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Estimating the Error of Quasi-Monte Carlo Integrations — Experimental Studies in Financial Problems —

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1 Introduction

The problem of pricing financial derivatives is one of great interests in recent years among scientists and practitioners. In a considerable literature it is reported that quasi-Monte Carlo(QMC) method shows very rapid convegence in computing the price of derivatives. On the other hand it is rare to see the invetigation on the error of the computed value by QMC method. In this paper we report some experimental results on the error estimation of computed value of derivatives by QMC methods.

All the problems in this paper are posed as the multi-dimensional integral over the unit cube:

$$\int_{[0,1]^s} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{1}$$

QMC method uses a low-discrepancy point sequence $\{\boldsymbol{x}_i\} \subset [0,1)^s$ and computes the approximate value of (1) by

$$I_N = \frac{1}{N}\sum_{i=1}^{N} f(\boldsymbol{x}_i). \tag{2}$$

For the error of QMC method, we have the following Koksma-Hlawka inequality.

$$\left|\frac{1}{N}\sum_{i=1}^{N} f(\boldsymbol{x}_i) - \int_{[0,1]^s} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right| \leq V(f)D_N^*,$$

where $V(f)$ is the total variation of f in the sense of Hardy and Krause, and D_N^* is the star discrepancy of the sequence $\{\boldsymbol{x}_i\}$. Koksma-Hlawka inequality is a basis of the superiority of QMC method to Monte Carlo(MC) method, because if we use a low-discrepancy point sequence, D_N^* (and also the absolute error of integral) goes to 0 with the rate $\mathrm{O}((\log N)^s/N)$ asymptotically as $N \to \infty$, which exhibits a striking contrast to the convergence rate $\mathrm{O}(1/\sqrt{N})$ of MC.

However we cannot make an error estimation with Koksma-Hlawka inequality, because it is usually impossible to calculate the total variation $V(f)$. Recently several works on the error analysis of QMC integrations have been developed. These approaches apply statistical error estimation methods to QMC integrations. Since low-discrepancy point sequences are deterministic, the point sequences must be selected from some probability space in order to do a statistical error estimation,. We need a probabilistic structure on the point sequences.

In this paper we select two methods and compare their efficiencies. The first method uses scrambled sequences, which were proposed by Owen [6]. The second one uses randomly shifted sequences, which are based on the idea of Cranley and Patterson [1] for good lattice points methods. A part of authors of this paper examined two methods by various test functions, and reported that both the methods give satisfactory error estimates from a practical point of view [3]. In this paper we apply two methods to a "real-world problem," pricing financial derivatives, and reach substantially the same conclusion as our previous work.

2 Randomized QMC

2.1 (t, s)-sequence

We use (t, s)-sequence as a low-discrepancy point sequence. Let us recall the definition of (t, s)-sequence [4]. A subset E of $I^s = [0, 1)^s$ of the form

$$E = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1) b^{-d_i})$$

with $a_i, d_i \in \mathbf{Z}$, $d_i \geq 0$, $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$ is called an *elementary interval in base* b.

Definition 1 *Let t and m be nonnegative integers and $t \leq m$. A (t, m, s)-net in base b is a set of b^m points in I^s such that every elementary interval of volume b^{t-m} contains exactly b^t points of the point set.*

Definition 2 *Let $t \geq 0$ be an integer. An infinite sequence $z_0, z_1, \ldots$ of points in I^s is a (t, s)-sequence in base b if, for all integers $k \geq 0$ and $m \geq t$, the point set consisting of the z_n with $kb^m \leq n < (k + 1)b^m$ is a (t, m, s)-net in base b.*

We summarize the construction of (t, s)-sequence in base b according to Niederreiter [4]. The n-th point $\boldsymbol{z}_n = (z_n^{(1)}, \ldots, z_n^{(s)})$ is given by

$$z_n^{(j)} = \sum_{k=1}^{\infty} y_{nk}^{(j)} b^{-k},$$

where

$$y_{nk}^{(j)} = \sum_{r=1}^{\infty} c_{kr}^{(j)} a_r(n) \pmod{b}.$$

Here $n = \sum_{r=1}^{\infty} a_r(n) b^{r-1}$ is the b-adic expansion of n. We call the matrix $C^{(j)} = (c_{kr}^{(j)})$ the generator matrix of the j-th coordinate of (t, s)-sequence.

