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Short time behaviors of curves and surfaces moved by surface diffusion

Kazuo Ito 伊藤一男
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060-0810, Japan

1 Introduction and main results

We consider the following geometric evolution law of the form

$$\begin{cases} V=-\Delta_{\Gamma(t)}H & \text{on } \Gamma(t),\ t>0, \\ \Gamma(0)=\Gamma_0. \end{cases} \tag{1}$$

Here t denotes the time variable and $\Gamma(t)$ denotes an unknown evolving hypersurface in $\mathbf{R}^n$ with $n\geq 2$ (of course, $\Gamma(t)$ is a curve when $n=2$); Γ_0 is a given initial hypersurface (curve) in $\mathbf{R}^n$. For each $t>0$ and $x\in\Gamma(t)$, the quantities $V=V(t,x)$ and $H=H(t,x)$ denote the outward normal velocity and the outward mean curvature of $\Gamma(t)$ at x, respectively. The operator $\Delta_{\Gamma(t)}$ stands for the Laplace-Beltrami operator on $\Gamma(t)$. In particular $\Delta_{\Gamma(t)}=\partial_s^2$ when $n=2$, where s is the arc-length parameter of $\Gamma(t)$. The law (1) is called the *surface diffusion flow equation.*

The solution $\Gamma(t)$ of (1) describes motion of interface in a binary alloy system. Equation (1) was first proposed by Mullins [15] to explain thermal grooving in material sciences. Also Davi and Gurtin [5] derived (1) from a view point of thermodynamics and continuum mechanics (see also Cahn and Taylor [1]). Recently, J. W. Cahn, C. M. Elliott and A. Novick-Cohen [2] linked (1) with the Cahn-Hilliard equation with a concentration dependent mobility via formal singular limit.

Parametrization of (1) tells us that (1) is a nonlinear fourth order parabolic equation. Generally speaking, behaviors of solutions of fourth order equations are less known than those of second order equations.

The purpose here is to study the qualitative behavior of the solution $\Gamma(t)$ of (1) in a short time.

Several mathematical and numerical studies for (1) show remarkable characteristic phenomena, for example, loss of embeddedness and loss of convexity. The loss of both embeddedness and convexity reflects the fact that a fourth order parabolic equation does not fulfill the maximum or comparison principle. In fact, this principle is satisfied in the second order models without nonlocal effect such as the mean curvature flow equation

$$\begin{cases} V = H \quad \text{on } \Gamma(t),\, t > 0, \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{2}$$

The maximum principle prevents the developments of self-intersections (Grayson [11] for $n = 2$) and preserves convexity (Gage and Hamilton [8] for $n = 2$ and Huisken [12] for $n \geq 3$) of solutions of (2).

The loss of embeddedness for (1) was conjectured by Elliott and Garcke [6], numerically established by Escher, Mayer and Simonett [7]. When $n = 2$ Giga and the author [9] mathematically proved this phenomena. The result is as follows.

Theorem 1 (*Loss of embeddedness*). *There is an embedded initial closed curve Γ_0 in $\mathbf{R}^2$ such that the smooth solution $\Gamma(t)$ of (1) starting from Γ_0 loses its embeddedness during a time interval (t_0, t_1) with $t_0 > 0$ determined by Γ_0.*

Recently this result is extended to higher dimensional version by Mayer and Simonett [14].

On the other hand J. Escher proposed a conjecture in the conference "Nonlinear Evolution Equation" held in the end of June of 1997 in Oberwolfach that the convexity of the solution $\Gamma(t)$ of (1) is not necessarily preserved. This phenomenon was also suggested by numerical studies by B. D. Coleman, R. S. Falk and M. Moakher [3, 4]. The rigorous proofs for the loss of convexity were obtained by Giga and the author [10] for closed curves and by the author [13] for compact hypersurfaces. The result is as follows.

