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WAVE FRONT SETS OF SOLUTIONS TO ELASTIC WAVE PROPAGATION PROBLEMS IN STRATIFIED MEDIA

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1. Introduction

We consider elastic wave propagation problems in plane-stratified media $\mathbf{R}^3$ with the planer interface $x_3 = 0$. This problem is formulated as an elastic mixed or initial-interfece problem in a stratified media.

An elastic equation has two speeds. Pressure or Primary wave (for simplicity called P wave) and Share or Secondary wave (S wave). P wave is a longitudinal wave and S wave is a transversal wave. In general the speed of P wave is grater than that of S wave. In plane-stratified media problem, a lower half-space $\mathbf{R}^3_-$ called *Medium I* has P_1 and S_1 waves and an upper half-space $\mathbf{R}^3_+$ called *Medium II* has P_2 and S_2 waves. The speed of P_1 (resp. P_2) wave is grater than that of S_1 (resp. S_2) wave. So the order relation of the speeds of P_1, P_2, S_1, and S_2 waves are six cases. Here we assume P_2, S_2, P_1, S_1 waves in order of speed since it is the most complex case.

We put unit impulse Dirac's delta in the lower half-space Medium I. Then P_1 incident wave which speed is faster than S_1 incident wave bumps against the interface and causes P_1 and S_1 reflected waves in Medium I and P_2 and S_2 refracted waves in Medium II as in Figure 1.

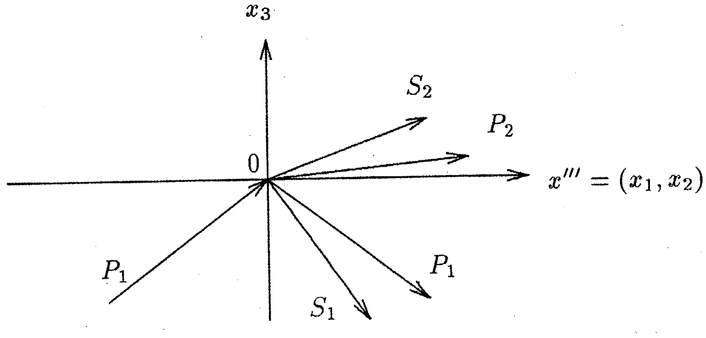

Figure 1 Reflected waves and refracted waves

Moreover when time goes on, lateral waves in other words glancing waves or total reflected (or refracted) waves arise. In Figure 2, dotted arrows show P_2-P_1 and P_2-S_1 lateral waves in Medium I, and P_2-S_2 lateral wave in Medium II for

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P_1 incident wave. P_2-P_1 lateral wave means that the wave originally should have been P_2 reflected wave tends to total reflection, then becomes source and causes P_1 reflected wave.

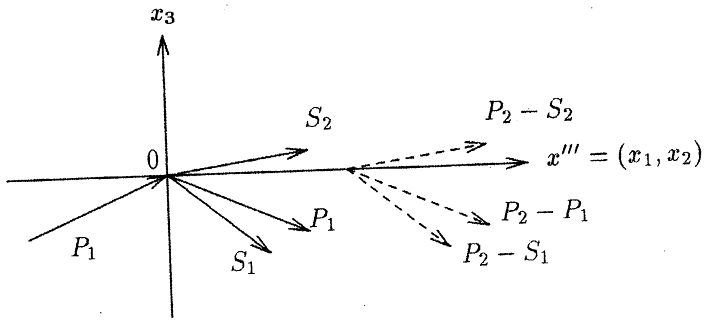

Figure 2 Lateral waves

We have 11th kind of lateral waves in all. It is a characteristic of our elastic wave propagation problems in stratified media. If half-space problem which has two speeds, there exist only one kind of lateral wave. If plane-stratified media problem that each medium has one speed, there exist only one kind of lateral wave. Thus this elastic wave propagation problems in plane-stratified media has many lateral waves.

