

## $P$ (locally-finite)-embedding and related topics

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All spaces are assumed to be  $T_1$ -spaces. Let  $X$  be a space and  $A$  a subspace. Let  $\gamma$  and  $\kappa$  be infinite cardinal numbers. Recently, in Dydak's paper [3],  $A$  is said to be  $P^\gamma$ (locally-finite)-embedded in  $X$  if for every locally finite partition  $\{p_\alpha : \alpha < \gamma\}$  of unity on  $A$ , there exists a locally finite partition  $\{q_\alpha : \alpha < \gamma\}$  of unity on  $X$  such that  $q_\alpha|_A = p_\alpha$  for every  $\alpha < \gamma$ .  $A$  is said to be  $P$ (locally-finite)-embedded in  $X$  if  $A$  is  $P^\gamma$ (locally-finite)-embedded in  $X$  for every  $\gamma$ . In our previous paper [19, Theorem 3.1], the following result was shown:

**Theorem 0** [19]. *Let  $X$  be a space and  $A$  a subspace. Then,  $A$  is  $P^\gamma$ (locally-finite)-embedded in  $X$  if and only if for every locally finite cover  $\{U_\alpha : \alpha < \gamma\}$  of cozero-sets of  $A$ , there exists a locally finite cover  $\{V_\alpha : \alpha < \gamma\}$  of cozero-sets of  $X$  such that  $V_\alpha \cap A = U_\alpha$  for every  $\alpha < \gamma$ .*

Przymusiński-Wage proved Theorem 0 in [17, Theorem 2] assuming that  $X$  is normal and  $A$  is closed in  $X$ .

In this report, related to Theorem 0, we denote two topics. One is related to Katětov spaces or functional Katětov spaces (these notions were studied by Katětov in [10] and defined by Przymusiński-Wage in [17]); the condition of the "if" part in Theorem 0 is closely related to functionally Katětov spaces. Another is related to "controlling extension" which was studied by Frantz in [5]; our key lemma [19, Lemma 3.2] to prove Theorem 0 is closely related to this notion.

$A$  is said to be  $C^*$  (respectively,  $C$ )-embedded in  $X$  if every continuous real-valued bounded (respectively, real-valued) function on  $A$  can be continuously extended over  $X$ .  $A$  is said to be well-embedded in  $X$  if every zero-set disjoint from  $A$  is completely separated from  $A$ . It is well-known that  $A$  is  $C$ -embedded in  $X$  if and only if  $A$  is  $C^*$ - and well-embedded in  $X$  (see [1] or [6]).

### 1. Characterizations of $P$ (locally-finite)-embedding and (functionally) Katětov spaces by products

In [10] or [17], the following four properties are studied:

(1)  $X$  is collectionwise normal and countably paracompact (respectively, normal and countably paracompact);

(2)  $X$  is normal and every locally finite open (respectively, countable locally finite open) cover of any closed subspace  $A$  of  $X$  can be extended to be a locally finite open cover of  $X$ ;

(3)  $X$  is normal and every locally finite cozero-sets (respectively, countable locally finite cozero-sets) cover of any closed subspace  $A$  of  $X$  can be extended to be a locally finite open - or equivalently, cozero-sets - cover of  $X$ ;

(4)  $X$  is collectionwise normal (respectively, normal).

In [17], a space  $X$  with the property (2) is said to be *Katětov* (respectively, *countably Katětov*) and a space  $X$  with the property (3) is said to be *functionally Katětov* (respectively, *countably functionally Katětov*). Katětov [10] proved that (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4), and Przymusiński-Wage showed in [17] any of these implications above need not be reversed.

Every  $P^\gamma$ (locally-finite)-embedded subset is  $P^\gamma$ -embedded (see below for the definition) [3],  $P^\omega$ -embedding equals to  $C$ -embedding [1]. Hence, we can say that  $X$  is functionally Katětov (respectively, countably functionally Katětov) if and only if for every closed subset  $A$  of  $X$  is  $P$ (locally-finite)-embedded (respectively,  $P^\omega$ (locally-finite)-embedded) in  $X$ .