Theorem 2 (*Loss of convexity*; [10] *for $n = 2$*, [13] *for $n \geq 3$*). *There is a strictly convex closed compact initial hypersurface Γ_0 such that the smooth solution $\Gamma(t)$ of (1) starting from Γ_0 loses its convexity during a time interval (t_0, t_1) with $t_0 > 0$ determined by Γ_0.*

Let us briefly mention the organization of this note. In Section 2 we state a parametrization of (1) near a fixed compact reference hypersurface Σ, which shows that (1) is described by a fourth order nonlinear parabolic equation of a unknown signed distance function from Σ. This equation can be given in a quite intrinsic manner. Section 3 is devoted to explain the idea of the proof of Theorem 1 by specifying Σ as a dumbbell like closed curve immersed in $\mathbf{R}^2$. Section 4 explains the idea of the proof of Theorem 2. The main part is to give a way of constructing a deformation for smooth compact strictly convex hypersurfaces which lose convexity in a short time without loosing their smoothness when they move by their surface diffusion.

2 Parametrization

Following [7], we introduce a parametrization for (1). Let Σ be a smooth, compact, closed, embedded, oriented hypersurface in $\mathbf{R}^n$ and let $\{U_\beta, \psi_\beta\}_{\beta=1}^{\bar{\beta}}$ be an atlas on Σ. For $s \in U_\beta \subset \Sigma$, $\psi_\beta(s) = (u_\beta^1, \cdots, u_\beta^{n-1}) \in U'_\beta := \psi_\beta(U_\beta) \subset \mathbf{R}^{n-1}$ is called the local coordinate of s. Let z be the induced metric on Σ from the Euclidean metric in $\mathbf{R}^n$ and let h be the second fundamental quantity of Σ. In local coordinates they are written as

$$z = \sum_{i,j=1}^{n-1} z_{\beta,ij} du_\beta^i \otimes du_\beta^j, \quad h = \sum_{i,j=1}^{n-1} h_{\beta,ij} du_\beta^i \otimes du_\beta^j$$

at $s \in U_\beta$, where

$$z_{\beta,ij} = \frac{\partial s}{\partial u_\beta^i} \cdot \frac{\partial s}{\partial u_\beta^j}, \quad h_{\beta,ij} = \frac{\partial^2 s}{\partial u_\beta^i u_\beta^j} \cdot \nu(s)$$

and $\nu(s)$ is the outward normal of Σ at s. Hereafter, if any confusion may not be caused, then using Einstein's convention and omitting the index β we often simply write them as

$$z = z_{ij} du^i \otimes du^j, \quad h = h_{ij} du^i \otimes du^j.$$

Throughout this paper we regard Σ as a Riemannian manifold with the metric z. We call Σ the reference hypersurface.

Let $\rho : [0,T) \times \Sigma \to \mathbf{R}$ be a smooth scalar field whose absolute value is small enough and we assume that $\Gamma(t)$, $t \in [0,T)$, is close to Σ and is written in terms of $\rho(t,s)$ with $\rho(0,s) = \rho_0(s)$ as

$$\Gamma(t) = \{s + \rho(t,s)\nu(s) \in \mathbf{R}^n; s \in \Sigma\}.$$

We define two geometric quantities of Σ:

$$\begin{aligned} &w(r) = w_{ij}(r) du^i \otimes du^j := (z_{ij} - 2h_{ij} r + z^{kl} h_{ki} h_{lj} r^2) du^i \otimes du^j \quad \text{for } r \in \mathbf{R}, \\ &\sigma(\rho, d\rho) := w(\rho) + d\rho \otimes d\rho, \end{aligned}$$

where $(z^{ij}) = (z_{ij})^{-1}$ and $d\rho = \frac{\partial \rho}{\partial u^i} du^i$. Note that $w(\rho)$ and $\sigma(\rho, d\rho)$ are also metrics on Σ as long as $|\rho|$ is small enough. Thus we can consider the geometric operators acting on scalar fields Ψ on Σ such as $\mathrm{grad}_{w(f)}$, $\mathrm{Hess}_{w(f)}$, $\Delta_{w(f)}$, and $\Delta_{\sigma(f,df)}$ for small scalar fields f on Σ. They are given in local coordinates by