Faure sequence in the original form is given by setting

$$C^{(j)} = (P^\top)^{j-1} \pmod{b},$$

where $P = (p_{kr})$, so-called Pascal matrix, is given by $p_{kr} = \binom{k-1}{r-1}$.

Generalized Faure sequence presented in [7] is given by setting

$$C^{(j)} = A^{(j)} (P^\top)^{j-1} \pmod{b},$$

where $A^{(j)}$ is an appropriately chosen lower triangular nonsingular matrix.

Sobol' sequence in the original form is given by setting

$$C^{(j)} = \begin{pmatrix} E_{e_j} & & W & & \\ & E_{e_j} & & W & \\ & & E_{e_j} & & \ddots \\ & & & \ddots & \\ O & & & & \ddots \end{pmatrix},$$

where E_{e_j} is the $e_j \times e_j$ unit matrix, and e_j is the degree of the j-th (in the order of nondecreasing degree) primitive polynomial p_j over GF(2). The matrix W is determined as follows. Let $p_j(x) = x^{e_j} + b_{e_j-1}x^{e_j-1} + \cdots + b_0$ $(b_0 \neq 0)$. For $1 \leq k \leq e_j$, we denote the k-th row vector of $C^{(j)}$ by $\boldsymbol{c}_k = (c_{k1}, c_{k2}, \ldots)$. Obviously the first e_j elements of $\boldsymbol{c}_k$ are those of the k-th row vector of $e_j \times e_j$ unit matrix E_{e_j}, so $c_{k1} = 0, \ldots, c_{kk} = 1, c_{k,k+1} = 0, \ldots, c_{k,e_j} = 0$. The elements c_{kr} for $r > e_j$, which are the elements of the k-th row of W, are determined by the recurrence relation:

$$c_{kr} + b_{e_j-1}c_{k,r-1} + \cdots + b_0 c_{k,r-e_j} = 0 \pmod 2. \tag{3}$$

Some generalization of Sobol' sequence is obtained by replacing E_{e_j} by an arbitrary nonsingular $e_j \times e_j$ matrix $F^{(j)}$. Naturally the elements of each row in the generator matrix must be changed to satisfy the relation (3). Hence a generalization of Sobol' sequence is given by setting

$$C^{(j)} = \begin{pmatrix} F^{(j)} & & W' & & \\ & F^{(j)} & & W' & \\ & & F^{(j)} & & \ddots \\ & & & \ddots & \\ O & & & & \ddots \end{pmatrix}.$$

This generator matrix can be interpreted as the product of a block diagonal matrix with $F^{(j)}$ as block diagonal elements and the original generator matrix:

$$C^{(j)} = \begin{pmatrix} F^{(j)} & & & \\ & F^{(j)} & & O \\ & & F^{(j)} & \\ & & & \ddots \\ O & & & & \ddots \end{pmatrix} \begin{pmatrix} E_{e_j} & & W & & \\ & E_{e_j} & & W & \\ & & E_{e_j} & & \ddots \\ & & & \ddots & \\ O & & & & \ddots \end{pmatrix}.$$

2.2 Statistical Error Estimation

We introduce statistical error estimation methods for numerical integrations using randomized (t, s)-sequence. The basic idea of statistical error estimation methods is a combination of MC and QMC. We construct a space of point sequences with some probabilistic structure. This is done by randomizing a (t, s)-sequence. The error estimate is obtained by means of choosing several point sequences from the probability space and calculating the standard deviation of the computed values by those sequences.