Let  $X$  be a space and  $A$  a subspace.  $A$  is said to be  $P^\gamma$ -embedded in  $X$  if every normal open cover  $\mathcal{U}$  of  $A$  with  $|\mathcal{U}| \leq \gamma$  can be extended to a normal open cover of  $X$ .

First we give some remarks about the difference between  $P$ (locally-finite)-embedding and  $P$ -embedding. It is well-known that the following conditions are equivalent:

(1)  $A$  is  $P^\gamma$ -embedded in  $X$ ;

(2) For every locally finite cover  $\{U_\alpha : \alpha < \gamma\}$  of cozero-sets of  $A$ , there exists a locally finite cover  $\{V_\alpha : \alpha < \gamma\}$  of cozero-sets of  $X$  such that  $V_\alpha \cap A \subset U_\alpha$  for every  $\alpha < \gamma$ ;

(3) For every locally finite cover  $\{U_\alpha : \alpha < \gamma\}$  of cozero-sets of  $A$ , there exists a  $\sigma$ -locally finite cover  $\{V_\alpha^n : \alpha < \gamma, n \in \mathbb{N}\}$  of cozero-sets of  $X$  such that  $\{V_\alpha^n : \alpha < \gamma\}$  is locally finite for each  $n \in \mathbb{N}$  and  $(\bigcup_{n \in \mathbb{N}} V_\alpha^n) \cap A \subset U_\alpha$  for every  $\alpha < \gamma$ .

Theorem 0 shows  $P^\gamma$ (locally-finite)-embedding is characterized as the condition replaced " $V_\alpha \cap A \subset U_\alpha$ " by " $V_\alpha \cap A = U_\alpha$ " on the above (2). Related to this, even if we replace " $(\bigcup_{n \in \mathbb{N}} V_\alpha^n) \cap A \subset U_\alpha$ " by " $(\bigcup_{n \in \mathbb{N}} V_\alpha^n) \cap A = U_\alpha$ " on the above (3), it is not equal to  $P^\gamma$ (locally finite)-embedding. In fact the statement replaced so equals  $P^\gamma$ -embedding. We may say that  $P^\gamma$ (locally

finite)-embedding is not the property concerning to extensions of normal open covers.

A product space  $X \times Y$  is said to be *rectangularly normal* if for every closed subspace  $A$  of  $X$  and closed subspace  $B$  of  $Y$ ,  $A \times B$  is  $C$ -embedded in  $X \times Y$  ([16]). Let  $\mathcal{C}$  be a class of spaces.  $A$  is said to be  $\pi_{\mathcal{C}}$ -embedded in  $X$  if  $A \times Y$  is  $C^*$ -embedded in  $X \times Y$  for every  $Y \in \mathcal{C}$  ([14]).

On (2) in the following proposition, the case  $Y$  is compact Hausdorff was shown in [19, Theorem 3.4]. When  $Y = I$ , it is an affirmative answer to a problem posed by Dydak in [3, Problem 13.16] (see [19]).

**Proposition 1.1.** *Let  $X$  be a space and  $A$  a subspace. Then, the following statements hold.*

(1) *Let  $A$  be a compact Hausdorff subspace of a Tychonoff space  $X$ . Then for any space  $Y$ ,  $A \times Y$  is  $P$ (locally-finite)-embedded in  $X \times Y$ .*

(2) *Let  $A$  be a  $P^\gamma$ (locally-finite)-embedded in  $X$  and  $Y$  be a locally compact paracompact Hausdorff space  $Y$  with weight  $w(Y) \leq \gamma$ . Then  $A \times Y$  is  $P^\gamma$ (locally-finite)-embedded in  $X \times Y$ .*

As an application of Proposition 1.1, we give a homotopy-type extension theorem. The case of  $P^\gamma$ -embedding was proved in [11, Theorem 3.4]. The “(1)  $\Rightarrow$  (3)” was already shown by using [3, Lemma 13.2] and [19, Theorem 3.4].

