6 cusps and the moduli space of irreducible plane curves of degree 12 with 24 cusps and 24 nodes respectively. Note that the genus of a generic curve in $\mathcal{M}$ (respectively in $\widetilde{\mathcal{M}}$) is 1 (resp. 7). By the class formula ([N] or [O2]), it is easy to see that for a generic $C\in\widetilde{\mathcal{M}}$, the dual curve C^* is also in $\widetilde{\mathcal{M}}$. We consider the mapping

$$\pi:\mathbf{P}^2\to\mathbf{P}^2,\quad (X,Y,Z)\mapsto(X^2,Y^2,Z^2)$$

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which is a 4-fold covering branched along the coordinate axes $\{X = 0\} \cup \{Y = 0\} \cup \{Z = 0\}$. Take a generic curve $C \in \mathcal{M}$ and let $F(X, Y, Z)$ be the defining homogeneous polynomial of degree 6. As C^* has three nodes, C has three bi-tangent lines. We denote by $\mathcal{M}^{nml}$ the subset of $\mathcal{M}$ which consists of curves $C \in \mathcal{M}$ whose three bitangent lines are $X = 0$, $Y = 0$ and $Z = 0$. We define a mapping $\psi : \mathcal{M}^{nml} \to \widetilde{\mathcal{M}}$ as follows. Let $C \in \mathcal{M}^{nml}$ and let $F(X, Y, Z)$ be the defining homogeneous polynomial. We define $\psi(C) := \pi^{-1}(C)$. Note that $\psi(C)$ is defined by $\widetilde{F}(X, Y, Z) := F(X^2, Y^2, Z^2)$. Each cusp of C produces 4 cusps on $\psi(C)$. Thus $\psi(C)$ has 24 cusps. Each node of C also gives 4 nodes on $\psi(C)$, thus we get 12 nodes on $\psi(C)$ which are mapped onto the nodes of C. As the restriction of π to the affine chart $\{Z \neq 0\}$ is the composition of double coverings $(x, y) \mapsto (x, y^2)$ and $(x, y) \mapsto (x^2, y)$, each simple tangent on the coordinate axis $X = 0$, $Y = 0$ gives 2 nodes on $\psi(C)$. This is the same for the simple tangents for $Z = 0$. Thus there are 12 nodes on $\psi(C)$ which are on the three coordinate axes and they are mapped to simple tangents on coordinate axis by π. Thus $\psi(C)$ has 24 nodes. Thus $\psi(C) \in \widetilde{\mathcal{M}}$.

Now for $C \in \mathcal{M}$, we define $\bar{\psi}(C)$ as $\psi(C^g)$ by choosing a $g \in G$ such that $C^g \in \mathcal{M}^{nml}$. The ambiguity for the choice of $g \in G$ are in the stabilizer $G_{\mathcal{M}^{nml}}$ of $\mathcal{M}^{nml}$ which is a direct product of $\mathfrak{S}_3$ (the permutations of coordinates) and $\mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C}^*$ (scalar multiplications). Thus the polynomial $\widetilde{F}(X, Y, Z)$ is also unique up to a $G_{\mathcal{M}^{nml}}$ action, and therefore $\widetilde{F}(X, Y, Z)$ is also unique up to a $G_{\mathcal{M}^{nml}}$ action. Thus $\bar{\psi} : \mathcal{M}/G \to \widetilde{\mathcal{M}}/G$ is well-defined.

Recall that a polynomial $F(X, Y, Z)$ is called *even* in X (respectively *symmetric* in X, Y) if $F(-X, Y, Z) = F(X, Y, Z)$ (resp. $F(Y, X, Z) = F(X, Y, Z)$). Thus the polynomial $F(X^2, Y^2, Z^2)$ is even in X, Y, Z.

Assume that $C \in \mathcal{M}$ is defined by $F(X, Y, Z) = 0$. If F is a even polynomial in the variable X (respectively a symmetric polynomial in X, Y), then 6 cusps are stable by the involution $(X, Y, Z) \mapsto (-X, Y, Z)$ (respectively $(X, Y, Z) \mapsto (Y, X, Z)$). Then there exists a homogeneous polynomial $F_2(X, Y, Z)$ of degree 2 which is even in X (respectively symmetric in X, Y) such that the conic $F_2(X, Y, Z) = 0$ passes through the 6 cusps of C. By the criterion of Degtyarev [D], the sextic $F(X, Y, Z) = 0$ is of torus type.