In this paper we prove the above physical situation mathematically by using a expression of an inner estimate of singularities. The main technical tool of our analysis is a localization method.

We gave an inner estimate of the location of singularities of the reflected and refracted Riemann functions by making use of the localization method [7,8]. This method is first studied by M. F. Atiyah, R. Bott, and L. Gårding [1] and L. Hörmander [2] for initial value problem, then studied by M. Matsumura [5], M. Tsuji [9], and S. Wakabayashi [10,11] for half-space mixed problem.

In this paper, we give an outer estimate of wave front sets of the incident, reflected and refracted Riemann functions by making use of the localization method. Atiyah-Bott-Gårding [1] studied the outer estimate of wave front sets of solutions to initial value problem. Wakabayashi studied for half-space mixed problem [12], and for more general case [13]. We analysis an outer estimate of wave front sets of the Riemann functions to the elastic mixed problem based on Wakabayashi's theorem [12, Theorem 4.2]. Combining the inner estimate and the outer estimate, we obtain the exact wave front sets of the elastic mixed problem in stratified media.

I would like to express my gratitude to Professor Seiichiro Wakabayashi for his invaluable suggestions.

2. Formulation of Problems

We consider elastic wave propagation problems in the following plane-stratified media $\mathbf{R}^3$ with the planar interface $x_3 = 0$:

$$(\lambda(x_3), \mu(x_3), \rho(x_3)) = \begin{cases} (\lambda_1, \mu_1, \rho_1) & \text{for} \quad x_3 < 0, \\ (\lambda_2, \mu_2, \rho_2) & \text{for} \quad x_3 > 0. \end{cases}$$

Here the constants λ_1, λ_2, μ_1, μ_2 are called the Lamé constants and the constants

ρ_1, ρ_2 are densities. We shall denote the lower half-space $\mathbf{R}^3_-$ by *Medium I* and the upper half-space $\mathbf{R}^3_+$ by *Medium II*, respectively, as in Figure 3.

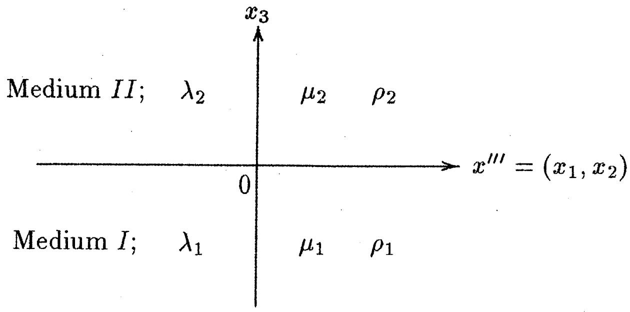

Figure 3 Stratified media

We assume that

$$(1.1) \qquad \lambda_i + \mu_i > 0, \quad \mu_i > 0, \quad \rho_i > 0, \quad i = 1, 2.$$

(1.1) is the natural assumption in practical situation. From the roots of the characteristic equations of $P^I(D)$ and $P^{II}(D)$ which are defined below 3×3 matrix valued hyperbolic partial differential operators in Medium I and Medium II, respectively, we obtain two speeds correspond to Pressure or Primary wave (for simplicity called P wave) and Share or Secondary wave (S wave) on each medium. P wave is a longitudinal wave and S wave is a transversal wave. c_{p_1} denotes the speed of P wave in Medium I and c_{s_1} denotes the speed of S wave in Medium I. c_{p_2} and c_{s_2} denote the speed of P and S wave in Medium II, respectively. They are given by

$$c^2_{p_i} = \frac{\lambda_i + 2\mu_i}{\rho_i}, \quad c^2_{s_i} = \frac{\mu_i}{\rho_i}, \quad i = 1, 2.$$

By assumption (1.1), the speed of P wave is greater than that of S wave in each medium. On account of this, these are six cases of the order relation of the speeds of $\{c_{p_1}, c_{s_1}, c_{p_2}, c_{s_2}\}$. Here we assume that

$$c_{s_1} < c_{p_1} \le c_{s_2} < c_{p_2},$$

since if we put an unit impulse Dirac's delta in Medium I, it is the case that the most number of lateral waves are appeared. The other cases can be treated in a similar manner (cf. [6, Section 3]).

