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Existence and Asymptotic Behavior for Solutions of the Equations of Motion of Compressible Viscous Fluid

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We consider the equation which describes the motion of compressible viscous fluid. The equation is given by the following system of four equations for the density ρ and the velocity $\mathbf{v} = {}^T(v_1, v_2, v_3)$:

$$\begin{cases} \rho_t + \mathrm{div}\,(\rho\,\mathbf{v}) = 0, \\ \rho(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}) - \mathrm{div}\,\mathbb{T}(\mathbf{v}, p) = 0, \end{cases} \tag{1.1}$$

where ${}^T(v_1, v_2, v_3)$ is the transposed (v_1, v_2, v_3) and $p = p(\rho)$ the pressure. By $\mathbb{T}(\mathbf{v}, p)$ denote the stress tensor of the form

$$\mathbb{T}(\mathbf{v}, p) = \{\mu(\partial_{x_j} v_j + \partial_{x_j} v_j) + (\nu - \mu)\delta_{ij}\mathrm{div}\,\mathbf{v} - \delta_{ij} p\}_{i,j=1,2,3},$$

where μ and ν ($\nu > \frac{1}{3}\mu > 0.$) are constant viscosity coefficients.

We consider the initial boundary value problem (IBVP) of (1.1) in the region $t \geqq 0, x \in \Omega$ where Ω is an domain in $\mathbb{R}^3$ with compact smooth boundary $\partial\Omega$. The boundary condition is supposed by

$$\mathbf{v}\,|_{\partial\Omega} = 0, \tag{1.2}$$

and the initial condition is given by

$$(\rho, \mathbf{v})(0, x) = (\rho_0, \mathbf{v}_0)(x) \quad \text{in } \Omega. \tag{1.3}$$

I. Global in time existence theorem.

The first results concerning the global in time existence for the Cauchy problem for the equations of viscous compressible and heat conducting fluids were obtained by Matsumura and Nishida [11]. The corresponding results for an initial boundary value problem for the same equations was also showed by Matsumura and Nishida [12]. They considered a half space and an exterior domain, and they assumed that initial density, velocity and temperature are from $H^3(\Omega)$. Valli in [15] improved the results for barotropic case showing global in time existence for an arbitrary bounded domain $\Omega \subset \mathbb{R}^3$, for initial density and velocity from $H^2(\Omega)$. Recently, Kawashita [6] considered the Cauchy problem in $\mathbb{R}^3$ and proved the unique existence of the solutions for initial data from $H^2(\mathbb{R}^3)$. The global in time existence results from [6,11,12,15] follows from some a priori estimate which proof depends heavily on the L_2 approach. However, the results are not sharp in the L_2 framework, where by sharp we mean that the existence results can not be proved with less regularity imposed on the data.

The following result is joint work with prof. W.Zajaczkowski[1]. Our result is sharp for the L_2-approach and this is the reason why the fractional derivatives spaces have not been used. However this pressured us to use the Lagrangian coordinates which are not appropriate for problems in fixed domains.

Let Ω be an bounded domain with smooth compact boundary $\partial\Omega$. From (1.1) and (1.3) it follows that the total mass of the fluid in Ω is conserved,

$$\int_\Omega \rho\, dx = M = \int_\Omega \rho_0\, dx.$$

We give the existence of global in time solutions which are close to the equilibrium solution,

$$\mathbf{v}_e = 0, \rho_e = \frac{M}{|\Omega|},$$

where $|\Omega| = \operatorname{vol}\Omega$. The proof basis on a local existence result from [1] and on the prolongation technique from [14]. To recall the result from [1] we have to introduce the Lagrangian coordinates which are initial data to the following Cauchy problem

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{v}(t,x), \quad \mathbf{x}|_{t=0} = \xi \in \Omega.$$

Integrating above equation, we obtain the relation between the Eulerian $\mathbf{x}$ and the Lagrangian ξ coordinates

$$\mathbf{x} = \xi + \int_0^t \mathbf{u}(\xi,\tau)\, d\tau \equiv \mathbf{x}_{\mathbf{u}}(\xi,t) \equiv \mathbf{x}(\xi,t),$$

where $\mathbf{u}(\xi,t) = \mathbf{v}(\mathbf{x}(\xi,t),t)$. Moreover, we introduce $\eta(\xi,t) = \rho(\mathbf{x}(\xi,t),t)$, $q(\xi,t) = p(\eta(\xi,t))$. To prove global existence we have to control a variation of the solution in a neighborhood of the equilibrium solution. For this purpose we introduce