Now we take a generic $C \in \mathcal{M}^{nml}$ and consider the dual curve $\psi(C)^*$ and let $\widetilde{G}(X^*, Y^*, Z^*)$ be a defining homogeneous polynomial of degree 12, where (X^*, Y^*, Z^*) is the dual coordinates of (X, Y, Z). As $\widetilde{F}(X, Y, Z)$ is even in X, Y, Z, so is $\widetilde{G}(X^*, Y^*, Z^*)$ in X, Y, Z

Proposition 2.1. *$\psi(C)^*$ has 4 nodes on each coordinate axis $X^* = 0$, $Y^* = 0$ or $Z^* = 0$.*

Proof. Let $C = \{F(X, Y, Z) = 0\}$ and let us consider the discriminant polynomial $\Delta_Y F(X, Z)$. This is a homogeneous polynomial of degree 30 ([O1]). We assume that the singularities of the sextic $F(X, Y, Z) = 0$ are not on the coordinate axis. Assume that $P := (\alpha, \beta, \gamma) \in C$ is a singular point of C with Milnor number μ and multiplicity m. Then $\Delta_Y F(X, Z)$ has a linear term $(\gamma X - \alpha Z)^\rho$ with $\rho \geq \mu + m - 1$ and the equality holds if the line $\gamma Y - \beta Z = 0$ is generic with respect to C (see [O2]). Thus to each cusp (respectively to each node), there is an associated linear term with multiplicity 3 (resp. with multiplicity 2). The factor $X = 0$ and $Z = 0$ has also multiplicity 2 in $\Delta_Y F(X, Z) = 0$, as they are bi-tangent lines. Assume C is generic in $\mathcal{M}$. Then the sum of degrees is 18+6+4=28 by the above consideration. Thus there exists two simple tangent lines of the form $X - \eta_1 Z = 0$ and $X - \eta_2 Z = 0$ for some $\eta_1, \eta_2 \neq 0$. Then four lines $X = \pm\sqrt{\eta_i}Z, i = 1, 2$ are bitangent lines for the curve $\psi(C)$. This implies that $(1, 0, \pm\sqrt{\eta_i})$, $i = 1, 2$ are nodes of the dual curve $\psi(C)^*$. Thus the coordinate axis $Y^* = 0$ contains 4 nodes of $\psi(C)^*$. By the same argument, $X^* = 0$ and $Z^* = 0$ contains also 4 nodes respectively. □

Definition 2.2. For $C \in \mathcal{M}^{nml}$, we define a polynomial of degree 6 by $G(X^*, Y^*, Z^*) := \widetilde{G}(\sqrt{X^*}, \sqrt{Y^*}, \sqrt{Z^*})$ and we define $\iota(C)$ by the sextics defined by $G(X^*, Y^*, Z^*) = 0$. For $C \in \mathcal{M}$, take $g \in G$ so that $C^g \in \mathcal{M}^{nml}$ and we define an involution $\bar{\iota} : \mathcal{M}/G \to \mathcal{M}/G$ by $\bar{\iota}(C) = \iota(C^g)$.

Claim 1. $\bar{\iota}(C) \in \mathcal{M}$ for a generic $C \in \mathcal{M}$ and $\bar{\iota}$ is an involution which preserves the type of sextics, that is we have the commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{M}/G & \xrightarrow{\bar{\iota}} & \mathcal{M}/G & \quad & \mathcal{M}_{torus}/G & \xrightarrow{\bar{\iota}} & \mathcal{M}_{torus}/G \\
\downarrow \bar{\psi} & & \downarrow \bar{\psi} & & \downarrow \bar{\psi} & & \downarrow \bar{\psi} \\
\widetilde{\mathcal{M}}/G & \xrightarrow{dual} & \widetilde{\mathcal{M}}/G & & \widetilde{\mathcal{M}}_{torus}/G & \xrightarrow{dual} & \widetilde{\mathcal{M}}_{torus}/G
\end{array}$$