**Corollary 1.2.** *Let  $X$  be a space and  $A$  its subspace. Then, the following statements are equivalent :*

- (1)  *$A$  is  $P^\gamma$ (locally-finite)-embedded in  $X$ ;*
- (2)  *$(X \times B) \cup (A \times Y)$  is  $P^\gamma$ (locally-finite)-embedded in  $X \times Y$  for every compact Hausdorff space  $Y$  with weight  $w(Y) \leq \gamma$  and every closed subspace  $B$  of  $Y$ ;*
- (3)  *$(X \times \{0\}) \cup (A \times I)$  is  $P^\gamma$ (locally-finite)-embedded in  $X \times I$ .*

A space  $X$  is said to be a  $P$ -space if every  $G_\delta$ -set of  $X$  is open.

**Proposition 1.3.** *Let  $X$  be a space and  $A$  a subspace. Assume  $A$  be a  $P$ -space. Then,  $A$  is  $P^\gamma$ -embedded in  $X$  if and only if  $A$  is  $P^\gamma$ (locally-finite)-embedded in  $X$ .*

**Corollary 1.4.** *Let  $X$  be a collectionwise normal  $P$ -space. Then,  $X$  is functionally Katětov.*

Related to the Corollary 1.4, Rudin’s Dowker space is collectionwise normal  $P$ -space but not (countably) Katětov ([17, Example 2]).

From another point of view, we have the following result:

**Theorem 1.5.** *Let  $X$  be a space and  $A$  a subspace. Then,  $A$  is  $P^\gamma$ (locally-finite)-embedded in  $X$  if and only if  $A$  is  $P^\omega$ (locally-finite)-embedded in  $X$  and for every locally finite collection  $\{U_\alpha : \alpha < \gamma\}$  of cozero-sets of  $A$  with finite order, there exists a locally finite collection  $\{V_\alpha : \alpha < \gamma\}$  of cozero-sets of  $X$  such that  $U_\alpha \subset V_\alpha$  for every  $\alpha < \gamma$ .*

**Corollary 1.6.** *A space  $X$  is functionally Katětov if and only if  $X$  is countably functionally Katětov and for every closed subspace  $A$  of  $X$  and every locally finite collection  $\{U_\alpha : \alpha < \gamma\}$  of cozero-sets of  $A$  with finite order, there exists a locally finite collection  $\{V_\alpha : \alpha < \gamma\}$  of cozero-sets of  $X$  such that  $U_\alpha \subset V_\alpha$  for every  $\alpha < \gamma$ .*

Here we pose two fundamental problems as follows:

**Problem 1.7.** *Let  $A$  be a  $P^\omega$ (locally-finite)- and  $P^\gamma$ -embedded subspace of  $X$ . Then, is  $A$   $P^\gamma$ (locally-finite)-embedded in  $X$  ?*

**Problem 1.8.** *Let  $X$  be a countably functionally Katětov and collectionwise normal. Then, is  $X$  functionally Katětov ?*

Theorem 1.5 or Corollary 1.6 may be regarded as a partial answer to these problems. If Problem 1.7 is affirmative, then Problem 1.8 is also affirmative. Problem 1.8 is motivated by a Przymusiński-Wage's question [17, Question 3], "Let  $X$  be countably Katětov and collectionwise normal. Then, is  $X$  Katětov ?"

Let  $J(\gamma)$  be the hedgehog with  $\gamma$  spines (e.g.[4]). Let  $J_0(\kappa) = \{\theta\} \cup \{(\lambda, 1/n) : n \in \mathbb{N}, \lambda < \kappa\}$  be a closed subspace of the hedgehog with  $\gamma$  spines  $J(\gamma)$  (see [16]). A subspace  $A$  of  $X$  is called  $F_\kappa$ -set if it is the union of  $\kappa$  many closed sets in  $X$ .