$$\rho_\sigma = \rho - \rho_e, \;\; p_\sigma = p - p_e,$$

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where $p_e = p(\rho_e)$. Then (1.1) implies

$$\begin{cases} q_{\sigma t} + p_\eta \eta \mathrm{div}_{\mathbf{u}}(\mathbf{u}) = 0 & \text{in } \Omega^T \\ \eta \mathbf{u}_t - \mathrm{div}_{\mathbf{u}} \mathbb{T}_{\mathbf{u}}(\mathbf{u}, q_\eta) = 0 & \text{in } \Omega^T = (0,\infty)\times\Omega, \\ \mathbf{u} = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ q_\sigma|_{t=0} = q_{\sigma 0} \equiv q_0 - p_e, \quad \mathbf{u}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega, \end{cases}$$

where $q_\sigma = q - p_e$, and

$$\eta_t + \eta \mathrm{div}_{\mathbf{u}} \mathbf{u} = 0 \quad \text{in } \Omega^T,$$
$$\eta|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega,$$

where $\nabla_{\mathbf{u}} = \xi_{ix}\partial_{\xi_i}$, $\mathbb{T}_{\mathbf{u}}(\mathbf{u}, q_\sigma) = -q_\sigma I + \mathbb{D}_{\mathbf{u}}(\mathbf{u})$, I is the unit matrix and the operator $\mathbb{D}_{\mathbf{u}}$, $\mathrm{div}_{\mathbf{u}}$ are obtained from $\mathbb{D}(\mathbf{v}) = \{\mu(\partial_{x_j} v_j + \partial_{x_j} v_j) + (\nu - \mu)\delta_{ij} \mathrm{div}\, \mathbf{v}\}_{i,j=1,2,3}$ and div replacing ∇ by $\nabla_{\mathbf{u}}$.

Finally we introduce the notation and spaces in this section. By $H^{k+\alpha, k/2+\alpha/2}(\Omega^T)$, we denote a Hilbert space with the norm

$$\begin{aligned} \|u\|^2_{H^{k+\alpha,k/2+\alpha/2}(\Omega^T)} &= \sum_{|\beta|+2i\le k} \|\partial_x^\beta \partial_t^i u\|^2_{L_2(\Omega^T)} \\ &+ \sum_{|\beta|=k} \int_0^T \int_\Omega \int_\Omega \frac{|\partial_x^\beta u(t,x) - \partial_{x'}^\beta u(t,x')|^2}{|x-x'|^{3+2\alpha}} dx dx' dt \\ &+ \int_\Omega \int_0^T \int_0^T \frac{|\partial_t^{[k/2]} u(t,x) - \partial_{t'}^{[k/2]} u(t',x)|^2}{|t-t'|^{1+\alpha+k-2[k/2]}} dx dt dt', \end{aligned}$$

where $k \in \mathbb{N}\cup\{0\}, \alpha \in (1/2, 1)$ and $[n]$ the integer parts of n. Similarly we can introduce the norm $H^{k+\alpha}(\Omega)$. Then we have

Theorem 1. *Assume that the bounded domain Ω is not rotationally symmetric, ρ_0, $\mathbf{v}_0 \in H^{1+\alpha}(\Omega)$, $\frac{1}{\rho_0} \in L_\infty(\Omega), p \in C^2, \alpha \in (\frac{1}{2}, 1)$ and $\mathbf{x} = \mathbf{x}_{\mathbf{u}}(\xi, t)$ determines the transformation between the Eulerian and Lagrangian coordinates. Assume that $\|\mathbf{v}_0\|_{H^{1+\alpha}(\Omega)}$, $\|\rho_0 - \rho_e\|_{H^{1+\alpha}(\Omega)}$ are sufficiently small. Then there exists a global solution to the problem* (1.1),(1.2) *and* (1.3) *such that*

$$\mathbf{u} \in H^{2+\alpha, 1+\alpha/2}(\Omega^t), \quad \eta \in H^{1+\alpha, 1/2+\alpha/2}(\Omega^t), \quad t \in \mathbb{R}^+,$$

where $\mathbf{u}(\xi, t) = \mathbf{v}(\mathbf{x}_{\mathbf{u}}(\xi, t), t)$, and $\eta(\xi, t) = \rho(\mathbf{x}_{\mathbf{u}}(\xi, t), t)$.