Let $x = (x_0, x_1, x_2, x_3) = (x', x_3) = (x_0, x'') = (x_0, x''', x_3)$ in $\mathbf{R}^4$. The variable x_0 will play a role of time, and $x'' = (x_1, x_2, x_3)$ will play that of space. ξ is a real dual variable of x and is equal to $(\xi_0, \xi_1, \xi_2, \xi_3) = (\xi', \xi_3) = (\xi_0, \xi'') = (\xi_0, \xi''', \xi_3)$ in $\mathbf{R}^4_\xi$. We use the differential symbol $D_j = i^{-1}\partial/\partial x_j$ $(j = 0, 1, 2, 3)$, where $i = \sqrt{-1}$. We shall denote by $\mathbf{R}^n_-$ the half-space $\{x = (x_1, \cdots, x_n) \in \mathbf{R}^n |\ x_n < 0\}$ and by $\mathbf{R}^n_+$ the half-space $\{x = (x_1, \cdots, x_n) \in \mathbf{R}^n |\ x_n > 0\}$, and also use the notation $|x| = \sqrt{x_1^2 + \cdots x_n^2}$.

Let $u(x) = {}^t(u_1(x), u_2(x), u_3(x)) \in \mathbf{R}^3$ be the displacement vector at time x_0 and position x''. The propagation problems of elastic waves in the stratified media is formulated as mixed (initial-interface value) problem:

$$\begin{cases} P^I(D)u(x) = f(x), \quad x_0 > 0,\ x'' = (x_1, x_2, x_3) \in \mathbf{R}^3_-, \\ P^{II}(D)u(x) = f(x), \quad x_0 > 0,\ x'' = (x_1, x_2, x_3) \in \mathbf{R}^3_+, \\ u(x)|_{x_3=-0} = u(x)|_{x_3=+0}, \quad x_0 > 0,\ x''' \in \mathbf{R}^2, \\ B^I(D)u(x)|_{x_3=-0} = B^{II}(D)u(x)|_{x_3=+0}, \quad x_0 > 0,\ x''' \in \mathbf{R}^2, \\ D_0^k u(x)|_{x_0=0} = g_k(x''), \quad k = 0, 1, \quad x'' \in \mathbf{R}^3. \end{cases}$$

Here

$$P^I(D)u = -D_0^2 E u + \frac{\lambda_1 + \mu_1}{\rho_1} \nabla_{x''}(\nabla_{x''} \cdot u) + \frac{\mu_1}{\rho_1} \Delta_{x''} u,$$

is a 3×3 matrix valued second order hyperbolic differential operator with constant coefficients where E is a 3×3 identity matrix,

$$(B^I(D)u)_k = i\lambda_1(\nabla_{x''} \cdot u)\delta_{k3} + 2\mu_1 \varepsilon_{k3}(u), \quad k = 1, 2, 3,$$

are the k-th component of symmetric stress tensors of $B^I(D)u$ where

$$\varepsilon_{k3}(u) = i/2\,(D_3 u_k + D_k u_3), \quad k = 1, 2, 3,$$

are strain tensors. The $P^{II}(D)u$ and $B^{II}(D)u$ are defined by replacing λ_1, μ_1, ρ_1 by λ_2, μ_2, ρ_2, respectively.