**Theorem 1.9** (Przymusiński [16, Proposition 2.2]). *Let  $X$  be a normal space and  $A$  a closed subspace. Then the following statements are equivalent :*

- (1)  $A \times J(\kappa)$  is  $C^*$ -embedded in  $X \times J(\kappa)$ ;
- (2)  $A \times J_0(\kappa)$  is  $C^*$ -embedded in  $X \times J_0(\kappa)$ ;
- (3) every countable locally finite cover of open  $F_\kappa$ -sets of  $A$  can be extended to a locally finite open cover of  $X$ .

In [16, Proposition 2.2], " $C^*$ -embedding" in (1) and (2) of the above theorem is written as " $C$ -embedding". However he actually proved  $C^*$ -embedding of them. If we use [7, Theorem 1.1] or [18, Theorem 1.1],  $C$ -embedding of (1) or (2) is implied by  $C^*$ -embedding.

Theorem 1.9 suggests us that the difference of  $P^\gamma$ (locally-finite)-embedding and  $P^\omega$ (locally-finite)-embedding doesn't appear the numbers of spines of the hedgehog. Extending Theorem 1.9, we give a characterization of  $P$ (locally-finite)-embedding.

Let  $\gamma$  be an infinite cardinal number and  $\kappa$  a cardinal number. Let  $J_\gamma(\kappa) = \{p\} \cup \{\langle \alpha, \beta \rangle : \alpha < \gamma, \beta < \kappa\}$  be a space satisfying that  $p$  has basic neighborhoods of the form

$$\{p\} \cup \{\langle \alpha, \beta \rangle : \alpha \in \gamma - \delta, \beta < \kappa\}; \quad \delta \in \gamma^{<\omega}$$

and other points are isolated. Notice that, for each  $\beta < \kappa$ ,  $\{p\} \cup \{\langle \alpha, \beta \rangle : \alpha < \gamma\}$  can be seen as the one point compactification of the discrete space with cardinality  $\gamma$ . Note that  $J_\omega(\kappa)$  can be regarded as the space  $J_0(\kappa)$ . (For the space  $J_\gamma(\kappa)$ , see also Remark 1.13.)

$P^\gamma$ (locally-finite)-embedding is characterized as follows:

**Theorem 1.10.** *Let  $X$  be a space and  $A$  a subspace. Then, the following statements are equivalent :*

- (1)  $A$  is  $P^\gamma$ (locally-finite)-embedded in  $X$  ;
- (2)  $A \times J_\gamma(\omega)$  is  $C^*$ -embedded in  $X \times J_\gamma(\omega)$ ;
- (3)  $A \times J_\gamma(\omega)$  is  $P^\gamma$ -embedded in  $X \times J_\gamma(\omega)$ .

The case of  $\gamma = \omega$ , we have more general observation as follows:

**Theorem 1.11.** *Let  $X$  be a space and  $A$  a subspace. Then the following statements are equivalent :*

- (1)  $A$  is  $P^\omega$ (locally-finite)-embedded in  $X$ ;
- (2)  $A \times J_0(\omega)$  is  $C^*$  (or equivalently  $C$ )-embedded in  $X \times J_0(\omega)$ ;
- (3)  $A \times J(\omega)$  is  $C^*$  (or equivalently  $C$ )-embedded in  $X \times J(\omega)$ ;
- (4) for some non-locally compact metric space  $Y$ ,  $A \times Y$  is  $C^*$  (or equivalently  $C$ )-embedded in  $X \times Y$ ;
- (5)  $A \times Y$  is  $C^*$  (or equivalently,  $C$ )-embedded in  $X \times Y$  for every separable metric space  $Y$  satisfying that  $Y - Y_1$  is locally compact for some closed discrete subspace  $Y_1$ .