Proof. We may assume that $C \in \mathcal{M}^{nml}$. By the above consideration, we have seen that the dual curve $\psi(C)^*$ of $\psi(C)$ is defined by a polynomial $G(X^*, Y^*, Z^*)$ of degree 12 which is even in each of the three variables and it has 24 cusps and 12 nodes outside of coordinate axis and 4 nodes on each coordinate axis. Thus $\iota(C)$ has 6 cusps and 3 nodes. Note that nodes of $\psi(C)^*$ on the coordinate axes are mapped on simple tangents on the corresponding coordinate axes of $\iota(C)$. Thus the curve $\iota(C)$, defined by $g(\sqrt{x^*}, \sqrt{y^*}) = 0$, belongs to $\mathcal{M}^{nml}$. Finally we will show that ι keeps the type of the curve. As the curves $\{\bar{\iota}(C); C \in \mathcal{M}_{torus}/G\}$ are topologically equivalent, the image is contained in a connected component. Thus it is enough to show that there exists a $C \in \mathcal{M}_{torus}/G$ such that $\bar{\iota}(C) \in \mathcal{M}_{torus}/G$. To see this, it is enough to take $C \in \mathcal{M}_{torus}^{nml}$ whose defining polynomial $F(X, Y, Z)$ is symmetric in each of X, Y. Then $\widetilde{F}(X, Y, Z)$ is also symmetric in X, Y. This implies also that $\widetilde{G}(X^*, Y^*, Z^*)$ and $G(X^*, Y^*, Z^*)$ symmetric in X^*, Y^*. By the Degtyarev's criterion, this implies that $\iota(C)$ is a sextic of torus type. The following example shows that $\bar{\iota}(C) \neq C^*$ in general. □

Thus we have proved the following:

Theorem 2.3. *There exists an involution $\bar{\iota}$ on the quotient moduli space $\mathcal{M}/G$ such that $\bar{\iota}$ is different from the dual curve operation and $\bar{\iota}$ preserves the types of the sextics, that is $\bar{\iota}(C) \in \mathcal{M}_{torus}/G \iff C \in \mathcal{M}_{torus}/G$.*

Example 2.4. Let $C \in \mathcal{M}_{torus}^{nml}$ be the sextic defined by the symmetric polynomial:
$f := -684(x^3y + xy^3) - 1055(x^3 + y^3) + 2235(x^2 + y^2) - 2178(x + y) + \frac{819}{16}(x^5y + y^5x) + \frac{1767}{16}(x^4y^2 + x^2y^4) + \frac{881}{8}y^3x^3 + \frac{405}{16}(x^6 + y^6) - \frac{873}{8}(x^5 + y^5) + \frac{2001}{4}(x^4 + y^4) - \frac{971}{8}(x^4y + xy^4) - \frac{6947}{2}y^2x^2 + 2268 + 1038(x^2y + xy^2) - 4883yx - \frac{375}{2}(x^2y^3 + x^3y^2)$.

Then $\psi(C)$ is defined by $f(x^2, y^2)$ and $\psi(C)^*$ is defined by $g(x^{*2}, y^{*2}) = 0$ and $\iota(C)$ is the sextic defined by the symmetric polynomial

$g(x^*, y^*) := 908294x^{*2}y^{*2} - 354000(x^*y^{*2} + x^{*2}y^*) + 302745(y^{*4} + x^{*4}) + 529284(x^{*4}y^{*2} + y^{*4}x^{*2}) - 396458(x^*y^{*4} + y^*x^{*4}) - 722148(x^{*3}y^{*2} + y^{*3}x^{*2}) + 11340(y^{*6} + x^{*6}) - 109170(x^{*5} +$

$y^{*5})+86296x^*y^*+482724(x^{*3}y^*+y^{*3}x^*)-158508(y^*x^{*5}+y^{*5}x^*)+103096y^{*3}x^{*3}-22230(x^*+y^*) - 203920(y^{*3} + x^{*3}) + 90570(y^{*2} + x^{*2}) + 2025$

The dual curve C^* of C is defined by the following symmetric polynomial and we can easily check that $\bar{\iota}(C) \neq C^*$.

$h(x^*, y^*) := 3(x^{*4} + y^{*4}) + 14(x^{*3} + y^{*3}) + 3(x^{*2} + y^{*2}) + 4(y^*x^{*4} + x^*y^{*4}) + 36(y^*x^{*3} + x^*y^{*3}) + 6(y^*x^{*2} + x^*y^{*2}) - 2y^*x^* + 12(y^{*2}x^{*4} + x^{*2}y^{*4}) + 84(y^{*2}x^{*3} + x^{*2}y^{*3}) + 14y^{*2}x^{*2} + 88y^{*3}x^{*3} + 4y^{*4}x^*$