If we put unit impulse Dirac's delta $\delta(x - y)$ with $x_0 \geq y_0$ and $y_3 < 0$, that is, put it in Medium I, then the Riemann function of this elastic mixed problem is given by the following:

$$G(x, y) = \begin{cases} E^I(x - y) - F^I(x, y) & \text{for} \quad x_3 < 0, \\ F^{II}(x, y) & \text{for} \quad x_3 > 0. \end{cases}$$

We call $E^I(x-y)$, $F^I(x,y)$, and $F^{II}(x,y)$ the incident, reflected, and refracted Riemann functions, respectively, because these are corresponding to incident, reflected, and refracted waves, respectively. $E^I(x)$ is the fundamental solution in Medium I describing an incident wave defined by

$$E^I(x) = (2\pi)^{-4} \int_{\mathbf{R}^4_\xi} e^{ix\cdot(\xi + i\eta)} P^I(\xi + i\eta)^{-1} d\xi, \qquad \eta \in -\gamma_0\vartheta - \Gamma(\det P^I, \vartheta),$$

where γ_0 is a positive real number, ϑ and $\Gamma(\det P^I, \vartheta)$ are defined Definition 1.4 below, and $P^I(\xi + i\eta)^{-1}$ is a 3×3 inverse matrix. Taking partial Fourier-Laplace transform with respect to x' for the mixed problem, we obtain a interface value problem for ordinary differential equation with parameters. Then taking partial inverse Fourier-Laplace transform for the solution, we obtain explicit expressions of reflected and refracted Riemann functions $F^I(x, y)$ and $F^{II}(x, y)$.

We define a wave front set $WF(u)$ and a analytic wave front set $WF_A(u)$ (cf. [3], [4], [11]).

Definition 1.1. Let $u(x) \in \mathcal{D}'(X)$. Then the wave front set $WF(u)$ is defined as the complement in $X \times (\mathbf{R}^n \setminus \{0\})$ of the collection of the points (x^0, ξ^0) such that there exist a conic neighborhood Δ of ξ^0 in $\mathbf{R}^n \setminus \{0\}$ and $\phi \in C_0^\infty(X)$ such that $\phi(x^0) \neq 0$ and

$$|\mathcal{F}[\phi u](\xi)| \leq C_N(1+|\xi|)^{-N} \quad \text{when} \quad \xi \in \Delta \text{and} \quad N = 0, 1, 2, \cdots .$$

Here $\mathcal{F}$ denotes the Fourier transform.

For the definition of a analytic wave front set $WF_A(u)$, we prepare that there exist a bounded sequence $\{\phi_N\}$ in C_0^∞ such that $\phi_N = 1$ on a fixed neighborhood of $x^{0'}$ in X, independent of N, and

$$|D^\alpha \phi_N| \leq C(CN)^{|\alpha|} \quad \text{for} \quad |\alpha| \leq N.$$

Definition 1.2. Let $u(x) \in \mathcal{D}'(X)$. Then the analytic wave front set $WF_A(u)$ is defined as the complement in $X \times (\mathbf{R}^n \setminus \{0\})$ of the collction of the points (x^0, ξ^0) such that for some sequence $\{\phi_N\}$ of the above type there exists a conic neighborhood Δ of ξ^0 in $\mathbf{R}^n \setminus \{0\}$ with

$$|\mathcal{F}[\phi_N u](\xi)| \leq C(CN)^N(1+|\xi|)^{-N} \quad \text{when} \quad \xi \in \Delta, \quad N = 0, 1, 2, \cdots .$$

By Definition 1.1 and Definition 1.2, we obtain

$$WF(u) \subset WF_A(u),$$

and $WF(u)$ and $WF_A(u)$ are closed subsets of $X \times (\mathbf{R}^n \setminus \{0\})$.

We define a localization of polynomials according to Atiyah-Bott-Gårding (cf. [1]):

Definition 1.3. Let $P(\xi)$ be a polynomial of degree $m \geq 0$ and develop $\nu^m P(\nu^{-1}\xi + \eta)$ in ascending power of ν:

$$\nu^m P\left(\nu^{-1}\xi + \eta\right) = \nu^p P_\xi(\eta) + O(\nu^{p+1}) \qquad \text{as} \quad \nu \to 0,$$

where $P_\xi(\eta)$ is the first coefficient that does not vanish identically in η. The polynomial $P_\xi(\eta)$ is the localization of P at ξ, the number p is the multiplicity of ξ relative to P.

