数理解析研究所講究録
1126 巻 2000 年 61-65

COHOMOLOGICAL DIMENSION AND RESOLUTION

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In this note we introduce the joint work [Ko-Y2] with A. Koyama (Osaka Kyoiku University).

To investigate dimension theory from the view point of algebraic topology, P.S.Alexandroff [Al_1] introduced cohomological dimension theory. It is really a powerful tool of analyzing dimension of product spaces and decomposition spaces, and has much connection with many areas of topology. Next the following Edwards theorem [Ed] was a turning point of recent development of the theory. The details can be found in [W].

Edwards Theorem. *For a compactum X with* $\text{c-dim}_{\mathbf{Z}} X \leq n$ *there exists an n-dimensional compactum Z and a cell-like map $f : Z \to X$*

We note that a map $f : Z \to X$ between compacta is *cell-like* if all point inverses $f^{-1}(x)$ have trivial shape. Edwards and Walsh clarified a relation between cohomological dimension and the topology of manifolds. Namely, the Edwards Theorem gives the exact connection between the *Alexandroff's long standing problem* [Al_2], of whether there exists an infinite-dimensional compactum whose integral cohomological dimension is finite, and the *cell-like mapping problem*, of whether a cell-like map on a finite-dimensional manifold can raise dimension. Although the Alexandroff problem was solved by Dranishnikov [Dr_1], their main idea, called *Edwards-Walsh resolution*, was also a key tool of the solution. In fact, he constructed an infinite-dimensional compactum X with $\text{c-dim}_{\mathbf{Z}} X = 3$. Hence we know that there is a cell-like map $f : I^7 \to Y$ with $\dim Y = \infty$. Moreover, following Dranishnikov's idea and applying the Sullivan Conjecture [Mi], Dydak and Walsh [D-W_2] constructed an infinite-dimensional compactum X with $\text{c-dim}_{\mathbf{Z}} X = 2$. Hence there is a cell-like map $f : I^5 \to Y$ with $\dim Y = \infty$.

1991 *Mathematics Subject Classification.* 55M10.
Key words and phrases. Cohomological dimension, cell-like resolution, Edwards-Walsh resolution.

Such compacta and their cell-like resolutions are applied to solve various problems. For example, existence of infinite-dimensional cohomology manifolds, [Dr$_1$], [Dr$_2$], existence of a linear metric space which is not an ANR [Ca], and etc.

On the other hand, Dranishnikov [Dr$_3$] constructed a cell-like map $f : I^6 \to Y$ with $\dim Y = \infty$ by constructing an exotic compactum X with $\dim X = \infty$ and $\text{c-dim}_{\mathbf{Z}/p} X \leq 2$ and $\text{c-dim}_{\mathbf{Z}_{[\frac{1}{p}]}} X \leq 2$. Note that those inequalities imply the inequality $\text{c-dim}_{\mathbf{Z}} X \leq 3$. Then he showed and essentially used the following cell-like resolution theorem:

Dranishnikov Cell-like Resolution Theorem. *If a compactum X has cohomological dimension $\text{c-dim}_{\mathbf{Z}/p} X \leq n, \text{c-dim}_{\mathbf{Z}_{[\frac{1}{p}]}} X \leq n$ for some prime number p, where $n > 1$, then there exists an $(n+1)$-dimensional compactum Z with $\text{c-dim}_{\mathbf{Z}/p} Z \leq n, \text{c-dim}_{\mathbf{Z}_{[\frac{1}{p}]}} Z \leq n$ and a cell-like map $f : Z \to X$.*

Testing constructions of acyclic resolutions in [Ko-Y], we can see that it is difficult to investigate acyclic resolutions for cohomological dimensions with respect to both a torsion group and a torsion free group. In that sense Dranishnikov Cell-like Resolution Theorem seems to be interesting.

We direct our attention to properties which the Dranishnikov infinite-dimensional compactum X in [Dr$_3$, Theorem 1] has. Namely, it satisfies inequalities $\text{c-dim}_{\mathbf{Z}/p} X \leq 2$ and $\text{c-dim}_{\mathbf{Z}_{(q)}} X \leq 2$ for all prime numbers $q \neq p$. For any integers $1 \leq m_p, m_q < n$, by [Dr$_2$], there exists an n-dimensional compactum Z such that $\text{c-dim}_{\mathbf{Z}/p} Z = m_p$ and $\text{c-dim}_{\mathbf{Z}_{(q)}} Z = m_q$. Hence, if $m_p, m_q \geq 2$, we can obtain the infinite-dimensional compactum $X \vee Z$ having the property that $\text{c-dim}_{\mathbf{Z}} X \vee Z = n, \text{c-dim}_{\mathbf{Z}/p} X \vee Z = m_p$ and $\text{c-dim}_{\mathbf{Z}_{(q)}} X \vee Z = m_q$. On the other hand, Dydak-Walsh, [D-W$_2$, Theorem 2] constructed an infinite-dimensional compactum Y such that $\text{c-dim}_{\mathbf{Z}} Y = 2$ and $\text{c-dim}_{\mathbf{Q}} Y = \text{c-dim}_{\mathbf{Z}/p} Y = 1$ for every prime number p. Hence, if $m_q \geq 2$, we also have the infinite-dimensional compactum $Y \vee Z$ having the property that $\text{c-dim}_{\mathbf{Z}} Y \vee Z = n, \text{c-dim}_{\mathbf{Z}/p} Y \vee Z = 1$ and $\text{c-dim}_{\mathbf{Z}_{(q)}} Y \vee Z = m_q$. However, since one of key tools of Dranishnikov's construction is the fact that $\widetilde{K}^*_{\mathbf{C}}(K(\mathbf{Z}/p, 2); \mathbf{Z}/p) = \widetilde{K}^*_{\mathbf{C}}(K(\mathbf{Z}_{[\frac{1}{p}]}, 2); \mathbf{Z}/p) = 0$, and for the Dydak-Walsh compactum Y, by Bockstein theorem, $\text{c-dim}_{\mathbf{Z}_{(q)}} Y = 2$ for at least one prime number q, both compacta cannot help to construct an infinite-dimensional compactum W such that $\text{c-dim}_{\mathbf{Z}} W < \infty$ and $\text{c-dim}_{\mathbf{Z}_{(q)}} W = 1$ for some prime number q. Note that we cannot decide the prime number q so that $\text{c-dim}_{\mathbf{Z}_{(q)}} = 2$.