By Theorem 1.11, we have the following result:

**Corollary 1.12.** *If  $A$  is  $\pi_{\mathcal{M}_\omega}$ -embedded in  $X$ , then  $A$  is  $P^\omega$ (locally-finite)-embedded in  $X$ .*

Machael's Example (see [4, 5.1.32]) shows that Corollary 1.12 can not be reversed.

**Remark 1.13.**  $\pi_{\mathcal{M}}$ -embedding need not imply  $P^\gamma$ -embedding in the case  $\gamma > \omega$  (for example, consider Bing's H; see [4, 5.5.3]). Namely, Corollary 1.12 does not hold in the case of the general cardinality. As an explanation of this, let us comment the test space  $J_\gamma(\omega)$  for  $P^\gamma$ (locally-finite)-embedding.  $J_\gamma(\kappa)$  can be regarded as a special subspace of  $\gamma$ -many of discrete spaces with cardinality  $\kappa$ . Let

$$\sigma_1(D(\kappa)^\gamma) = \{(x_\alpha) \in D(\kappa)^\gamma : |\{\alpha < \gamma : x_\alpha \neq 0\}| \leq 1\},$$

where  $D(\kappa)$  is the set  $\kappa$  with discrete topology. Namely,  $\sigma_1(D(\kappa)^\gamma)$  is the  $\sigma_1$ -product of  $\gamma$ -many of discrete spaces with cardinality  $\kappa$  with the base point  $\theta = (0, 0, \dots)$ . Note that the space  $J_\gamma(\kappa)$  is homeomorphic to  $\sigma_1(D(\kappa)^\gamma)$ .

Next we give some conclusion by rectangular normality with  $J_\gamma(\kappa)$  as the following; (2) is in [16, Theorem 2.3], (4) is in [16, Theorem 2.4], and (5) and (6) can be easily shown by using the well-known fact (see [8, Lemma 4.4]) and [12, Theorem 1.5] or [13, Theorem 3].

**Theorem 1.14.** *Let  $X$  be a space. Then, the following statements hold.*

(1)  $X \times J_\gamma(\kappa)$  is rectangularly normal for every  $\kappa$  and every  $\gamma$  if and only if  $X$  is Katětov.

(2)  $X \times J_\omega(\kappa)$  is rectangularly normal for every  $\kappa$  if and only if  $X$  is countably Katětov.

(3)  $X \times J_\gamma(\omega)$  is rectangularly normal for every  $\gamma$  if and only if  $X$  is functionally Katětov.

(4)  $X \times J_\omega(\omega)$  is rectangularly normal if and only if  $X$  is countably functionally Katětov.

(5)  $X \times J_\gamma(1)$  is rectangularly normal for every  $\gamma$  if and only if  $X$  is collectionwise normal.

(6)  $X \times J_\omega(1)$  is rectangularly normal if and only if  $X$  is normal.

On the other hand, it is known that  $A$  is  $C$ -embedded in  $X$  if and only if  $A \times Y$  is  $C^*$ (or equivalently,  $C$ )-embedded in  $X \times Y$  for every locally compact (separable) metric space  $Y$  (see [9]). It shows that the separability of  $Y$  is not essential. [17, Example 2] and the following theorem shows, in these case, the separability of  $Y$  is essential.

**Theorem 1.15.** *Let  $X$  be a space. Then the following statements hold.*

(1)  $X$  is countably functionally Katětov if and only if for every separable metric space  $Y$  satisfying that  $Y - Y_1$  is locally compact for some closed discrete subspace  $Y_1$ ,  $X \times Y$  is rectangularly normal.

(2)  $X$  is countably Katětov if and only if for every metric space  $Y$  satisfying that  $Y - Y_1$  is locally compact for some closed discrete subspace  $Y_1$ ,  $X \times Y$  is rectangularly normal.

