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Strong solutions of Cauchy problems for compressible Navier-Stokes equations

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§0. Introduction

In this article, we consider existence and uniqueness of strong small solutions of compressible Navier-Stokes equation

$$\left\{\begin{array}{ll}\partial_t\rho+\operatorname{div}(\rho v)=0 & \text{in}\quad \mathbf{R}_+\times\mathbf{R}^n,\\ \rho(\partial_t v+v\cdot\nabla_x v)+\nabla_x P(\rho)=L_\rho v & \text{in}\quad \mathbf{R}_+\times\mathbf{R}^n,\end{array}\right. \tag{0.1}$$

near $\rho(t,x)=\rho_0$, $v(t,x)=0$ with some positive constant ρ_0. In (0.1), $P(\rho)$ is pressure which depends only upon the density of fluid smoothly and the operator $L_\rho v={}^t((L_\rho v)_1,(L_\rho v)_2,\cdots,(L_\rho v)_n)$ is defined as $(L_\rho v)_j=\sum_{i=1}^n\partial_{x_i}(\sigma_{ij}(v))$ for any $j=1,\cdots,n$, where $\sigma_{ij}(v)=\lambda(x,\rho)(\operatorname{div}v)\delta_{ij}+\mu(x,\rho)\{\partial_{x_i}v_j+\partial_{x_j}v_i\}$ are the stress tensor of viscous fluid.

We assume that the viscosity coefficient $\mu(x,\rho)$ and the second viscosity coefficient $\lambda(x,\rho)$ are functions in x and ρ belongings to $\mathcal{B}^\infty(\mathbf{R}^n\times\mathbf{R})$. We also assume that

(A.0)
$$P'(\rho_0)>0$$

(A.1)
$$\inf_{(x,\rho)\in\mathbf{R}^n\times\mathbf{R}}\mu(x,\rho)>0,\qquad \inf_{(x,\rho)\in\mathbf{R}^n\times\mathbf{R}}(\lambda(x,\rho)+\frac{2}{n}\mu(x,\rho))\geq 0.$$

For simplicity, we write the operator L_ρ as divergence form:

$$L_\rho v = \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x,\rho)\partial_{x_j} v).$$

In the above, a_{ij} are n×n-matrices whose the (p,q)-components are given by $a_{ipjq}(x,\rho) = \lambda(x,\rho)\delta_{ip}\delta_{jq} + \mu(x,\rho)(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{jp})$. From now on, we use these notations.

For the equations of compressible viscous and heat conductive fluids (the equation of compressible viscous fluid (0.1) taking the heat effect into account), Matsumura and Nishida [**8**] and [**9**] show uniqueness and global existence of the small solutions near $\rho(t,x) = \rho_0$, $v(t,x) = 0$ in the Sobolev space H^3 in the case that fluid stay in whole space $\mathbf{R}^3$ or an exterior domain in $\mathbf{R}^3$. They also show the solution goes to the stationary solution as $t \to \infty$.

For these solutions, many authors investigate the asymptotic behaviour of the solutions as $t \to \infty$ if the viscosity coefficients are constants and the initial data are also small in some other function spaces (cf. [**4**], [**7**], [**10**] in Cauchy problem, [**1**], [**6**] in exterior problem).

To solve (0.1), one of the difficulty is how to show that the density ρ is positive in the second equation in (0.1). The simplest way to keep positivity of the density is to show that L^∞-norm of $\rho(t,\cdot) - \rho_0$ is small enough uniformly in t. In [**8**] and [**9**], from smallness of $\rho(0,\cdot) - \rho_0$, and $v(0,\cdot)$ in H^3 Sobolev space, they obtain a priori estimates of H^2 norm of $\rho(t,\cdot) - \rho_0$ and $v(t,\cdot)$. From these a priori estimates, we can control L^∞ norm of ρ by using usual L^2 Sobolev inequality.

In their a priori estimates, we need one more derivative in the initial data than that, which is nesessary to estimate L^∞ norm of ρ by usual L^2-Sobolev inequality. Thus, it seems to be interesting problem considering whether we can control the L^∞ norm of ρ by using smallness of the initial differences from stationary state in H^2 spaces. In this note, we treat this problem.