3. Normal forms of the moduli $\mathcal{N}$

We consider the submoduli $\mathcal{N}^{(1)}$ of the sextics whose cusps are at $O := (0,0), A := (1,1)$ and $B := (1,-1)$. Under the action of G, every sextic in $\mathcal{N}$ can be represented by a curve in $\mathcal{N}^{(1)}$. Consider the stabilizer group $G^{(1)} := \{g \in G; g(\mathcal{N}^{(1)}) = \mathcal{N}^{(1)}\}$. By an easy computation, we see that $G^{(1)}$ is the semi-direct product of the group $G_0^{(1)}$ and and a finite group $\mathcal{K}$ where $\mathcal{K}$ is a finite linear subgroup of G, isomorphic to the permutation group $\mathcal{S}_3$, and $G_0^{(1)}$ is defined by

$$G_0^{(1)} := \{M = \begin{pmatrix} a_1 & a_2 & 0 \\ a_2 & a_1 & 0 \\ a_1 - a_3 & a_2 & a_3 \end{pmatrix} \in G; a_3(a_1^2 - a_2^2) \neq 0\}$$

which fix singular points pointwise. Note thet $G_0^{(1)}$ is normal in $G^{(1)}$. The isomorphism $\mathcal{K} \cong \mathcal{S}_3$ is given by identifying $g \in \mathcal{K}$ as the permutation of three singular locus O, A, B. We will study the normal forms of the quotient moduli $\mathcal{N}/G \cong \mathcal{N}^{(1)}/G^{(1)}$.

Lemma 3.1. *For a given line $L := \{y = bx\}$ with $b^2 - 1 \neq 0$, there exists $M \in G_0^{(1)}$ such that L^M is given by $x = 0$.*

Proof. By an easy computation, the image of L by the action of M^{-1}, where M is as above, is defined by $(a_1 - ba_2)y + (a_2 - ba_1)x = 0$. Thus we take $a_1 = ba_2$. Then $a_1^2 - a_2^2 = a_2^2(b^2 - 1) \neq 0$ by the assumption. □

Lemma 3.2. *The tangent cone at O is not $y \pm x = 0$ for $C \in \mathcal{N}^{(1)}$.*

Proof. Assume for example that $y - x = 0$ is the tangent cone of C at O. The intersection multiplicity of the line $L_1 := \{y - x = 0\}$ and C at O is 4 and thus $L_1 \cdot C \geq 7$, an obvious contradiction to Bezout theorem. □

Let $\mathcal{N}^{(2)}$ be the subspace of $\mathcal{N}^{(1)}$ consisting of curves whose tangent cone at O is given by $x = 0$. Let $G^{(2)}$ be the stabilizer of $\mathcal{N}^{(2)}$. By Lemma 3.1 and Lemma 3.2, we have the isomorphism :

Corollary 3.3. $\mathcal{N}^{(1)}/G^{(1)} \cong \mathcal{N}^{(2)}/G^{(2)}$.

It is easy to see that $G^{(2)}$ is generated by the group $G_0^{(2)} := G^{(2)} \cap G_0^{(1)}$ and an element τ of order two defined by $\tau(x,y) = (x,-y)$. Note that

$$G_0^{(2)} = \{M = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ a_1 - a_3 & 0 & a_3 \end{pmatrix} \in G_0^{(1)}; \quad a_1 a_3 \neq 0\}$$

For $C \in \mathcal{N}^{(2)}$, we associate complex numbers $b(C), c(C) \in \mathbf{C}$ which are the directiections of the tangent cones of C at A, B respectively. This implies that the lines $y-1 = b(C)(x-1)$ and $y+1 = c(C)(x-1)$ are the tangent cones of C at A and B respectively. We have shown that $C \in \mathcal{N}_{torus}^{(2)}$ if and only if $b(C)+c(C) = 0$ and C is not of torus type if and only if $c(C)^2 + 3c(C) - b(C)c(C) + 3 - 3b(C) + b(C)^2 = 0$ (§4, [O2]).

We consider the subspaces:

$$\mathcal{N}_{torus}^{(3)} := \{C \in \mathcal{N}_{torus}^{(2)}; b(C) = 1\}, \quad \mathcal{N}_{gen}^{(3)} := \{C \in \mathcal{N}_{gen}^{(2)}; b(C) = c(C) = \sqrt{-3}\}$$

and we put $\mathcal{N}^{(3)} := \mathcal{N}_{torus}^{(3)} \cup \mathcal{N}_{gen}^{(3)}$.

Remark . The common solution of the both equations: $b+c = c^2+3c-bc+3-3b+b^2 = 0$ is $(b,c) = (1,-1)$ and in this case, C degenerates into two non-reduced lines $(y^2-x^2)^2 = 0$ and a conic.