Moreover we introduce the following:

Definition 1.4. $\Gamma = \Gamma(P, \vartheta)$ is the component of $\mathbf{R}^n_\eta \setminus \{\eta \in \mathbf{R}^n_\eta,\ P(\eta) = 0\}$ which contains $\vartheta = (1, 0, \cdots, 0) \in \mathbf{R}^n$. Moreover $\Gamma' = \Gamma'(P, \vartheta) = \{x \in \mathbf{R}^n |\ x \cdot \eta \geq 0,\ \eta \in \Gamma\}$ is the dual cone of Γ and is called the propagation cone.

3. Results

We obtain the exact wave front sets of the elastic mixed problem in stratified media by combining an inner estimate and an outer estimate. First we mention about the results of the incident Riemann function, namely fundamental solution of Medium I $E^I(x)$. This proposition is a version of the theorem proved by Atiyah-Bott-Gårding [1, Theorem 4.10] adopted the present context.

Proposition. *For $\xi^0 \in \mathbf{R}^4_\xi \setminus \{0\}$ satisfying* $(\det P^I_j)(\xi^0) = 0$ *($j \in \{p_1, s_1\}$), that is,*

$$(\det P^I_{p_1})(\xi^0) = \xi_0^{0^2} - c^2_{p_1} |\xi^{0''}|^2 = 0,$$

or

$$(\det P^I_{s_1})(\xi^0) = \xi_0^{0^2} - c^2_{s_1} |\xi^{0''}|^2 = 0,$$

then we have

$$\lim_{\nu \to \infty} \nu e^{-i\nu x \cdot \xi^0} E^I(x) = E^I_{j\xi^0}(x), \quad j \in \{p_1, s_1\},$$

in the distribution sense with respect to $x \in \mathbf{R}^4$, where

$$E^I_{j\xi^0}(x) = (2\pi)^{-4} \int_{\mathbf{R}^4_\xi} e^{ix\cdot(\xi+i\eta)} \frac{(\mathrm{cof} P^I)_{\xi^0}(\xi + i\eta)}{(\det P^I_j)_{\xi^0}(\xi + i\eta)} d\xi$$
$$\text{for } \eta \in -\vartheta - \Gamma(\det P^I_j, \vartheta) \text{ and } j \in \{p_1, s_1\}$$

with a positive real s large enough. Moreover we have

$$WF(E^I(x)) = \bigcup_{\xi^0 \neq 0} \left(\mathrm{supp}\, E^I_{p_1\xi^0}(x) \cup \mathrm{supp}\, E^I_{s_1\xi^0}(x)\right) \times \{\xi^0\},$$

and

$$\mathrm{supp}\, E^I_{j\xi^0}(x) = \left(\Gamma_{j\xi^0}\right)' = \Big\{ x \in \mathbf{R}^4 : x \cdot \eta \geq 0 \text{ for any } \eta \in \Gamma_{j\xi^0}$$
$$u = \Gamma\big((\det P^I_j)_{\xi^0}(\eta), \vartheta\big) \Big\}, \quad \vartheta = (1,0,0,0), \quad j \in \{p_1, s_1\}.$$

In general, $\mathrm{supp} E^I_{j\xi^0}(x) \subset (\Gamma_{j\xi^0})'(j = \{p_1, s_1\})$, more precisely $ch[\mathrm{supp}\, E^I_{j\xi^0}(x - y)] = \left(\Gamma_{j\xi^0}\right)'$, where ch denotes convex hull. However in our problem we obtain $\mathrm{supp} E^I_{j\xi^0}(x) = (\Gamma_{j\xi^0})'$.