$$\begin{aligned} &\mathrm{grad}_{w(f)} \Psi = w^{ij}(f) \frac{\partial \Psi}{\partial u^j} \frac{\partial}{\partial u^i}, \\ &\mathrm{Hess}_{w(f)} \Psi = \left(\frac{\partial^2 \Psi}{\partial u^i \partial u^j} - W_{ij}^k(f, df) \frac{\partial \Psi}{\partial u^k}\right) du^i \otimes du^j, \\ &\Delta_{w(f)} \Psi = w^{ij}(f) \left(\frac{\partial^2 \Psi}{\partial u^i \partial u^j} - W_{ij}^k(f, df) \frac{\partial \Psi}{\partial u^k}\right), \\ &\Delta_{\sigma(f,df)} \Psi = \sigma^{ij}(f, df) \left(\frac{\partial^2 \Psi}{\partial u^i \partial u^j} - \gamma_{ij}^k(f, df, \nabla df) \frac{\partial \Psi}{\partial u^k}\right), \end{aligned}$$

where $(w^{ij}(f)) := (w_{ij}(f))^{-1}$, $(\sigma^{ij}(f,df)) := (\sigma_{ij}(f,df))^{-1}$; $W_{ij}^k(f,df)$ and $\gamma_{ij}^k(f,df,\nabla df)$ with $i,j,k = 1,2,\cdots,n-1$ are the Christoffel symbols of $w(f)$ and $\sigma(f,df)$, respectively. We also define

$$L = L(\rho, d\rho) := (1 + w(\rho)[\mathrm{grad}_{w(\rho)} \rho, \mathrm{grad}_{w(\rho)} \rho])^{1/2}, \tag{3}$$

$$w'(r) = w'_{ij}(r) du^i \otimes du^j := (-2h_{ij} + 2z^{kl} h_{ki} h_{lj} r) du^i \otimes du^j,$$

$$w'(r) w^*(r) := w'_{ij}(r) w^{ij}(r) \tag{4}$$

for $r \in \mathbf{R}$ with $|r|$ small enough.

Then, as computed in [7] we can have the partial differential equation of $\rho(t,s)$ described by local coordinates which parametrizes (1). But this equation can be rewritten in more intrinsic way by making further computations. We present here the equation of $\rho(t,s)$ in an intrinsic manner without detailed computations:

$$\begin{cases} \rho_t = -L\Delta_{\sigma(\rho,d\rho)}\Big[L^{-3}\{L^2\Delta_{w(\rho)}\rho - \mathrm{Hess}_{w(\rho)}\rho[\mathrm{grad}_{w(\rho)}\rho, \mathrm{grad}_{w(\rho)}\rho] \\ \qquad -\frac{1}{2}w'(\rho)[\mathrm{grad}_{w(\rho)}\rho, \mathrm{grad}_{w(\rho)}\rho] - \frac{1}{2}L^2 w'(\rho)w^*(\rho)\}\Big], \quad t>0,\ s\in\Sigma, \\ \rho(0,s) = \rho_0(s), \quad s\in\Sigma. \end{cases} \tag{5}$$

3 Idea of the proof of Theorem 1

Here we summarize an idea of the proof of Theorem 1. The detailed proof is given in [9].

First let us point out that for the case $n=2$ the equation (5) is simply written in the form

$$\begin{cases} \rho_t + J^{-4}\rho_{uuuu} + P\rho_{uuu} + Q = 0, \ 0<t<T, \ u\in\mathbf{T}, \\ \rho(0,u) = \rho_0(u), \ u\in\mathbf{T}. \end{cases} \tag{6}$$

Here $\mathbf{T} := \mathbf{R}/2l\mathbf{Z}$ and $2l$ is the total length of the reference curve Σ; we choose u as the arc-length parameter of Σ. J is the line element of $\Gamma(t)$ and its explicit form is

$$J = (\rho_u^2 + (1-h\rho)^2)^{1/2}.$$

P and Q are polynomials with arguments $(1-h\rho)^{-1}$, J^{-1}, h, h_u, h_{uu}, h_{uuu}, ρ, ρ_u, and ρ_{uu}. We note that h toghther with its derivatives h_u, h_{uu}, h_{uuu} is continuous and bounded on $\mathbf{T}$ since Σ is smooth.