Lemma 3.4. *Assume that $C \in \mathcal{N}^{(2)}$. Then there exists $C' \in \mathcal{N}^{(3)}$ and an element $g \in G^{(2)}$ such that $C^g = C'$ and such a C' is unique. This implies that*

$$\mathcal{N}_{torus}/G \cong \mathcal{N}_{torus}^{(2)}/G^{(2)} \cong \mathcal{N}_{torus}^{(3)}, \quad \mathcal{N}_{gen}/G \cong \mathcal{N}_{gen}^{(2)}/G^{(2)} \cong \mathcal{N}_{gen}^{(3)}$$

Proof. Assume that $C \in \mathcal{N}_{torus}^{(1)}$, $b+c=0$. Consider an element $g \in G_0^{(1)}$,

$$g^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-a_3 & 0 & a_3 \end{pmatrix}$$

The image L_A^g is given by $y - x + xa_3 - a_3 - bxa_3 + ba_3 = 0$. Thus we can solve the equation $a_3(1-b) - 1 = 0$ in a_3 uniquely as $a_3 = 1/(1-b)$ as $b \neq 1$. Thus $g \in G_0^{(1)}$ is unique if it fixes the singular points pointwise and thus C' is also unique. It is easy to see that the stabilizer of $\mathcal{N}_{torus}^{(3)}$ is the cyclic group of order two generated by τ, as C' is even in y (see the normal form below) and $C'^{\tau} = C'$ for any $C' \in \mathcal{N}_{torus}^{(3)}$. Thus we have $\mathcal{N}_{torus}^{(2)}/G^{(2)} \cong \mathcal{N}_{torus}^{(3)}$.

Consider the case $C \in \mathcal{N}_{gen}^{(2)}$. Then the images of the tangent cones at A, B by the action of g are given by $y-x+xa_3-a_3-bxa_3+ba_3 = 0$ and $y+x-xa_3+a_3-cxa_3+ca_3$ respectively. Assume that $b(C^g) = c(C^g)$. Then we need to have $a_3(1-b)-1 = a_3(-1-c)+1$, which has a unique solution in a_3, if ($\star$) $b-c-2 \neq 0$. Assume that $c^2+3c-bc+3-3b+b^2 = 0$ and $b-c-2 = 0$. Then we get $(b,c) = (1,-1)$ which is excluded as it corresponds to non-reduced sextic. Thus the condition ($\star$) is always satisfied. Put $(b',c') := (b(C^g), c(C^g))$. They satisfy the equality $c'^2 + 3c' - b'c' + 3 - 3b' + b'^2 = 0$ and $b' = c'$. Thus we have either $b' = c' = \sqrt{-3}$ or $b' = c' = -\sqrt{-3}$. However in the second case, we can take the automorphism $(x,y) \to (x,-y)$ to change into the first case. Thus $b' = c' = \sqrt{-3}$ and $C^g \in \mathcal{N}_{gen}^{(3)}$ as desired. □

3.1. Normal forms of curves of torus type. In [O2], we have shown that a curve in $\mathcal{N}_{torus}^{(1)}$ is defined by a polynomial $f(x,y)$ which is defined by a sum $f_2(x,y)^3 + sf_3(x,y)^2$ where $f_2(x,y)$ is a smooth conic passing through O, A, B, $f_3(x,y) = (y^2 - x^2)(x-1)$ and $s \in \mathbf{C}^*$.

Proposition 3.5. *The direction of the tangent cones at O, A and B are the same with the tangent line of the conic $f_2(x,y) = 0$ at those points.*

This is immediate as the multiplicity of $f_3(x,y)^2$ at O, A, B are 4. See also Lemma 23 of [O2]. Assume that $C \in \mathcal{N}_{torus}^{(3)}$, that is, the tangent cones of C at O, A and B are given by $x = 0$, $y - 1 = 0$ and $y + 1 = 0$ respectively. Thus the conic $f_2(x,y) = 0$ is also uniquely determined as $f_2(x,y) = y^2 + x^2 - 2x$. This is the circle with radius 1, centered at (1,0). Therefore $\mathcal{N}_{torus}^{(3)}$ is one-dimensional and it has the representaion

$$C_s: \quad f_{torus}(x,y,s) := f_2(x,y)^3 + sf_3(x,y)^2 = 0 \tag{3.6}$$

For $s \neq 0, 27$, C_s is a sextic with three (3,4) cusps, while C_{27} obtains a node. As is easy to see, if $g \in G^{(2)}$ fixes the tangent lines $y \pm 1 = 0$, then $g = e$ or τ and $C_s^\tau = C_s$. Thus $C_s \neq C_t$ if $s \neq t$.