Secondly we mention about main result. Since we take a partial Fourier-Laplace transform with respect to x' of $\delta(x - y)$ regarding y' as a parameter, y' appears only in the form $x' - y'$. So we put

$$\tilde{F}^\iota(x, y_3) = F^\iota(x, 0, y_3), \qquad \iota = \{I, II\}.$$

The following Main Theorem shows the exact wave front sets of $\tilde{F}^I(x, y_3)$ and $\tilde{F}^{II}(x, y_3)$.

Main Theorem. *For $\xi^0 \in \mathbf{R}^4_\xi \setminus \{0\}$ satisfying $(\det P^I_j)(\xi^0) = 0$ ($j \in \{p_1, s_1\}$) we have the following:*

(1) For the reflected Riemann function $\tilde{F}^I(x, y_3)$, we have

(3.1)
$$\lim_{\nu \to \infty} \nu e^{-i\nu\{x'\cdot\xi^{0'} + x_3\tau^-_k(\xi^{0'}) - y_3\xi^0_3\}} \tilde{F}^I(x, y_3) = \tilde{F}^I_{j\xi^0 k}(x, y_3),$$
$$(j, k) = \{(p_1, p_1), (p_1, s_1), (s_1, p_1), (s_1, s_1)\}$$

and if $\xi^{0'}$ *are zeros of* $\tau_m^+(\zeta')$*, that is,* $\xi^{0'}$ *satisfy* $|\xi^{0'''}| = \xi_0^0/c_m$ *(*$m \in \{p_1, p_2, s_2\}$*), then we have*

(3.2)

$$\lim_{\nu\to\infty}\left\{\nu^{\frac{3}{2}}e^{-i\nu\{x'\cdot\xi^{0'}+x_3\tau_k^-(\xi^{0'})-y_3\xi_3^0\}}\tilde F^I(x,y_3) - \nu^{\frac{1}{2}}\tilde F^I_{j\xi^0k}(x,y_3)\right\} = \tilde F^I_{j\xi^0km}(x,y_3),$$
$$(j,k,m) = \{(p_1,p_1,p_2),(p_1,p_1,s_2),(p_1,s_1,p_2),(p_1,s_1,s_2),(s_1,p_1,p_2),$$
$$(s_1,p_1,s_2),(s_1,s_1,p_2),(s_1,s_1,s_2),(s_1,s_1,p_1)\}$$

in the distribution sense with respect to $(x,y_3) \in \mathbf{R}_-^4 \times \mathbf{R}_-$.

Moreover we have

$$WF(\tilde F^I(x',x_3,y_3)) = WF_A(\tilde F^I(x',x_3,y_3)) = \bigcup_{\xi^0\neq 0}$$
$$\Bigg[\bigcup_{\substack{(j,k)=\{(p_1,p_1),(p_1,s_1),\\(s_1,p_1),(s_1,s_1)\}}}\left(\operatorname{supp}\tilde F^I_{j\xi^0k}(x',x_3,y_3)\times\{(\xi^{0'},\tau_k^-(\xi^{0'}),-\xi_3^0)\}\right)$$
$$\bigcup\bigcup_{\substack{(j,k,m)=\\\{(p_1,p_1,p_2),(p_1,p_1,s_2),(p_1,s_1,p_2),\\(p_1,s_1,s_2),(s_1,p_1,p_2),(s_1,p_1,s_2),\\(s_1,s_1,p_2),(s_1,s_1,s_2),(s_1,s_1,p_1)\}}}\left(\operatorname{supp}\tilde F^I_{j\xi^0km}(x',x_3,y_3)\times\{(\xi^{0'},\tau_k^-(\xi^{0'}),-\xi_3^0)\}\right)\Bigg]$$