Now, we formulate our problem. Since we treat small solutions near $\rho = \rho_0$, $v = 0$, we change ρ to $\rho_0(1+\rho)$ and $\rho_0^{-1}a_{ij}(x,\rho_0(1+\rho))$ to $a_{ij}(x,\rho)$ in (0.1). By this reduction, our problem is just to solve the following equation:

$$(0.2) \qquad \begin{cases} \partial_t\rho + \operatorname{div} v = F_0(\rho, v) & \text{in} \quad \mathbf{R}_+ \times \mathbf{R}^n, \\ \partial_t v + c^2\nabla_x\rho - \dfrac{1}{1+\rho}L_\rho v = F(\rho,v) & \text{in} \quad \mathbf{R}_+ \times \mathbf{R}^n, \end{cases}$$

with initial condition

$$(0.3) \qquad \rho(0,x) = \rho^0(x), \qquad v(0,x) = v^0(x).$$

In (0.2), $c = \sqrt{P'(\rho_0)}$ and $F_0(\rho, v)$ and $F(\rho, v)$ are defined as

$$F_0(\rho, v) = -\mathrm{div}\ (\rho v),$$
$$F(\rho, v) = -(v \cdot \nabla_x v + \frac{1}{1+\rho}\nabla_x(Q(\rho)\rho^2)),$$

respectively with some $Q(\rho) \in C^\infty(\mathbf{R})$. Note that the operator L_ρ in (0.2) satisfies assumption (A.1).

By the works [**8**] and [**9**], we have uniqueness and global existence of the solutions of (0.2) and (0.3) with small initial data ρ^0, v^0 in the Sobolev space H^3. We can show unique existence of time global small solution only if we assume that ρ^0, v^0 are small enough in H^2 space in the three dimensional case.

For the space dimension n, we set $s_0 = [n/2]+1$ ($[x]$ means the largest integer not greater than $x \in \mathbf{R}$).

THEOREM 0.1. *Under the assumptions* (A.0) *and* (A.1), *there exists a constant* $\epsilon_0 > 0$ *such that problem* (0.2) *and* (0.3) *has a unique solution* $(\rho(t,x), v(t,x))$ *globally in time which satisfies*

$$\rho \in C([0,\infty) : H^{s_0}(\mathbf{R}^n)) \cap C^1([0,\infty) : H^{s_0-1}(\mathbf{R}^n))$$
$$v \in C([0,\infty) : H^{s_0}(\mathbf{R}^n)) \cap C^1([0,\infty) : H^{s_0-2}(\mathbf{R}^n))$$
$$\lim_{t\to\infty}\{\| \rho(t,\cdot) \|^2_{H^{s_0}(\mathbf{R}^n)} + \| v(t,\cdot) \|^2_{H^{s_0}(\mathbf{R}^n)}\} = 0$$

if the initial data $(\rho^0, v^0) \in H^{s_0}(\mathbf{R}^n)$ *satisfies*

$$\| \rho^0 \|^2_{H^{s_0}(\mathbf{R}^n)} + \| v^0 \|^2_{H^{s_0}(\mathbf{R}^n)} \leq \epsilon_0.$$

Thus, the initial value problem for compressible Navier-Stokes equation (0.2) and (0.3) for small data ρ^0 and v^0 has unique global small strong solution in $H^{s_0}(\mathbf{R}^n)$, which gives an generalization of Matsumura and Nishida [**8**].

In the case of constant viscosity coefficients, existence of global weak solutions for (0.2) and (0.3) with small initial data having less regularities than that in [**9**] are shown by Hoff [**2**] in an isothermal case (i.e, $P(\rho) = C\rho$, where $C > 0$ is a constant) and [**3**] in the polytoropic case (i.e. $P(\rho) = C\rho^\gamma$, where $C > 0$ and $\gamma > 1$ are constants). In [**2**] and [**3**], the class of the initital data contains discontinuous functions, however, the solutions satisfy the equation in a weak sense. The class of the initial data in Theorem 0.1 is smaller than that in [**2**] and [**3**]. Instead of that, we have the unique strong solution in usual L^2-sense for the equation with the pressure $P(\rho)$ and the viscosity coefficients of general type.

To make sure the term $\frac{1}{1+\rho}$ in the second equation in (0.2), we need to keep the norm $\| \rho \|_{L^\infty}$ small enough. To accomplish this, we use the usual L^2-Sobolev inequality. Indeed, we can obtain the estimates which shows the norm $\| \rho \|_{H^{s_0}}$ is controlled by the same norm of initial data if they are small enough in the function space $H^{s_0}(\mathbf{R}^n)$. This is one of the main difficulty of our problem.