3.2. Normal form of sextics of non-torus type. For the moduli of non-torus type sextic $\mathcal{N}_{gen}$, we start from the expression given in §4.1, [O2]. We may assume $b = c = \sqrt{-3}$. Then the parametrization is given by

$$f_{gen}(x,y,s) := f_0(x,y) + sf_3(x,y)^2, \quad f_3(x,y) = (y^2 - x^2)(x-1)$$

where s is equal to a_{06} in [O2] and f_0 is the sextic given by

$$\begin{aligned} f_0(x,y) &:= y^6 + y^5(6\sqrt{-3} - 6\sqrt{-3}x) + y^4(35 - 76x + 38x^2) \\ &+ y^3(-24\sqrt{-3}x + 36\sqrt{-3}x^2 - 12\sqrt{-3}x^3) + y^2(-94x^2 + 200x^3 - 103x^4) \\ &+ y(24\sqrt{-3}x^3 - 42\sqrt{-3}x^4 + 18\sqrt{-3}x^5) + 64x^3 - 133x^4 + 68x^5 \end{aligned} \tag{3.7}$$

Let $D_s := \{f_{gen}(x,y,s) = 0\}$ for each $s \in \mathbf{C}$. Observe that $D_0 = \{f_0(x,y) = 0\}$ is a sextic with three (3,4)-cusps and of non-torus type. For the computational reason, we take the substitution $y \mapsto y\sqrt{-3}$ to make the equationto be defined over rational numbers: Then $f_0(x,y)$ and $f_3(x,y)$ change into:

$$\begin{aligned} f_0(x,y) &:= -27y^6 + (-162 + 162x)y^5 + (315 - 684x + 342x^2)y^4 \\ &+ (-216x + 324x^2 - 108x^3)y^3 + (282x^2 - 600x^3 + 309x^4)y^2 \\ &+ (-54x^5 + 126x^4 - 72x^3)y + 68x^5 + 64x^3 - 133x^4 \\ f_3(x,y) &:= -(x-1)(3y^3 + x^2) \end{aligned} \tag{3.8}$$

Summerizing the discussion, we have

Theorem 3.9. *The quotient moduli space $\mathcal{N}/G$ is one dimensional and consists of two components.*
(1) The component $\mathcal{N}_{torus}/G$ has the normal forms represented by the family of sextics $C_s = \{f(x,y,s) = 0\}$ where $f(x,y,s) = f_2(x,y)^3 + sf_3(x,y)^2$ for $s \in \mathbf{C}^*$ and $s \neq 0, 27$ where

$$f_2(x,y) = y^2 + x^2 - 2x, \quad f_3(x,y) = (y^2 - x^2)(x-1)$$

The curve C_{54} is a unique curve in $\mathcal{N}/G$ which is self-dual.
(2) The component $\mathcal{N}_{gen}/G$ of sextics of non-torus type has the normal form: $f_{gen}(x,y,s) = f_0(x,y) + sf_3(x,y)^2$ where f_3 is as above and the sextic $f_0(x,y) = 0$ is contained in $\mathcal{N}_{gen}$. This component has no self-dual curve.

Proof of Theorem 3.9. We need only prove the assertion for the dual curves. The proof will be done by a direct computation of dual curves using the method of §2, [O2] and the above parametrizations. We use Maple V for the practical computation. Here is the recipe of the proof. Let X^*, Y^*, Z^* be the dual coordinates of X, Y, Z and let $(x^*, y^*) := (X^*/Z^*, Y^*/Z^*)$ be the dual affine coordinates.

(1) Compute the defining polynomials of the dual curves C_s^* and D_s^* respectively, using the method of Lemma 2.4, [O2]. Put $g_{torus}(x^*, y^*, s)$ and $g_{gen}(x^*, y^*, s)$ the defining polynomials.

(2) Let $G_\varepsilon(X^*, Y^*, Z^*, s)$ be the homogenization of $g_\varepsilon(x^*, y^*, s)$, $\varepsilon =$ *torus* or *gen*. Compute the discriminant polynomials $\Delta_{Y^*}(G)$ which is a homogeneous polynomial in X^*, Z^* of degree 30 (cf. Lemma 2.8, [O1]). Recall that the multiplicity of the pencil $X^* - \eta Z^* = 0$ passing through a singular point is generically given by $\mu + m - 1$ where μ, m are the Milnor number and the multiplicity of the singularity ([O2]). Thus the contribution from a (2,3)-cusp (respectively from a (3,4)-cusp) is 3 (resp. 8). Thus if C_s^* has three (3,4) cusps, it is necessary that $\Delta_{Y^*}(G) = 0$ has three linear factors with multiplicity at least 8.