The general scheme of the methods is as follows. We select point sequences $\{\boldsymbol{x}_i^{(j)}\}$, $j = 1, \ldots, M$, independently from a probability space, and compute the value of (2) using the first N elements of each sequence:

$$S^{(j)} = \frac{1}{N} \sum_{i=1}^{N} f(\boldsymbol{x}_i^{(j)}), \quad j = 1, \ldots, M. \tag{4}$$

Then we calculate the estimate of $\int_{[0,1]^s} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$ by

$$\hat{I} = \frac{1}{M} \sum_{j=1}^{M} S^{(j)}. \tag{5}$$

The error of the numerical integration is estimated using the variance of the evaluated values.

$$\hat{\sigma}^2 = \frac{1}{M(M-1)} \sum_{j=1}^{M} (S^{(j)} - \hat{I})^2. \tag{6}$$

We will consider the following two probabilistic structures.

Scrambling [6].

Let $\{\boldsymbol{z}_i\}$ be a (t, s)-sequence in base b. Suppose $\boldsymbol{z}_i = (z_i^1, \ldots, z_i^s)$ and $z_i^j = \sum_{k=1}^{\infty} z_{ijk} b^{-k}$ for integers $0 \leq z_{ijk} < b$. A scrambled sequence $\{\boldsymbol{x}_i\}$, $\boldsymbol{x}_i = (x_i^1, \ldots, x_i^s)$ is defined as $x_i^j = \sum_{k=1}^{\infty} x_{ijk} b^{-k}$, where $\{x_{ijk}\}$ is a random permutation applied to $\{z_{ijk}\}$. Specifically x_{ijk} are determined as follows.

$$\begin{aligned}
x_{ij1} &= \pi_j(z_{ij1}), \\
x_{ij2} &= \pi_{j z_{ij1}}(z_{ij2}), \\
&\vdots \\
x_{ijk} &= \pi_{j z_{ij1} z_{ij2} \ldots z_{ij,k-1}}(z_{ijk}), \\
&\vdots
\end{aligned}$$

Here each π is a random permutation over $\{0, 1, \ldots, b-1\}$. In the second line the subscript z_{ij1} means that the permutation depends on the value of z_{ij1}. In the same way $\pi_{j z_{ij1} z_{ij2} \ldots z_{ij,k-1}}$ is a permutation depending on the values of $z_{ij1}, \ldots, z_{ij,k-1}$. All permutations are mutually independent.

Random Shifting [1].

Let $\{\boldsymbol{z}_i\}$ be a (t, s)-sequence in base b and $\boldsymbol{u}$ be an s-dimensional random vector uniformly distributed over a unit cube. A randomly shifted sequence $\{\boldsymbol{x}_i\}$ is given by

$$\boldsymbol{x}_i = \boldsymbol{z}_i + \boldsymbol{u} \pmod{\mathbf{1}},$$

where (mod $\mathbf{1}$) means the componentwise (mod 1) operation.

3 Pricing Derivative Securities

Three kind of financial derivatives are used for numerical experiments. In all the experiments we use generalized Faure sequence and generalized Sobol' sequence described in Sect. 2.

3.1 European Call Option

First we deal with a basic case, European call option. Let $S(t)(0 \le t \le T)$ be the underlying asset price which obeys the Black-Scholes model:

$$\mathrm{d}S(t) = S(t)r\mathrm{d}t + \sigma S(t)\mathrm{d}B(t), \quad S(0) = S_0, \tag{7}$$

where r is the riskless interest rate, σ is the volatility, and $B(t)$ is the standard Brownian motion. The price of European call option with a strike price K can be written as follows.