Since this space is the largest integer order L^2-Sobolev space contained in the space $L^\infty(\mathbf{R}^n)$, we can say Theorem 0.1 is best possible in the direction of finding strong solutions by the usual integer order L^2-Sobolev imbedding theorem but not using so called smoothing effect.

In this note, we only give the outline of the proof of Theorem 0.1. The detail is discussed in Kawashita [**5**].

§1. Prolongation of local solutions

We intend to show Theorem 0.1 by prolonging time local solutions. To do this, we need a result on local existense of solutions (cf. Theorem 1.2 below). As is in Matsumura and Nishida [**9**], it is essential to be able to take existence time of solutions independent of initial data. A priori estimates (cf. Theorem 1.1 below) is used to ensure this independence.

For $t_0 \le t \le t_1$, $E > 0$ and integer $l \ge s_0$, we say a pair of functions (ρ, v) belongs to the space $Y_l(t_0, t_1; E)$ if and only if $\rho(t,x)$ and $v(t,x)$ satisfy

$$(1.1) \qquad \begin{cases} \rho \in L^\infty([t_0,t_1] : H^l(\mathbf{R}^n)) \cap Lip([t_0,t_1] : H^{l-1}(\mathbf{R}^n)) \\ v \in L^\infty([t_0,t_1] : H^l(\mathbf{R}^n)) \cap Lip([t_0,t_1] : H^{l-2}(\mathbf{R}^n)) \\ \nabla_x v \in L^2([t_0,t_1] : H^l(\mathbf{R}^n)) \\ \sup\limits_{t_0 \le t \le t_1} \{ \| \rho(t,\cdot) \|^2_{H^{s_0}(\mathbf{R}^n)} + \| v(t,\cdot) \|^2_{H^{s_0}(\mathbf{R}^n)} \} \le E \end{cases}$$

where $Lip([t_0,t_1] : H^l(\mathbf{R}^n))$ describes the function space consisting of $H^l(\mathbf{R}^n)$-valued Lipschitz continuous function on $[t_0, t_1]$. We denote by $CY_l(t_0, t_1; E)$ the function space defined by replacing L^∞ to C and Lip to C^1 in (1.1). For (ρ, v) we set

$$\begin{aligned} \| \{\rho, v\} \|^2_{(l;t_0,t_1)} &= \sup_{t_0 \le t \le t_1} \{ \| \rho(t,\cdot) \|^2_{H^l} + \| v(t,\cdot) \|^2_{H^l} \} \\ &\qquad + \int_{t_0}^{t_1} \| \nabla_x v(s,\cdot) \|^2_{H^l} \, ds, \\ N_l(t_0,t_1;\rho,v) &= \| \{\rho, v\} \|^2_{(l;t_0,t_1)} + \int_{t_0}^{t_1} \| \nabla_x \rho(s,\cdot) \|^2_{H^{l-1}} \, ds. \end{aligned}$$

We denote by $X_l(t_0, t_1; E)$ the function space consisting of functions $(\rho, v) \in CY_l(t_0, t_1; E)$ satisfying $N_l(t_0, t_1; \rho, v) \leq E$.

THEOREM 1.1. (a priori estimate) *Under the same assumption as in Theorem 0.1, there exists a sufficiently small constant $E_0 > 0$ and a constant $C_0 > 0$ such that for any solution $(\rho, v) \in X_{s_0}(0, t_0; E_0)$ of (0.2) in $0 \leq t \leq t_0$ with some $t_0 > 0$, the following a priori estimate holds:*

$$N_{s_0}(0, t; \rho, v) \leq C_0 N_{s_0}(0, 0; \rho, v) \qquad \text{for any } 0 \leq t \leq t_0.$$

THEOREM 1.2. (local existence) *Under the same assumption as in Theorem 0.1, there exist a sufficiently small constant $E_1 > 0$, a constant $T > 0$ and $C_1 > 0$ such that if a solution $(\rho, v) \in X_{s_0}(0, t_0; E_0)$ of (0.2) in $0 \leq t \leq t_0$ for some $t_0 \geq 0$ satisfies $N_{s_0}(t_0, t_0; \rho, v) \leq E_1$, it can be prolonged as a unique solution of (0.2) in $0 \leq t \leq t_0 + T$ belonging to the space $X_{s_0}(0, t_0 + T; C_1 N_{s_0}(0, 0; \rho, v))$. Here, $E_0 > 0$ is the constant in Theorem 1.1.*