We show that there is an evolving closed curve which pinches in finite time, even if initial curve is simple. Let us explain our idea of the proof. We specify the reference curve Σ as a dumbbell like curve symmetric with respect to both x-axis and y-axis and its neck is so thin so that it is just a segment on the x-axis. It is normalized by setting $s(0) = s(l) =$ the origin $(0,0)$. Let $\Gamma_0 = \{s(u) + \rho_0(u)\nu(u) \in \mathbf{R}^2; u\in\mathbf{T}\}$ with $\rho_0(u) > 0$ be symmetric with respect to both x-axis and y-axis and assume that $\rho_0(u)$ takes its global isolated minimum at $u = 0$ and l. We then establish a unique local existence result in L^2-framework whose precise statement is omitted here. Moreover we give a fact that by symmetry of the equation (6), the solution $\Gamma(t) = \{s(u) + \rho(t,u)\nu(u); u\in\mathbf{T}\}$ stays symmetric with respect to both x-axis and y-axis. In particular, $\rho_u(t,0) = 0$ and $\rho_{uuu}(t,0) = 0$. Thus if $\rho(t,u)$ solves (6), then

$$\rho_t(0,0) = -\partial_u^4\rho(0,0) + 3(\partial_u^2\rho(0,0))^3.$$

Thus, by the fundamental theorem of calculus,

$$\begin{aligned} \rho(t,0) &= \rho(0,0) + \rho_t(0,0)t + \int_0^t\int_0^\tau \rho_{\sigma\sigma}(\sigma,0)\,d\sigma d\tau \\ &\le \rho(0,0) + (-\partial_u^4\rho(0,0) + 3(\partial_u^2\rho(0,0))^3)t + t^2\cdot \sup_{t\in[0,\bar{t}],u\in\mathbf{T}} |\rho_{tt}(t,u)|, \end{aligned} \tag{7}$$

where $\bar{t}$ is taken so that $\rho(t,u)$ exists for $[0,\bar{t}]$. Roughly speaking, if $\rho(0,0)$ is sufficiently small and $-\partial_u^4\rho(0,0) + 3(\partial_u^2\rho(0,0))^3 < 0$, then $\rho(t,0)$ may be nagative for t between two roots of the quadratic equation of t: the R.H.S. of (7)= 0, which will imply a pinching of $\Gamma(t)$.

We shall state our result rigorously in the following. To do this, we define a special (C^∞) reference curve Σ. This is parametrized by

$$s(u) = (s^1(u), s^2(u)) \quad \text{for } u \in \mathbf{T} = \mathbf{R}/(2l\mathbf{Z})$$

satisfying

$$\begin{cases} s^1(u) = -s^1(-u), & 0 \le u \le l, \\ s^2(\eta) = s^2(-u), & 0 \le u \le l, \\ s(u) = (u, 0), & 0 \le u \le l/4, \\ s^1_u(u) > 0, & 0 \le u \le l/2, \\ s^1(l/2+u) = s^1(l/2-u), & 0 \le u \le l/2, \\ s^2(u) > 0, & l/4 < u < l/2, \\ s^2(l/2+u) = -s^2(l/2-u), & 0 \le u \le l/2, \end{cases}$$

where u is taken as the arclength parameter of Σ. We define a set of functions in $\mathbf{T}$ depending on positive parameters N and ε:

$$\begin{aligned} D_0(N,\varepsilon) \;=\; & \{\rho_0 : \text{smooth}; \rho_0(-u) = \rho_0(u) = \rho_0(l-u), \quad \rho_0(u) > 0 \quad (\forall u \in \mathbf{T}), \\ & \|\rho_0\|_{H^9(\mathbf{T})} \le N, \quad \rho_0(0) \le \varepsilon, \quad \rho_0^{(4)}(0) - 3\rho_0''(0)^3 > 0, \\ & \rho_0(u) \text{ attains its global minimum at } u = 0\}. \end{aligned}$$