and

$$\operatorname{supp}\tilde F^I_{j\xi^0k}(x,y) = \left(\Gamma_{j\xi^0}\right)_k^I \equiv \Bigg\{(x,y_3)\in\mathbf{R}_-^4\times\mathbf{R}_- :$$
$$\left(x'+x_3\mathrm{grad}_\xi\tau_k^-(\xi^{0'})\right)\cdot\eta' - y_3\eta_3 \geq 0 \text{ for any } \eta\in\Gamma_{j\xi^0}\Bigg\},$$
$$(j,k) = \{(p_1,p_1),(p_1,s_1),(s_1,p_1),(s_1,s_1)\}$$

for ξ^0 *satisfying* $\tilde F^I_{j\xi^0k}(x,y_3)\neq 0$,

$$\operatorname{supp}\tilde F^I_{j\xi^0km}(x,y) = \left(\Gamma_{j\xi^0m}\right)_k^I \equiv \Bigg\{(x,y_3)\in\mathbf{R}_-^4\times\mathbf{R}_- :$$
$$\left(x'+x_3\mathrm{grad}_\xi\tau_k^-(\xi^{0'})\right)\cdot\eta' - y_3\eta_3 \geq 0 \text{ for any } \eta\in\Gamma_{j\xi^0m}\Bigg\},$$
$$(j,k,m) = \{(p_1,p_1,p_2),(p_1,p_1,s_2),(p_1,s_1,p_2),(p_1,s_1,s_2),(s_1,p_1,p_2),$$
$$(s_1,p_1,s_2),(s_1,s_1,p_2),(s_1,s_1,s_2),(s_1,s_1,p_1)\}$$

for ξ^0 *satisfying* $\tilde F^I_{j\xi^0km}(x,y_3)\neq 0$.

(2) For the refracted Riemann function $\tilde F^{II}(x,y_3)$*, we have*

(3.3)
$$\lim_{\nu\to\infty}\nu e^{-i\nu\{x'\cdot\xi^{0'}+x_3\tau_k^+(\xi^{0'})-y_3\xi_3^0\}}\tilde F^{II}(x,y_3) = \tilde F^{II}_{j\xi^0k}(x,y_3),$$

$$(j,k)=\{(p_1,p_2),(p_1,s_2),(s_1,p_2),(s_1,s_2)\}$$

and if $\xi^{0'}$ *are zeros of* $\tau_m^+(\zeta')$ $(m \in \{p_2\})$*, then we have*

(3.4)
$$\lim_{\nu\to\infty}\left\{\nu^{\frac{3}{2}}e^{-i\nu\{x'\cdot\xi^{0'}+x_3\tau_k^+(\xi^{0'})-y_3\xi_3^0\}}\tilde{F}^{II}(x,y_3)-\nu^{\frac{1}{2}}\tilde{F}^{II}_{j\xi^0k}(x,y_3)\right\}=\tilde{F}^{II}_{j\xi^0km}(x,y_3),$$
$$(j,k,m)=\{(p_1,s_2,p_2),(s_1,s_2,p_2)\}$$

in the distribution sense with respect to $(x,y_3)\in\mathbf{R}^4_+\times\mathbf{R}_-$.

Moreover we have

$$WF(\tilde{F}^{II}(x',x_3,y_3))=WF_A(\tilde{F}^{II}(x',x_3,y_3))=\bigcup_{\xi^0\neq 0}$$
$$\left[\bigcup_{\substack{(j,k)=\{(p_1,p_2),(p_1,s_2),\\(s_1,p_2),(s_1,s_2)\}}}\left(\operatorname{supp}\tilde{F}^{II}_{j\xi^0k}(x',x_3,y_3)\times\{(\xi^{0'},\tau_k^+(\xi^{0'}),-\xi_3^0)\}\right)\right.$$
$$\left.\bigcup\bigcup_{(j,k,m)=\{(p_1,s_2,p_2),(s_1,s_2,p_2)\}}\left(\operatorname{supp}\tilde{F}^{II}_{j\xi^0km}(x',x_3,y_3)\times\{(\xi^{0'},\tau_k^+(\xi^{0'}),-\xi_3^0)\}\right)\right]$$