(3-1) For the non-torus curves, it is not possible to get a degeneration into 3 (3,4)-cuspidal sextic.

(3-2) For the torus curves, we can see that $s = 54$ is the only possible parameter. Thus it is enough to show that $C_{54}^* \cong C_{54}$.

(4) The dual curve C_{54}^* of C_{54} is defined by the homogeneous polynomial

$$\begin{aligned} G(X^*, Y^*, Z^*) := \; & 128X^{*5}Z^* + 1376X^{*4}Z^{*2} - 192X^{*3}Y{*}^2Z^* + 4664X^{*3}Z^{*3} - 2X^{*2}Y^{*4} \\ & -1584X^{*2}Y^{*2}Z^{*2} + 7090X^{*2}Z^{*4} + 58X^*Y^{*4}Z^* - 3060X^*Y^{*2}Z^{*3} \\ & +5050X^*Z^{*5} + Y^{*6} + 349Y^{*4}Z^{*2} - 1725Y^{*2}Z^{*4} + 1375Z^{*6} \end{aligned}$$

We can see that C_{54}^* has also 3 (3,4)-cusps. Moreover we can see that $C_{54}{}^*$ is isomorphic to C_{54} as $(C_{54}^*)^A = C_{54}$ where

$$A = \begin{pmatrix} -4/3 & 0 & -5/3 \\ 0 & 1 & 0 \\ -5/3 & 0 & -13/3 \end{pmatrix}$$

3.3. Involution τ on C_{54}. For the later purpose, we change the coordinates of G so that the three cusps of C_s are at $O_Z := (0,0,1), O_Y := (0,1,0), O_X := (1,0,1)$. New normal form in affine spce is given by $f(x,y,s) = f_2(x,y)^3 + sf_3(x,y)^2$ where

$$f_2(x,y) := xy - x + y, \quad f_3(x,y) := -xy \tag{3.10}$$

and C_{54} is defined by $f(x,y) = (xy - x + y)^3 - 54x^3y^3 = 0$. In this coordinate, C_{54}^* is defined by

$$\begin{gathered} -28y^3 - 17x^4y^2 - 17x^2y^4 - 28x^3y^3 - 2y^5 + 1788x^3y + 1788x^2y - 17y^4 - 17x^4 \\ +262xy + 1788x^2y^3 - 1788xy^2 - 262xy^4 + 1788xy^3 - 1788x^3y^2 - 8166x^2y^2 + 28x^3 \\ +262x^4y - 2x^5y - 2xy^5 + 1 - 17y^2 - 17x^2 + 2x^5 + 2x - 2y + x^6 + y^6 = 0 \end{gathered}$$

It is easy to see that $(C_{54}^*)^{A_1} = C_{54}$ where

$$A_1 := \begin{pmatrix} -1/3 & 7/3 & -1/3 \\ 7/3 & -1/3 & 1/3 \\ -1/3 & 1/3 & -7/3 \end{pmatrix}$$

Let $F(X,Y,Z)$ be the homogenization of $f(x,y)$. Then the Gauss map induces an automorphism $\mathrm{dual}_C : C_{54} \to C_{54}^*$ which is defined by $(X,Y,Z) \mapsto (F_X, F_Y, F_Z)$, where F_X, F_Y, F_Z are partial derivatives. We define an isomorphism $\tau : C_{54} \to C_{54}$ by the composition of $\mathrm{dual}_{C_{54}}$ and the linear map $\varphi_{A_1} : C_{54}^* \to C_{54}$ which is defined by the multiplication by A_1 from the right. τ is given by the restriction of the rational mapping: $\Psi : \mathbf{C}^2 \to \mathbf{C}^2$, $(x,y) \mapsto (x_d, y_d)$ and

$$x_d := \frac{(-y^3+4x^2-x^2y^3+4x^3y^2-8x^3y-4x^2y^2-8xy-4xy^2-2xy^3+109x^2y+4y^2+4x^3)}{(-4y^3+x^2-4x^2y^3+4x^3y^2-8x^3y-109x^2y^2-2xy-4xy^2-8xy^3+4x^2y+y^2+4x^3)}$$

$$y_d := \frac{-(-4y^3+4x^2-4x^2y^3+x^3y^2-2x^3y-4x^2y^2-8xy-109xy^2-8xy^3+4x^2y+4y^2+x^3)}{(-4y^3+x^2-4x^2y^3+4x^3y^2-8x^3y-109x^2y^2-2xy-4xy^2-8xy^3+4x^2y+y^2+4x^3)}$$