II. Asymptotic behavior for solutions.

Concerning the decay rate of solutions in the Caucy problem case, Matsumura and Nishida [11] showed that if the $L_1(\mathbb{R}^3) \cap H^4(\mathbb{R}^3)$−norm of the initial data are sufficiently small, then

$$\|(\rho - \bar{\rho}_0, \mathbf{v})\|_{H^2(\mathbb{R}^3)} = O(t^{-\frac{3}{4}}) \quad \text{as} \quad t \to \infty.$$

Also Ponce [13] showed that if the $W_1^{s_0}(\mathbb{R}^3) \cap H^{s_0}(\mathbb{R}^3)$−norm ($s_0 \geqq 4$, integer) of the initial data are sufficiently small, then

$$\|\partial_x^\alpha(\rho - \bar{\rho}_0, \mathbf{v})\|_{L_p(\mathbb{R}^3)} = O(t^{-\frac{3}{2q}-\frac{|\alpha|}{2}}) \quad \text{as} \quad t \to \infty$$

where $p \geqq 2, 1/p+1/q = 1$ and $|\alpha| \leqq 2$. Recently, Hoff and Zumbrun [4,5], Liu and Wang [10] they showed that if the $L_1(\mathbb{R}^3) \cap H^4(\mathbb{R}^3)$-norm of the initial data are sufficiently small, then

$$\|(\rho - \bar{\rho}_0, \mathbf{v})\|_{L_p(\mathbb{R}^3)} = \begin{cases} O(t^{-\frac{3}{2}(1-\frac{1}{p})}) & 2 \leqq p \leqq \infty, \\ O(t^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}(\frac{2}{p}-1)}) & 1 \leqq p \leqq 2, \end{cases}$$

as $t \to \infty$. As was already stated, the case of half-space or exterior domain has been studied by Matsumura and Nishida [12]. They proved the global in time existence theorem for small initial data in $H^3(\Omega)$ and showed that the L_∞-norm of solutions vanishes as $t \to \infty$. Deckelnick [2,3] proved the following decay rate:

$$\|\,\partial_x^1(\rho, \mathbf{v})\|_{L_2(\Omega)} = O(t^{-\frac{1}{4}}) \quad \text{as} \quad t \to \infty,$$

$$\|\,\rho - \bar{\rho}_0\|_{L_\infty(\Omega)} = O(t^{-\frac{1}{8}}) \quad \text{as} \quad t \to \infty,$$

$$\|\,\mathbf{v}\|_{L_\infty(\Omega)} = O(t^{-\frac{1}{4}}) \quad \text{as} \quad t \to \infty.$$

But this rate is weaker compared with the decay rate obtained by Matsumura and Nishida [11] and Ponce [13] in Cauchy problem case, because the initial data are assumed to be in $H^3(\Omega)$ only.

The following result is joint work with Prof. Y.Shibata[2]. Our result gives an optimal rate in the case that the initial data belong to $L_1(\Omega)$, which is corresponding to the rate in the Cauchy problem case which was obtained by Matsumura and Nishida [11], Ponce [13], Hoff and Zumbrun [4,5] and Liu and Wang [10]. Moreover, Theorem 2 is slightly better than [11], [13] and [4,5] because we do not assume the smallness of $L_1(\Omega)$ norm of the initial data.

Theorem 2. *Let Ω be an exterior domain with smooth compact boundary. Assume that $\frac{\partial}{\partial\rho}p > 0$ near $\bar{\rho}_0$. Assume that $(\rho_0, \mathbf{v}_0)$ satisfies the suitable compatibility condition and $(\rho_0 - \bar{\rho}_0, \mathbf{v}_0) \in L_1(\Omega) \cap H^4(\Omega)$. Then, there exists an $\epsilon > 0$ such that if $\|(\rho_0 - \bar{\rho}_0, \mathbf{v}_0)\|_{H^4(\Omega)} \leqq \epsilon$ then the solution $(\rho, \mathbf{v})$ of* (IBVP) *:* (1.1), (1.2) *and* (1.3) *has the following asymptotic behavior as $t \to \infty$:*

$$\|(\rho - \bar{\rho}_0, \mathbf{v})(t)\|_{L_2(\Omega)} = O(t^{-\frac{3}{4}});$$
$$\|\partial_x(\rho, \mathbf{v})(t)\|_{H^1(\Omega)\times H^2(\Omega)} + \|\partial_t(\rho, \mathbf{v})(t)\|_{H^1(\Omega)\times H^2(\Omega)} = O(t^{-\frac{5}{4}});$$
$$\|(\rho - \bar{\rho}_0, \mathbf{v})(t)\|_{L_\infty(\Omega)} = O(t^{-\frac{3}{2}});$$
$$\|\partial_x(\rho, \mathbf{v})(t)\|_{L_p(\Omega)} = O(t^{-\frac{3}{2}}), \ \ 3 < p < \infty;$$

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$$\|(\rho-\bar{\rho}_0,\mathbf{v})(t)\|_{L_1(\Omega)}=O(t^{\frac{1}{2}}).$$

Moreover, if $\rho_0-\bar{\rho}_0\in W_1^1(\Omega)$*, then*

$$\|\,\partial_x(\rho-\bar{\rho}_0,\mathbf{v})(t)\|_{L_1(\Omega)}=O(1)\quad \mathit{as}\quad t\to\infty.$$

Here ϵ *depends on* p.