In [Ko-Y2], giving a localized version of Dydak-Walsh's idea, we construct the following infinite-dimensional compactum:

Theorem 1. *For each pair p, q of distinct prime numbers there exists an infinite-dimensional compactum X such that* $\text{c-dim}_{\mathbf{Z}} X = 2$ *and* $\text{c-dim}_{\mathbf{Z}/p} X = \text{c-dim}_{\mathbf{Z}_{(q)}} X = 1$

Hence we have the following formulation of exotic compacta:

Corollary. *For given prime numbers $p \neq q$ and given integers $1 \leq m_p, m_q < n$, there exists an infinite-dimensional compactum $X(p, q; m_p, m_q, n) = X$ such that* $\text{c-dim}_{\mathbf{Z}} X = n, \text{c-dim}_{\mathbf{Z}/p} X = m_p$ *and* $\text{c-dim}_{\mathbf{Z}_{(q)}} X = m_q$.

We call such a compactum type $(p, q; m_p, m_q, n)$.

Then related to the Edwards Theorem and the Dranishnikov Cell-like Resolution Theorem we naturally pose the following problem:

Cell-like Resolution Problem of type $(p, q; m_p, m_q, n)$. *Let p, q, be distinct prime numbers and let $1 \leq m_p, m_q < n$ be integers. For a compactum X of type $(p, q; m_p, m_q, n)$ does there exist an n-dimensional compactum Z with* $\text{c-dim}_{\mathbf{Z}/p} Z \leq m_p$ *and* $\text{c-dim}_{\mathbf{Z}_{(q)}} Z \leq m_q$ *and a cell-like map $f : Z \to X$?*

We do not know its general answer. However, applying our calculation in [Ko-Y] to the Dranishnikov Cell-like Resolution Theorem, we shall give a detailed proof of the theorem and affirmatively answer the problem of type $(p, q; n, n, n+1)$, where $n > 1$, as follows:

Theorem 2. *Let p, q be distinct prime numbers and let n be an integer > 1. Then for a compactum X of type $(p, q; n, n, n+1)$, there exists an $(n+1)$-dimensional compactum Z with* $\text{c-dim}_{\mathbf{Z}/p} Z \leq n$, $\text{c-dim}_{\mathbf{Z}_{(q)}} Z \leq n$ *and a cell-like map $f : Z \to X$.*

On the other hand, a theorem of Daverman [Da] essentially implies that for any subset Q of prime numbers an infinite-dimensional compactum X with $\text{c-dim}_{\mathbf{Z}} X = 2$ and $\text{c-dim}_{\mathbf{Z}_{(Q)}} X = 1$ cannot be a cell-like image of any 2-dimensional compactum Z with $\text{c-dim}_{\mathbf{Z}_{(Q)}} Z = 1$. Thus, Theorem 1 gives a negative answer to the Cell-like Resolution Problem of type $(p, q; 1, 1, 2)$ for any distinct prime numbers p, q.

In [Ko-Y] we discussed several types of acyclic resolutions. Related to those results we shall pose the following problem:

Problem 1. *Let p, q be distinct prime numbers. For a compactum X with* $\text{c-dim}_{\mathbf{Z}/p} X \leq n$ *and* $\text{c-dim}_{\mathbf{Z}_{(q)}} X \leq n$*, then does there exist an $(n+1)$-dimensional compactum Z and a $\mathbf{Z}/p$- and $\mathbf{Z}_{(q)}$- acyclic resolution ?*

Comparing our results the following problem seems to be interesting:

Problem 2. *If a compactum X has* $\text{c-dim}_{\mathbf{Z}} X \leq n+1$ *and* $\text{c-dim}_{\mathbf{Z}/p^\infty} X \leq k$*, where p is a prime number and $n \geq k \geq 1$, then does there exist an $(n+1)$-dimensional compactum Z with* $\text{c-dim}_{\mathbf{Z}/p^\infty} Z \leq k$ *and a cell-like map $f : Z \to X$?*

For basic results of cohomological dimension and a brief history of the theory we refer [D], [Dr_5], [K] and [Ku] to readers.

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