Related to Theorem 1.15, Przymusiński states in [15, Theorem 4] that  $X$  is countably Katětov if and only if for every closed subset  $A$  of  $X$  and every  $\sigma$ -locally compact metric space  $Y$ ,  $A \times Y$  is  $C^*$ -embedded in  $X \times Y$ . So Theorem 1.15 (2) is contained in his result. However he gives its proof for only the case of  $\dim Y = 0$ , and comments that “I have a very complicated proof that eliminates the assumption of  $\dim Y = 0$ ” and asks the reasonable simple way of eliminating the  $\dim Y = 0$ . The author does not know whether if the general case is true.

The following proposition seems to be a natural explanation of the fact that the collectionwise normal and countably paracompact implies Katětov by comparing (1) on Theorem 1.14. In other words, the normality of product with  $J_\gamma(\kappa)$  can not induce the difference among Katětov, functional Katětov and collectionwise normality.

**Proposition 1.16.** *Let  $X$  be a space,  $\kappa$  a cardinal number and  $\gamma$  an infinite cardinal number. The product space  $X \times J_\gamma(\kappa)$  is normal if and only if  $X$  is  $\gamma$ -collectionwise normal and countably paracompact.*

*Namely,  $X \times J_\gamma(\kappa)$  is normal if and only if  $X \times J_\gamma(1)$  is normal.*

All of our results related to  $J_\gamma(\kappa)$  in this report can be replaced by spaces of some class, but the details are omitted here.

## 2. Controlling extensions of continuous functions and $C$ -embedding

In [5], M. Frantz proved a theorem as follows:

**Theorem 2.1** (Frantz, [5]). *Let  $X$  be a normal space and  $A$  a closed subspace. Let  $f : A \rightarrow [c, d]$  be a continuous function with  $f^{-1}(\{c\}) \neq \emptyset$  and  $f^{-1}(\{d\}) \neq \emptyset$  and suppose  $C$  and  $D$  are disjoint closed  $G_\delta$ -sets of  $X$  satisfying  $C \cap A = f^{-1}(\{c\})$  and  $D \cap A = f^{-1}(\{d\})$ . Then  $f$  has a continuous extension  $g : X \rightarrow [c, d]$  such that  $C = g^{-1}(\{c\})$  and  $D = g^{-1}(\{d\})$ .*

According to [5], this result shows that the well-known Tietze-Urysohn extension theorem admits controlling the extended function so as to take on certain specified values. Extending Theorem 2.1, we show that controlling extension (here, “controlling extension” is used by means of Theorem 2.1) itself equals to  $C$ -embedding.

**Theorem 2.2.** *Let  $X$  be a space and  $A$  a subspace. Then  $A$  is  $C$ -embedded in  $X$  if and only if for every continuous function  $f : A \rightarrow [0, 1]$  and disjoint zero-sets  $Z_0, Z_1$  of  $X$  with  $Z_i \cap A = f^{-1}(\{i\})$  ( $i = 0, 1$ ), there exists a continuous extension  $g : X \rightarrow [0, 1]$  of  $f$  such that  $Z_i = g^{-1}(\{i\})$  ( $i = 0, 1$ ).*

For a function  $f : X \rightarrow \mathbb{R}$ ,  $\text{Coz}(f)$  means  $f^{-1}((-\infty, 0) \cup (0, \infty))$ . As applications of Theorem 2.2, we characterize  $C$ -embedding by various types of controlling extensions.