$$E[\max(S(T) - K, 0)]. \tag{8}$$

The expectation is taken over all Brownian motions starting at S_0. We apply Euler-Maruyama scheme to (7), obtaining the stochastic difference equation:

$$S_n = S_{n-1}(1 + r\Delta t + \sigma\sqrt{\Delta t}\xi_n), \quad n = 1, \ldots, s, \tag{9}$$

where $\Delta t = T/s$, and ξ_n are i.i.d. standard normal random variables. Then we approximate (8) by

$$\int_{\mathbf{R}^s} \max\left(S_0 \prod_{i=1}^{s}(1 + r\Delta t + \sigma\sqrt{\Delta t}\xi_i) - K, 0\right) \left(\frac{1}{\sqrt{2\pi}}\right)^s \prod_{j=1}^{s} \exp\left(-\frac{\xi_j^2}{2}\right) \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_s. \tag{10}$$

Under the change of variables $x_i = \Phi(\xi_i)$, where Φ is the standard normal distribution function, the integral becomes

$$\int_{[0,1]^s} \max\left(S_0 \prod_{i=1}^{s}(1 + r\Delta t + \sigma\sqrt{\Delta t}\Phi^{-1}(x_i)) - K, 0\right) \mathrm{d}x_1 \ldots \mathrm{d}x_s. \tag{11}$$

The parameters used in the experiment are as follows [2]:

current price, S_0	100
strike price, K	100
riskless interest rate, r	0.1
time to maturity, T	1
volatility, σ	0.3
s	360

We can find the exact value of this option for these parameters by the analytical solution. The value is approximately 16.734.

The result is shown in Figure. 1. In each figure, the abscissa is the number of points N in the underlying sequence, and the estimated values are indicated by dots. In the experiment we used 10 independent sequences ($M = 10$), so each estimate is the average of $S^{(j)}$, $j = 1, \ldots, 10$ (cf. (4), (5)). The vertical bar on the dot shows the estimated error $3\hat{\sigma}$ (cf. (6)). In the experiments, we observe that both scrambling method and random shifting method lead to good error estimates, and that randomized Faure sequences give smaller variances than Sobol' sequences. The difference between the variances given by scrambling and random shifting is not distinguishable.

3.2 Mortgage Backed Security

Next we consider another financial problem, mortgage-backed security(MBS). Here we use a simplified model provided by [5] for the experiment.

The underlying pool of mortgages has a thirty-year maturity and cash flows occur monthly. So there are 360 cash flows. For $1 \le k \le 360$ we set,

$$\begin{aligned} C&: \text{ the monthly payment on the underlying pool of mortgages.}\\ r_k&: \text{ the appropriate interest rate in month } k.\\ w_k&: \text{ the percentage prepaying in month } k.\\ a_{360-k+1}&: \text{ the remaining annuity after month } k.\end{aligned}$$

Then we recall that the remaining annuity a_k is given by

$$a_k = \sum_{i=0}^{k-1} v_0^i,$$

where $v_0 = 1/(1+r_0)$ and r_0 means the current monthly interest rate. In our setting C and r_0 (therefore also a_k) are constants; r_k and w_k are stochastic variables determined by

$$\log r_k - \log r_{k-1} = -\log 1.020201 + 0.2\xi_k, \tag{12}$$
$$w_k = 0.24 + 0.134\arctan(12.72 - 261.17 r_k), \tag{13}$$

respectively, where ξ_k are independent normal random variables. The cash flow in month k is

$$M_k = C(1-w_1)\cdots(1-w_{k-1})(1-w_k+w_k a_{360-k+1}). \tag{14}$$

To get the present value of this cash flow, we multiply M_k by the corresponding discount factor

$$u_k = \prod_{i=0}^{k-1} v_i,$$

where

$$v_i = \frac{1}{1+r_j}, \quad i = 1, \ldots, 359.$$

Summing up the present values of the cash flow for every month k, we obtain the present value of the security

$$V(\xi_1, \ldots, \xi_{360}) = \sum_{k=1}^{360} M_k u_k.$$

The objective is to estimate the expected value $E[V]$. As in the previous example, using the change of variables $x_k = \Phi(\xi_k)$, we see that

$$E[V] = \int_{[0,1]^{360}} V(\xi_1(x_1), \ldots, \xi_{360}(x_{360}))\mathrm{d}x_1 \ldots \mathrm{d}x_{360}.$$

In the experiment we set $C = 2000$, $r_0 = 0.075/12$. We have no analytical solution for this problem.