In order to prove Theorem 2, we shall use the decay property of solutions to the corresponding linearized problem. If we linearize the equation (1.1) at the constant state $(\bar{\rho}_0,0)$ and we make some linear transformation of the unknown function, then we have the following initial boundary value problem of the linear operators :

$$\begin{cases} \rho_t+\gamma\mathrm{div}\;\mathbf{v}=0 & \text{in } [0,\infty)\times\Omega,\\ \mathbf{v}_t-\alpha\Delta\mathbf{v}-\beta\nabla\mathrm{div}\;\mathbf{v}+\gamma\nabla\rho=0 & \text{in } [0,\infty)\times\Omega,\\ \mathbf{v}|_{\partial\Omega}=0, & \text{on } [0,\infty)\times\partial\Omega,\\ (\rho,\mathbf{v})|_{t=0}=(\rho_0,\mathbf{v}_0), & \text{in } \Omega, \end{cases}$$

where α, κ, γ and ω are positive constants and β is a nonnegative constant. Let $\mathbb{A}$ be the 4×4 matrix of the differential operators of the form :

$$\mathbb{A}=\begin{pmatrix} 0 & \gamma\mathrm{div}\\ \gamma\nabla & -\alpha\Delta-\beta\nabla\mathrm{div}\end{pmatrix}$$

with the domain :

$$\mathcal{D}_p(\mathbb{A})=\{\mathbb{U}=(\rho,\mathbf{v})\in W_p^1(\Omega)\times W_p^2(\Omega)\mid \mathbf{v}|_{\partial\Omega}=0\}$$

for $1<p<\infty$. Then, above equations are written in the form :

$$\mathbb{U}_t+\mathbb{A}\mathbb{U}=0\quad\text{for } t>0,\quad \mathbb{U}|_{t=0}=\mathbb{U}_0,$$

where $\mathbb{U}_0=(\rho_0,\mathbf{v}_0)$ and $\mathbb{U}=(\rho,\mathbf{v})$.

Moreover, if we apply some linear transformation to $(\rho-\bar{\rho}_0,\mathbf{v})$ (the resulting vector of functions being denoted by $\tilde{\mathbb{U}}=(\tilde{\rho},\tilde{\mathbf{v}})$, then we can reduce IBVP : (1.1), (1.2) and (1.3) to the problem :

$$\tilde{\mathbb{U}}_t+\mathbb{A}\tilde{\mathbb{U}}=\mathbb{F}(\tilde{\mathbb{U}})\quad\text{for } t>0,\quad \tilde{\mathbb{U}}|_{t=0}=\tilde{\mathbb{U}}_0$$

with suitable nonlinear term $\mathbb{F}(\tilde{\mathbb{U}})$. Therefore, in order to prove Theorem 2, we have to obtain the suitable decay property of solutions to the above linearized equations. We show that $\mathbb{A}$ generates an analytic semigroup $\{e^{-t\mathbb{A}}\}_{t\geqq 0}$ on $W_p^1(\Omega)\times L_p(\Omega)$, $1<p<\infty$ (cf. [7,8,9]). Then we show the L_p-L_q type estimate concerning the decay rate of $\{e^{-t\mathbb{A}}\}_{t\geqq 0}$. These L_p-L_q type estimate is proved by combination of the L_p-L_q type estimate in the $\mathbb{R}^3$ case and the local energy decay estimate of $\{e^{-t\mathbb{A}}\}_{t\geqq 0}$, via cut-off technique. To prove Theorem 2, we reduce IBVP : (1.1), (1.2) and (1.3) to the integral equation :

$$\tilde{\mathbb{U}}(t)=e^{-t\mathbb{A}}\tilde{\mathbb{U}}_0-\int_0^t e^{-(t-s)\mathbb{A}}\mathbb{F}(\tilde{\mathbb{U}}(s))\,ds.$$

Applying L_p-L_q type estimate and using the fact that the $H^4(\Omega)$-norm of solutions are bounded which were proved by Matsumura and Nishida [12], we have Theorem 2.

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