Observe that τ is defined over $\mathbf{Q}$. C_{54} has three flexes of order 2 at $F_1 := (1, -1/4, 1)$, $F_2 := (1/4, -1, 1)$, $F_3 := (4, -4, 1)$ and τ exchanges flexes and cusps:

$$\left\{ \begin{array}{l} \tau(O_X) = F_1, \tau(O_Y) = F_2, \tau(O_Z) = F_3, \\ \tau(F_1) = O_X, \tau(F_2) = O_Y, \tau(F_3) = O_Z \end{array} \right. \tag{3.11}$$

Furthermore we assert that

Proposition 3.12. *The morphism τ is an involution C_{54}.*

For the proof, we prepare a lemma. Let C be a given irreducible curve in $\mathbf{P}^2$ defined by a homogeneous polynomial $F(X,Y,Z)$ and let $B \in \mathrm{GL}(3,\mathbf{C})$. Then C^B is defined by $G(X,Y,Z) := F((X,Y,Z)B^{-1})$. Let $\mathrm{dual}_C : C \to C^*$ be the Gauss map which is defined by $(X,Y,Z) \mapsto (F_X(X,Y,Z), F_Y(X,Y,Z), F_Z(X,Y,Z))$.

Lemma 3.13. *Two curves $(C^B)^*$ and $(C^*)^{{}^tB^{-1}}$ coincide and the following diagram commutes.*

$$\begin{tikzcd} C \arrow[r, "dual_C"] \arrow[d, "\varphi_B"] & C^* \arrow[d, "\varphi_{{}^tB^{-1}}"] \\ C^B \arrow[r, "dual_{C^B}"] & (C^B)^* \end{tikzcd}$$

Proof. This is essentially the same as Lemma 2, [O2]. The assertion follows from the following equalities. Let $(a,b,c) \in C$.

$$\begin{aligned} \mathrm{dual}_{C^B}(\varphi_B(a,b,c)) &= (G_X(\varphi_B(a,b,c)), G_Y(\varphi_B(a,b,c)), G_Z(\varphi_B(a,b,c))) \\ &= (F_X(a,b,c), F_Y(a,b,c), F_y(a,b,c))^tB^{-1} = \varphi_{{}^tB^{-1}}(\mathrm{dual}_C(a,b,c) \quad \square \end{aligned}$$

Proof of Proposition 3.12. By the definition of τ, we have ($C := C_{54}$):

$$\tau \circ \tau = (\varphi_{{}^tA_1^{-1}} \circ \mathrm{dual}_C)^2 = (\mathrm{dual}_{CA_1} \circ \varphi_{A_1}) \circ (\varphi_{{}^tA_1^{-1}} \circ \mathrm{dual}_C) = \mathrm{id}$$

as A_1 is a symmetric matrix. □

Of course, the same assertion is true for C_{54} in the old normal form. C_{54} has another obvious involution $\iota : C_{54} \to C_{54}$ which is defined by $(x, y) \mapsto (x, -y)$ in the old normal form. For the application to arithmetic property of cubic curves, see [O3].

References

[B-K] E. Brieskorn and H. Knörrer, *Ebene Algebraische Kurven*, Birkhäuser (1981), Basel-Boston - Stuttgart.

[D] A. Degtyarev, *Alexander polynomial of a curve of degree six*, J. Knot Theory and its Ramification, Vol. 3, No. 4, 439-454, 1994

[Ko] K. Kodaira, *On compact analytic surfaces II*, Ann. of Math. 77 (1963) 563-626 and III, Ann. of Math. 78 (1963) 1-40.

[N] M. Namba, *Geometry of projective algebraic curves*, Decker, New York, 1984

[O1] M. Oka, *Flex Curves and their Applications*, Geometriae Dedicata, Vol. 75 (1999), 67-100

[O2] M. Oka, *Geometry of cuspidal sextics and their dual curves*, to appear in Advanced Studies in Pure Math. 2?, 1999?, Singularities and arrangements, Sapporo-Tokyo 1998.

[O3] M. Oka, *Moduli of Sextics and Elliptic Curves*, preprint 1999.

[W] R. Walker, *Algebraic curves*, Dover Publ. Inc., New York, 1949.

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