Note that closed curves Γ_0 parametrized by $s(u)+\rho_0(u)\nu(u)$ with $\rho_0 \in D_0(N,\varepsilon)$ are simple in $\mathbf{R}^2$. A typical result is:

Theorem 3 *(Pinching of evolving closed curves). For any $N > 0$ depending on Σ, there is an $\varepsilon_0 > 0$; for any $\varepsilon \in (0, \varepsilon_0)$ and any $\rho_0 \in D_0(N,\varepsilon)$, there are $t_0 \in (0, T_1(N))$ (where $T_1(N)$ is an existing time of the solution of (6)) and $t_1 (> t_0)$ such that for initial simple closed curve Γ_0 with parametrization*

$$\Gamma_0 = \{s(u) + \rho_0(u)\nu(u); u \in \mathbf{T}\},$$

the solution curve $\Gamma(t)$ with parametrization

$$\Gamma(t) = \{s(u) + \rho(t,u)\nu(u); u \in \mathbf{T}\}, \quad t \in [0, T_1(N)],$$

where $d \in D_{T_1(N)}(N)$ is the unique solution of (6), ceases to be simple for at least $t_0 < t < \min(t_1, T_1(N))$.

This result looks stronger than the one presented in [9] in the sense that ρ_0 is taken arbitrary but clearly the proof in [9] yields this result.

4 Idea of the proof of Theorem 2

Here we summarize the strategy to prove Theorem 2. The datailed proof is given in [10, 13]. For consistency of the descriptions we only consider the case that $\Gamma(t)$ are compact hypersurfaces, that is, $n \ge 3$. We set the refernce hypersurface Σ to be convex. Then we have:

Lemma 4 *Let Σ be a smooth, compact, convex, closed, embedded, oriented hypersurface in $\mathbf{R}^n$. Then,*
(i) $w(r)$ is a metric on Σ for $r \geq 0$,
(ii) $\sigma(f, df)$ is a metric on Σ for nonnegative (at least) C^1-scalar field $f = f(s)$ on Σ.

We state the local existence theorem of (5) for positive initial data. The differences between the local existence theorem in [7] and ours is that our theorem uses the convex reference hypersurface Σ, which enable us to treat large magnitude of positive initial data $\rho_0(s)$. This advantage is useful in the proof of Theorem 2. Another difference is that our local existence theorem is established in a different category from that of [7], that is, the local existence theorem in [7] is established in the framework of t-continuous and s-little Hölder continuous regularity whereas our theorem is in the framework of the type $C^{[m/4]+\alpha, m+4\alpha}(\Sigma)$ for an integer $m \geq 4$ and $0 < \alpha < 1/4$, where the symbol $[q]$ denotes the largest integer less than or equal to q. Our theorem also explicitly gives an information how the existence time depends on initial data, which is also useful in the proof of Theorem 2.

Theorem 5 *Let $m \geq 4$ be an integer and $0 < \alpha < 1/4$. Let $\rho_0 \in C^{m+4\alpha}(\Sigma)$ with $\rho_0(s) > 0$ for $s \in \Sigma$. Set*

$$m_0 = \min_{s \in \Sigma} \rho_0(s) > 0. \tag{8}$$

Then there are positive constants $T(||\rho_0||_{C^{m+4\alpha}(\Sigma)}, m_0)$ and $G(||\rho_0||_{C^{m+4\alpha}(\Sigma)})$ such that (5) has a unique solution $\rho(t, s)$ satisfying

$$\begin{aligned}
&\rho \in C^{[m/4]+\alpha, m+4\alpha}([0, T_0] \times \Sigma), \\
&||\rho||_{C^{[m/4]+\alpha, m+4\alpha}([0,T_0]\times\Sigma)} \leq G(||\rho_0||_{C^{m+4\alpha}(\Sigma)}), \\
&\min_{(t,s)\in[0,T_0]\times\Sigma} \rho(t, s) \geq m_0/2 > 0,
\end{aligned}$$

where $T_0 := T(||\rho_0||_{C^{m+4\alpha}(\Sigma)}, m_0)$.