Proof of Theorem 0.1. For the constants E_0, E_1, C_1 in Theorems 1.1 and 1.2, we set $\epsilon_0 = \max\{ E_0, E_1, E_0/C_1, E_1/C_1 \}$. If we take $\rho(0, x) = \rho^0(x)$ and $v(0, x) = v^0(x)$ small enough that $N_{s_0}(0, 0; \rho, v) \leq \epsilon_0$, by Theorem 1.2, we have a unique solution $(\rho, v) \in X_{s_0}(0, T; C_1 N_{s_0}(0, 0; \rho, v))$ for some fixed $T > 0$. By the definition of $\epsilon_0 > 0$, this implies

$$(\rho, v) \in X_{s_0}(0, T; E_0) \quad N_{s_0}(T, T; \rho, v) \leq N_{s_0}(0, T; \rho, v) \leq E_1.$$

Hence by Theorem 1.2, we can prolong this solution (ρ, v) by the time $2T$. This means that there is a unique solution $(\rho, v) \in X_{s_0}(0, 2T; C_1 N_{s_0}(0, 0; \rho, v))$. Repeating this step, we obtain Theorem 0.1.

§2. Outline of the proof

Our approach is based on various a priori estimates Theorem 1.1. Indeed, Theorem 1.2 is obtained by Theorem 1.1 and some estimate ensureing uniqueness of the solutions $(\rho, v) \in Y_{s_0}(0, T'; E)$ of (0.2).

THEOREM 2.1. *There exists a small constant $E_2 > 0$ such that for any solution (ρ, v) and $(\tilde{\rho}, \tilde{v}) \in Y_{s_0}(0, T'; E_2)$ of (0.2), the following estimate holds:*

$$\begin{aligned} &\| \rho(t, \cdot) - \tilde{\rho}(t, \cdot) \|_{H^{s_0 - 1}}^2 + \| v(t, \cdot) - \tilde{v}(t, \cdot) \|_{H^{s_0 - 1}}^2 \\ &\qquad \leq C\{\| \rho(0, \cdot) - \tilde{\rho}(0, \cdot) \|_{H^{s_0 - 1}}^2 + \| v(0, \cdot) - \tilde{v}(0, \cdot) \|_{H^{s_0 - 1}}^2\} \\ &\qquad\qquad \text{for any } 0 \leq t \leq T', \end{aligned}$$

where $C > 0$ a constant depending only on $\| \{\rho, v\} \|_{(s_0;0,T')}$, $\| \{\tilde{\rho}, \tilde{v}\} \|_{(s_0;0,T')}$ and T'. Thus, solution of (0.2) *in* $Y_{s_0}(0, T'; E_2)$ *are unique.*

Basically, we can obtain Theorems 1.1 and 2.1 by integrating by parts. In this procudere, we keep the L^2-inner product forms as possible and use integration by parts to homogenize the orders of the differentiations of each function in the integrated functions. This is one of the main idea to get a priori estimates. The complete proof is given in Kawashita [**5**].

Proof of Theorem 1.2. First we need to obtain local existence of solutions of (0.2) with initial data $(\rho^0, v^0) \in H^{s^0+4}$ satisfying $\| \rho^0 \|^2_{H^{s_0}} + \| v^0 \|^2_{H^{s_0}} \leq E_3$ for some fixed constant $E_3 > 0$ (cf. Proposition 3.1 in [**5**]).

Now, we remove regularities assumption to the initial data, that is the case $t_0 = 0$ in Theorem 1.2. For $E_2 > 0$ in Theorem 2.1 and $E_3 > 0$ in the above, we set $E_1 = \min\{E_2, E_3\}$.