and

$$\operatorname{supp}\tilde{F}^{II}_{j\xi^0k}(x,y_3)=\left(\Gamma_{j\xi^0}\right)_k^{II}\equiv\Big\{(x,y_3)\in\mathbf{R}^4_+\times\mathbf{R}_-:$$
$$\left(x'+x_3\mathrm{grad}_\xi\tau_k^+(\xi^{0'})\right)\cdot\eta'-y_3\eta_3\geq 0 \text{ for any } \quad \eta\in\Gamma_{j\xi^0}\Big\},$$
$$(j,k)=\{(p_1,p_2),(p_1,s_2),(s_1,p_2),(s_1,s_2)\}$$

for ξ^0 *satisfying* $\tilde{F}^{II}_{j\xi^0k}(x,y_3)\neq 0$,

$$\operatorname{supp}\tilde{F}^{II}_{j\xi^0km}(x,y_3)=\left(\Gamma_{j\xi^0m}\right)_k^{II}\equiv\Big\{(x,y_3)\in\mathbf{R}^4_-\times\mathbf{R}_-:$$
$$\left(x'+x_3\mathrm{grad}_\xi\tau_k^+(\xi^{0'})\right)\cdot\eta'-y_3\,\eta_3\geq 0 \text{ for any } \quad \eta\in\Gamma_{j\xi^0m}\Big\},$$
$$(j,k,m)=\{(p_1,s_2,p_2),(s_1,s_2,p_2)\}$$

for ξ^0 *satisfying* $\tilde{F}^{II}_{j\xi^0km}(x,y_3)\neq 0$.

Here $F^I_{j\xi^0k}(x,y_3)$, $F^I_{j\xi^0km}(x,y_3)$, $F^{II}_{j\xi^0k}(x,y_3)$, *and* $F^{II}_{j\xi^0km}(x,y_3)$ *are localizations in the sense of (3.1), (3.2), (3.3), and (3,4), respectively, and more precise expressions are given in (*[7],[8]*). Moreover*

$$\Gamma_{j\xi^0}=\Gamma\left((\det P_j^I)_{\xi^0}(\eta),\vartheta\right),\qquad \vartheta=(1,0,0,0),\quad j\in\{p_1,s_1\},$$
$$(\det P_j^I)_{\xi^0}(\eta),\vartheta)\cap\left\{\Gamma\left(\frac{\xi_0^0}{c_m^2}\eta_0-\xi_1^0\eta_1-\xi_2^0\eta_2,\vartheta'\right)\times\mathbf{R}_\eta\right\},$$

$$\vartheta' = (1,0,0), \quad j \in \{p_1, s_1\}, \quad m \in \{p_2\},$$

$$\tau_{p_1}^{\pm}(\xi') = \mathrm{sgn}(\mp\xi_0)\sqrt{\frac{\xi_0^2}{c_{p_1}^2} - |\xi^{0'''}|^2}, \quad \text{if} \quad \frac{\xi_0^2}{c_{p_1}^2} - |\xi^{0'''}|^2 \geq 0,$$

and $\tau_{p_1}^{\pm}(\xi')$ *is taken a branch of* $\sqrt{\frac{\xi_0^2}{c_{p_1}^2} - |\xi^{0'''}|^2}$ *such that* $\pm\mathrm{Im}\tau_{p_1}^{\pm}(\xi') > 0$ *if* $\frac{\xi_0^2}{c_{p_1}^2} - |\xi^{0'''}|^2 < 0$. $\tau_{s_1}^{\pm}(\xi')$, $\tau_{p_2}^{\pm}(\xi')$, *and* $\tau_{s_2}^{\pm}(\xi')$ *are defined as the same as* $\tau_{p_1}^{\pm}(\xi')$ *substituting* c_{p_1} *for* c_{s_1}, c_{p_2}, *and* c_{s_2}, *respectively.*

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