Remark 6 . *Here $T(M_0, m_0)$ is nonincreasing in M_0 and nondecreasing in m_0; $G(M_0)$ is nondecreasing in M_0.*

The essential task of the proof of Theorem 5 is to show that the linearized differential operator at initial data ρ_0 is sectorial in $C(\Sigma)$. Once this is verified, we can use the iteration method in the linearized equation to obtain the desired unique local solution of (5). The details are omitted.

To consider the proof of Theorem 2, it is useful to intuitively imagine that the hypersurface of the solution of (1) starting from an initial hypersurface with sufficiently weak convexity may easily creats a loss of convexity. From this observation, as a first step, we introduce a deformation depending on a small parameter $\varepsilon > 0$ for strictly convex hypersurfaces Γ_0. Let us denote by Γ_0^ε the deformed hypersurface. This deformation should be constructed to preserve the convexity everywhere and to weaken the convexity of the original surface Γ_0 as one of the principal curvatures of Γ_0^ε has the order $O(-\varepsilon)$ locally but its fourth order derivative stays negative away from 0. Also this deformation should guarantee that the distance function ρ_0^ε of Γ_0^ε from Σ is bounded in $C^{m+\alpha}(\Sigma)$ with respect to ε. Let us give a rough explanation on how to construct this deformation. Let us assume that Γ_0 is an axisymmetric rotational hypersurface about the x^1-axis with a generator $x^2 = f(x^1) \geq 0$ for $x \in [-1, 1]$. The function f should be smooth, even, and strictly concave, i.e., $f'' < 0$. Let $\delta > 0$ be a sufficiently small parameter and φ_δ be a smooth cut-off function such as

$$\varphi_\delta(x^1) = \begin{cases} 1 & \text{for } |x^1| < \delta, \\ 0 & \text{for } 2\delta < |x^1| < 1 \end{cases}$$

and $0 \leq \varphi_\delta \leq 1$. We combine the function $-\varepsilon - (x^1)^4/4!$ with f'' as follows

$$w^{\varepsilon,\delta}(x^1) := (-\varepsilon - \frac{(x^1)^2}{4})\varphi_\delta(x^1) + f''(x^1)(1 - \varphi_\delta(x^1)).$$

Moreover we set

$$v^{\varepsilon,\delta}(x^1) := \int_0^{x^1} w^{\varepsilon,\delta}(\xi)d\xi\varphi_{1/4}(x^1) + f'(x^1)(1 - \varphi_{1/4}(x^1)),$$

$$(M^{\varepsilon,\delta}f)(x^1) := f(\frac{1}{2}) + \int_0^{x^1} v^{\varepsilon,\delta}(\xi)d\xi.$$

Then we can easily check that the axisymmetric rotational hypersurface about the x^1-axis with the generator $x^2 = (M^{\varepsilon,\delta}f)(x^1)$ satisfies the desired properties if $\delta > 0$ is small enough (the size of δ is determined by f). This is the rough explanation for the way of constructing the deformation.

Then the unique smooth solution $\Gamma^\varepsilon(t)$ of (1) starting from Γ_0^ε exists for $t \in [0, T^\varepsilon]$ for some $T^\varepsilon > 0$. But we should be afraid that T^ε may shrink to 0 as $\varepsilon \to 0$. In the second step, using Theorem 5, we present a fact that there is a time $T > 0$ such that $T^\varepsilon \geq T$ for any sufficiently small ε depending on δ. This means that $\Gamma^\varepsilon(t)$ exists uniformly in ε. Finally, using the results of the previous two steps, we prove that if $\varepsilon > 0$ is sufficiently small, then one of the principal curvatures of $\Gamma^\varepsilon(t)$ becomes positive after a finite time, which means that $\Gamma^\varepsilon(t)$ loses its convexity.

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