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Topological groups, k-networks, and weak topology

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Let G be a topological group. Then, we give affirmative answers to (Q1), and partial answers to (Q2) and (Q3) in the following questions.

(Q1) (A) Let G have a σ-hereditarily closure-preserving k-network. Is G an $\aleph$-space ?

(B) Let G be a k-space with a star-countable k-network. Is G an $\aleph$-space ?

(Q2) Let G be the quotient s-image of a metric space. Is G paracompact (or, meta-Lindelöf) ?

(Q3) (A. V. Arhangelskiǐ). Let G be a sequential space. Does G contain no (closed) copy of S_{ω_1} ?

Let us recall some definitions which will be used in this paper.

A family $\{A_\alpha : \alpha \in I\}$ of subsets of a space X is *hereditarily closure-preserving* (simply, HCP) if $\bigcup\{clB_\alpha : \alpha \in J\} = cl(\bigcup\{B_\alpha : \alpha \in J\})$, whenever $J \subset I$ and $B_\alpha \subset A_\alpha$ for each $\alpha \in J$.

Let $\mathcal{P}$ be a cover of a space X. Then, $\mathcal{P}$ is a *k-network* for X, if whenever $K \subset U$ with K compact and U open in X, $K \subset \cup\mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. When a k-network $\mathcal{P}$ is a closed cover, then $\mathcal{P}$ is called a *closed* k-network.

Recall that a space is an *$\aleph$-space* (resp. *$\aleph_0$-space*) if it has a σ-locally finite k-network (resp. countable k-network).

Following [GMT], a space X *is determined by* a cover $\mathcal{C}$, if $F \subset X$ is closed in X iff $F \cap C$ is closed in C for every $C \in \mathcal{C}$. We use " X is determined by $\mathcal{C}$ " instead of the usual " X has the weak topology with respect to $\mathcal{C}$ ". Obviously, every space X is determined by any open cover, or any HCP closed cover of X.

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A space is a *k-space* (resp. *sequential space*) if it is determined by a cover of compact subsets (resp. compact metric subsets). As is well-known, every k-space (resp. sequential space) is precisely the quotient image of a locally compact space (resp. (locally compact) metric space).

A space X *has countable tightness* ($= t(X) \leq \omega$) if, whenever $x \in clA$, then $x \in clB$ for some countable subset B with $B \subset A$. As is well-known, $t(X) \leq \omega$ iff X is determined by a cover of countable subsets.

Let us recall canonical quotient spaces S_α, and the *Arens' space* S_2.

For an infinite cardinal α, S_α is the space obtained from the topological sum of α convergent sequences by identifying all the limit points to a single point. In paricular, S_ω is called the *sequential fan.*

Let $L = \{a_n : n \in \omega\}$ be an infinite sequence with a limit point $\infty \notin L$. Let L_n $(n \in \omega)$ be an infinite sequence with a limit point $a_n \notin L_n$. Then, S_2 is the space obtained from the topological sum of L and these L_n by identifying each $a_n \in L$ with the limit point a_n of L_n.

We assume that spaces are regular and T_1, and maps are continuous and onto.

Results

Lemma 1. ([JZ]) Let X be a space with a σ-HCP k-network. Then, X is an $\aleph$-space if and only if X contains no (closed) copy of S_{ω_1}.

Every Fréchet space X with a σ-HCP k-network (equivalently, X is a Lašnev space [F]) need not be an $\aleph$-space; see Example 16(1). But, we have the following among topological groups.

Theorem 2. Let G be a topological group. If G has a σ-HCP k-network, then G is an $\aleph$-space. (Affirmative answer to (A) in (Q1))

Corollary 3. Let G be a topological group which is the closed image of an $\aleph$-space. Then, G is an $\aleph$-space.

Remark 4. For a space X, the following decomposition theorems hold. (1) is due to [M] or [Ln], and (2) is due to [LT1].

(1) Let X be a space with a σ-HCP k-network. Then X, as well as every closed image of X, is decomposed into a σ-discrete space and an $\aleph$-space.

(2) Let X be a Fréchet space with a star-countable k-network (more gen-

generally, point-countable k-network of separable subsets). Then X is decomposed into a closed discrete space and a space which is the topological sum of $\aleph_0$-spaces. (The Fréchetness of X is essential; see Example 16(2)).

Let us consider topological groups having certain point-countable covers. The parenthetic part is due to [NT].

Lemma 5. Let $t(X) \leq \omega$. If X contains a copy of S_{ω_1} (resp. S_ω), then X contains a closed copy of S_{ω_1} (resp. S_ω).

For an infinite cardinal α, a space X is *α-compact* if every subset of cardinality α has an accumulation point in X. Clearly, Lindelöf spaces (resp. countably compact spaces) are ω_1-compact (resp. ω-compact).