**Theorem 2.3.** *Let  $X$  be a space and  $A$  a subspace. Then the following statements are equivalent :*

- (1)  $A$  is  $C$ -embedded in  $X$ ;
- (2) for every continuous function  $f : A \rightarrow [0, 1]$  and any zero-set  $Z$  of  $X$  with  $Z \cap A = f^{-1}(\{0\})$ , there exists a continuous extension  $g : X \rightarrow [0, 1]$  of  $f$  such that  $Z = g^{-1}(\{0\})$ ;
- (3) for every continuous function  $f : A \rightarrow [0, \infty)$  and any zero-set  $Z$  of  $X$  with  $Z \cap A = f^{-1}(\{0\})$ , there exists a continuous extension  $g : X \rightarrow [0, \infty)$  of  $f$  such that  $Z = g^{-1}(\{0\})$ ;
- (4) for every continuous function  $f : A \rightarrow \mathbb{R}$ , any real numbers  $r_1 < r_2 < \dots < r_n$  and any collection  $\{Z_i, Z_i^* : i = 1, 2, \dots, n\}$  of zero-sets of  $X$  satisfying that  $Z_i \cap Z_{i+1}^* = \emptyset$  ( $i = 1, \dots, n-1$ ),  $Z_i \cup Z_i^* = X$ ,  $f^{-1}((-\infty, r_i]) = Z_i \cap A$  and  $f^{-1}([r_i, \infty)) = Z_i^* \cap A$  ( $i = 1, 2, \dots, n$ ), there exists a continuous extension  $g : X \rightarrow \mathbb{R}$  of  $f$  such that  $g^{-1}((-\infty, r_i]) = Z_i$  and  $g^{-1}([r_i, \infty)) = Z_i^*$  for  $i = 1, 2, \dots, n$ ;
- (5) for every continuous function  $f : A \rightarrow \mathbb{R}$  and any cover  $\{Z^-, Z^+\}$  of zero-sets of  $X$  with  $f^{-1}((-\infty, 0]) = Z^- \cap A$  and  $f^{-1}([0, \infty)) = Z^+ \cap A$ , there exists a continuous extension  $g : X \rightarrow \mathbb{R}$  of  $f$  such that  $g^{-1}((-\infty, 0]) = Z^-$  and  $g^{-1}([0, \infty)) = Z^+$ ;
- (6) for every continuous function  $f : A \rightarrow \mathbb{R}$  and any cozero-set  $U$  of  $X$  with  $\text{Coz}(f) = U \cap A$ , there exists a continuous extension  $g : X \rightarrow \mathbb{R}$  of  $f$  such that  $\text{Coz}(g) \subset U$ .

Continuous real-valued functions  $f_\alpha$ 's ( $\alpha \in \Omega$ ) are said to be *pairwise disjoint* if  $|f_\alpha| \wedge |f_\beta| = 0$  for every  $\alpha, \beta \in \Omega$  with  $\alpha \neq \beta$ . Obviously,  $|f_\alpha| \wedge |f_\beta| = 0$  if and only if  $\text{Coz}(f_\alpha) \cap \text{Coz}(f_\beta) = \emptyset$ . The following result extends [5, Proposition 5].

**Proposition 2.4.** *Let  $X$  be a space and  $A$  a subspace. Then  $A$  is  $C$ -embedded in  $X$  if and only if for every collection  $\{f_i : i \in \mathbb{N}\}$  of pairwise disjoint real-valued continuous functions on  $A$ , there exists a collection  $\{g_i : i \in \mathbb{N}\}$  of pairwise disjoint real-valued continuous functions on  $X$  such that  $g_i|_A = f_i$  for each  $i \in \mathbb{N}$ .*

A subspace  $A$  of a space  $X$  is said to be  $T_z$ -embedded in  $X$  if every disjoint collection of cozero-sets of  $A$  can be extended to a disjoint collection of cozero-sets of  $X$  [2].



**Proposition 2.5.** *A is  $C$ - and  $T_z$ -embedded in  $X$  if and only if for every collection  $\{f_\alpha : \alpha \in \Omega\}$  of pairwise disjoint real-valued continuous functions on  $A$ , there exists a collection  $\{g_\alpha : \alpha \in \Omega\}$  of pairwise disjoint real-valued continuous functions on  $X$  such that  $g_\alpha|_A = f_\alpha$  for each  $\alpha \in \Omega$ .*

The next result extends [5, Proposition 6], where  $X$  is a metric space.