For the initial data ρ^0, $v^0 \in H^{s_0}(\mathbf{R}^n)$ with $\| \rho^0 \|^2_{H^{s_0}} + \| v^0 \|^2_{H^{s_0}} \leq E_1$, we take sequences ρ^0_j, $v^0_j \in H^\infty(\mathbf{R}^n)$ $(j = 1, 2, \cdots)$ satisfying $\rho^0_j \to \rho^0$ $v^0_j \to v^0$ in $H^{s_0}(\mathbf{R}^n)$ as $j \to \infty$, $\| \rho^0_j \|^2_{H^{s_0}} + \| v^0_j \|^2_{H^{s_0}} \leq E_1$ for any $j = 1, 2, \cdots$. By existence of local solution for small initial data with regularities, there exists a solution (ρ_j, v_j) of (0.2) and $\rho_j(0, x) = \rho^0_j(x)$ $v_j(0, x) = v^0_j(x)$. From Theorem 2.1, ρ_j and v_j converge to ρ and v in $C([0, T]; H^{s_0-1}(\mathbf{R}^n))$ strongly. We can assume that ρ_j and v_j converge to ρ and v in $L^\infty([0, T] : H^{s_0}(\mathbf{R}^n))$ weakly by choosing sub sequence if it is necessary. Since $s_0 \geq 2$, we can obtain (ρ, v) satisfy the equation (0.2).

By an a priori estimate (cf. §3 in [**5**]), $\nabla_x v_j \in L^2([0, T] : H^{s_0}(\mathbf{R}^n))$ $(j = 1, 2, \cdots)$ is uniformly bounded. Hence, we can also assume $\nabla_x v_j$ converges to a function in $L^2([0, T] : H^{s_0}(\mathbf{R}^n))$. This means $(\rho, v) \in Y_{s_0}(0, T; C_1 N_{s_0}(0, 0))$ for some fixed constant $C_1 > 0$. From this fact, we can obtain the case $t_0 = 0$ of Theorem 1.2. For detail, we refer the arguments in §3 of Kawashita [**5**].

To show the case $t_0 > 0$, assume that a solution $(\rho, v) \in X_{s_0}(0, t_0; E_0)$ of (0.2) satisfies $N_{s_0}(t_0, t_0) \leq E_1$. By uniqueness of the solution and the case $t_0 = 0$ in Theorem 1.2, we can prolong the solution (ρ, v) to $(\rho, v) \in X_{s_0}(t_0, t_0 + T; C_2 N_{s_0}(t_0, t_0))$ for some $T > 0$ and $C_2 > 0$ independent of t_0 and the original solution in $X_{s_0}(0, t_0; E_0)$. Since $N_{s_0}(0, t_0) \leq E_0$, Theorem 0.2 implies $N_{s_0}(0, t_0 + T) \leq (1 + C_2) C_0 N_{s_0}(0, 0)$. This completes the proof of Theorem 1.2.

REFERENCES

[1] K. Deckelnick, *L^2-decay for the compressible Navier-Stokes equations in unbounded domains*, Commun. in Partial Differential Equations **18** (1993), 1445–1476.

[2] D. Hoff, *Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytoropic equation of state and discontinuous initial data*, Arch. Rational Mech. Anal. **132** (1995), 1-14.

[3] D. Hoff, *Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data*, J. Diff. Eq. **120** (1995), 215-254.

[4] D. Hoff and K. Zumbrun, *Multi-dimensional diffusion waves for the Navier -Stokes equations of compressible flow*, Indiana Univ. Math. J. **44** (1995), 603-676.

[5] M. Kawashita, *On global solutions of Cauchy problems for compressible Navier -Stokes equations*, preprint.

[6] T. Kobayashi and Y. Shibata, *Decay estimates of solutions for the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in* $\mathbf{R}^3$, preprint.

[7] A. Matsumura and T. Nishida, *The initial value problem for the equation of motion of compressible viscous and heat-conductive fluids*, Proc. Japan Acad. Ser. A **55** (1979), 337–342.

[8] A. Matsumura and T. Nishida, *The initial value problem for the equation of motion of compressible viscous and heat-conductive gases*, J. Math. kyoto Univ. **20** (1980), 67–104.

[9] A. Matsumura and T. Nishida, *Initial boundary value problems for the equation of motion of compressible viscous and heat-conductive fluids*, Comm. Math. Phys. **89** (1983), 445–464.

[10] G. Ponce, *Global existence of small solutions to a class of nonlinear evolution equations*, Nonlinear Anal. TMA **9** (1985), 399–418.

[11] M. Taylor, "Partial Differential Equations III Nonlinear equations," Splinger, New York, 1996.

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