Corollary 6. Let $t(X) \leq \omega$. If X is determined by a point-countable (resp. point-finite) cover of ω_1-compact (resp. ω-compact) subsets, then X contains no copy of S_{ω_1} (resp. S_ω).

In particular, S_{ω_1} (resp. S_ω) can not be embedded into any ω_1-compact (resp. ω-compact) space of countable tightness.

Let us say that a cover $\mathcal{P}$ of X is a *cs-cover* of X if, for every infinite convergent sequence C in X, some $P \in \mathcal{P}$ contains at least two points of C. We note that S_{ω_1} has a point-countable cs-cover of two-point sets.

Theorem 7. Let G be a sequential group with a point-countable cs-cover of ω_1-compact subsets. Then, G contains no copy of S_{ω_1}. (Partial answer to (Q3)).

Corollary 8. Let G be sequential group with a point-countable k-network of ω_1-compact subsets. Then, G contains no copy of S_{ω_1}.

Lemma 9. Let G be a sequential topological group satisfying (a) and (b) below. Then, G is the topological sum of ω_1-compact subsets.

In particular, if each element of $\mathcal{F}$ is cosmic (resp. compact), then G is the topological sum of cosmic subspaces (resp. σ-compact subspaces). Here, a space is *cosmic* if it has a countable network.

(a) G contains no (closed) copy of S_{ω_1}.

(b) G has a point-countable cover $\mathcal{F}$ such that $\mathcal{F}^* = \{\cup\mathcal{F}' : \mathcal{F}' \subset \mathcal{F}, \mathcal{F}'$ is finite$\}$ determines G; and, any finite product of elements of $\mathcal{F}$ is ω_1-compact.

Theorem 10. Let G be a topological group. If G is a k-space with a point-countable k-network $\mathcal{P}$ of cosmic subspaces, then G is the topological sum of cosmic subspaces.

In particular, if G is a k-space with a star-countable k-network, then G is the topological sum of $\aleph_0$-subspaces. (Affirmative answer to (B) in (Q1)).

Remark 11. In the previous theorem, the property " G is a k-space " is essential. According to [Tk2], under (CH) there exists a countably compact topological group G in which every compact set is finite, but G is not metrizable (cf. [Tk1]). Hence, the topological group G has a star-countable k-network of singletons, but not even a σ-space.

Let us recall that every CW-complex, more generally, every space *dominated* by k-and-$\aleph_0$-subspaces is a k-space with a star-countable k-network ([IT]). (Conversely, every k-space with a star-countable k-network is a space *dominated* by k-and-$\aleph_0$-subspaces ([S])). Then, the following holds by Theorem 7 and [T3; Corollary 6].

Corollary 12. Let K be a topological group. If K is a CW-complex, then K is the topological sum of countable CW-subcomplexes.

In the previous corollary, " K is a topological group " is essential, and the topological group K need not be metrizable; see Example 16.

Now, every quotient finite-to-one image of a locally compact metric space need not be paracompact, nor even meta-Lindelöf; see [GMT; Example 9.3]. But, we have the following among topological groups.

Theorem 13. Let $f : X \to G$ be a quotient s-map such that X is a locally separable metric space. If G is a topological group, then G is a paracompact space (actually, G is the topological sum of cosmic subspaces). (Partial answer to (Q2)).

In the previous theorem, the topological group G need not be metrizable by Example 16(3).

Similarly, we have the following since G is determined by a point-countable cover of compact subsets.

Theorem 14. Let $f : X \to G$ be a quotient s-map such that X is a locally compact paracompact space. If G is a sequential topological group, then G is a paracompact space (actually, G is the topological sum of σ-compact subspaces).

Remark 15. Let G be a topological group. Then, G is metrizable if the following (a), (b), or (c) holds. (Cf. [LST]).

(a) G is a k-space with a point-countable k-network, and G contains no closed copy of S_ω, or no S_2.

(b) G is the quotient compact image of a metric space.

(c) G is a Fréchet space with a point-countable k-network. In particular, G is a Lašnev space, or a Fréchet space which is the quotient s-image of a metric space.

Example 16. (1) A Lašnev CW-complex K, but K is not an $\aleph$-space.

(2) A CW-complex K which is not Fréchet, and K has the following properties. (Cf. [LT1]).

(a) K contains no copy of S_ω.

(b) K has a point-countable *closed* k-network.

(c) K has a star-countable k-network of separable metric subsets.

(d) K can not be decomposed into a σ-discrete space and a space with a σ-HCP k-network, or star-countable *closed* k-network.

(3) A topological group G which is a countable CW-complex (hence, an $\aleph_0$-space), and G is the quotient countable-to-one image of a locally compact, separable metric space. But, G is not metrizable, not even Fréchet.

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