**Corollary 2.6.** *Let  $X$  be a hereditarily collectionwise normal space and  $A$  a closed subspace. Then, for every collection  $\{f_\alpha : \alpha \in \Omega\}$  of pairwise disjoint real-valued continuous functions on  $A$ , there exists a collection  $\{g_\alpha : \alpha \in \Omega\}$  of pairwise disjoint real-valued continuous functions on  $X$  such that  $g_\alpha|_A = f_\alpha$  for each  $\alpha \in \Omega$ .*

Next we comment to a Frantz's problem in [5]. Frantz proved the following result:

**Theorem 2.7** (Frantz, [5, Theorem 7]). *Let  $A$  be a closed subspace of a normal space  $X$ , and let  $f_1, \dots, f_n$  be a partition of unity on  $A$  subordinated to an open cover  $\{U_1, \dots, U_n\}$  of  $A$ . If  $\{\hat{U}_1, \dots, \hat{U}_n\}$  is an open cover of  $X$  such that  $\hat{U}_i \cap A = U_i$  for each  $i$ , then there exists a partition of unity  $\{\hat{f}_1, \dots, \hat{f}_n\}$  on  $X$  subordinated to  $\{\hat{U}_1, \dots, \hat{U}_n\}$  such that  $\hat{f}_i|_A = f_i$  for each  $i$ .*

Concerning this theorem, a problem is posed in [5, Remark p.68]:

*"Does Theorem 7 hold for an infinite partition of unity?"*

We can construct a counterexample of this problem by using any normal but not paracompact space. On the other hand, if we require the extended cover  $\{\hat{U}_\alpha : \alpha \in \Omega\}$  of  $X$  to be locally finite, we have a positive answer as the following:

**Proposition 2.8.** *Let  $A$  be a closed subspace of a normal space  $X$ , and let  $\{f_\alpha : \alpha \in \Omega\}$  be a partition of unity on  $A$  subordinated to a locally finite open cover  $\{U_\alpha : \alpha \in \Omega\}$  of  $A$ . If  $\{\hat{U}_\alpha : \alpha \in \Omega\}$  is a locally finite open cover of  $X$  such that  $\hat{U}_\alpha \cap A = U_\alpha$  for each  $\alpha \in \Omega$ , then there exists a partition of unity  $\{\hat{f}_\alpha : \alpha \in \Omega\}$  on  $X$  subordinated to  $\{\hat{U}_\alpha : \alpha \in \Omega\}$  such that  $\hat{f}_\alpha|_A = f_\alpha$  for each  $\alpha \in \Omega$ .*

Controlling extensions are useful to discuss extensions of continuous functions and extensions of locally finite cozero-sets cover. Finally, for an application, we give a proof of Theorem 0 by the condition (2) in Theorem 2.3; the proof is simpler than the original one in [19].

**Proof of Theorem 0.** The "only if" part is easy to see. Assume the condition of the "if" part to be satisfied. We note first  $A$  is  $C$ -embedded in  $X$  (e.g.

[1] or [8, Theorem 2.6]). Let  $\{f_\alpha : \alpha \in \Omega\}$  be a locally finite partition of unity on  $A$ . From the assumption, there exists a locally finite cover  $\{U_\alpha : \alpha \in \Omega\}$  of cozero-sets of  $X$  such that  $U_\alpha \cap A = f_\alpha^{-1}((0, 1])$  for every  $\alpha \in \Omega$ . By (2) of Theorem 2.3, there exists a continuous extension  $g_\alpha : X \rightarrow [0, 1]$  of  $f_\alpha$  such that  $g_\alpha^{-1}((0, 1]) = U_\alpha$  for every  $\alpha \in \Omega$ . It is easy to see that  $\sum_{\alpha \in \Omega} g_\alpha$  is continuous and positive-valued. Hence  $\{g_\alpha / \sum_{\beta \in \Omega} g_\beta : \alpha \in \Omega\}$  is the required locally finite partition of unity on  $X$ .  $\square$

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December 7, 1999