The result is shown in Figure 2. The notation used in the figure is the same as in the case of European call option. The error estimation is made in terms of 10 independent sequences ($M = 10$).

Randomized Faure sequences give smaller variances than randomized Sobol' sequences. As for Faure sequence, the difference between the variances given by scrambling and random shifting is not distinguishable. The result of randomized Sobol' sequences (Figure 2(a), (b)) is unsatisfactory. We observe the very slow convergence of estimated errors for these cases. We are not clear about the reason for this. However we give some heuristics, permuting coordinates, in order to improve the convergence. Specifically, we select first κ $(1 < \kappa \leq 360)$ variables $x_1, \ldots, x_\kappa$, then permute the order of variables:

$$x'_1 = x_{\sigma(1)}, \ldots, x'_\kappa = x_{\sigma(\kappa)},$$

with a randomly chosen permutation σ over $\{1, \ldots, \kappa\}$. The other variables $x_{\kappa+1}, \ldots, x_{360}$ remain in the same order:

$$x'_{\kappa+1} = x_{\kappa+1}, \ldots, x'_{360} = x_{360}.$$

We use new variables $\{x'_i\}_{i=1}^{360}$ for the computation. Figure 2(e) and (f) present the result for this heuristics, where we set $\kappa = 30$, and for a fixed order of variables determined by a randomly chosen permutation, 10 independent sequences are generated to estimate the error.

Insofar as we observe the experimental results, we can consider both methods, scrambling and shifting, give reliable error estimations.

3.3 Asian Option

The payoff of the Asian option depends on the arithmetic average of an asset price. We examine here the call option. The payoff at expiry is given by

$$\max(\frac{1}{T}\int_0^T S(t)\mathrm{d}t - K, 0),$$

where $S(t)$ is the asset price which follows (7), and K is the strike price. As in Sect. 3.1, discretization in time leads to an approximation of the option price.

$$\int_{\mathbf{R}^s} \max\left(\frac{1}{s}\sum_{n=1}^{s} S_n - K, 0\right)\left(\frac{1}{\sqrt{2\pi}}\right)^s \prod_{j=1}^{s} \exp\left(-\frac{\xi_j^2}{2}\right) \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_s,$$

where $S_n = S_n(\xi_1, \ldots, \xi_n)$ is defined as in (9), and $\Delta t = T/s$. After the change of variables $x_i = \Phi(\xi_i)$, we obtain the third test problem:

$$\int_{[0,1]^s} \max\left(\frac{1}{s}\sum_{n=1}^{s} \tilde{S}_n(x_1, \ldots, x_n) - K, 0\right) \mathrm{d}x_1 \ldots \mathrm{d}x_s, \tag{15}$$

where $\tilde{S}_n(x_1, \ldots, x_n) = S_n(\Phi^{-1}(x_1), \ldots, \Phi^{-1}(x_n))$. The parameters used in the experiment are as follows (we used the same parameters as in [2]):

current price, S_0	100
strike price, K	100
riskless interest rate, r	0.09
time to maturity, T	1
volatility, σ	0.5
s	52

The result is shown in Figure 3. Similarly to the case of MBS, the convergence of the error for Sobol' sequences is very slow. Applying the same heuristics, i.e. a permutation, we have the improved convergence, see Figure 3(e), (f).

4 Conclusion

We presented the result of some numerical experiments on the statistical error estimation of QMC integrations. The numerical experiments show that both methods lead to good error estimates. As for the magnitudes of errors given by two methods, the estimated error value of one method can become bigger or smaller than another in different experiments. Thus we cannot provide a definite conclusion on which method gives a shaper error estimate.

The implementation of scrambling method is very complicated and the method is very time-consuming. On the other hand, the random shifting method is simple and very fast. From a practical point of view, we consider random shifting method can be a good alternative of scrambling method.

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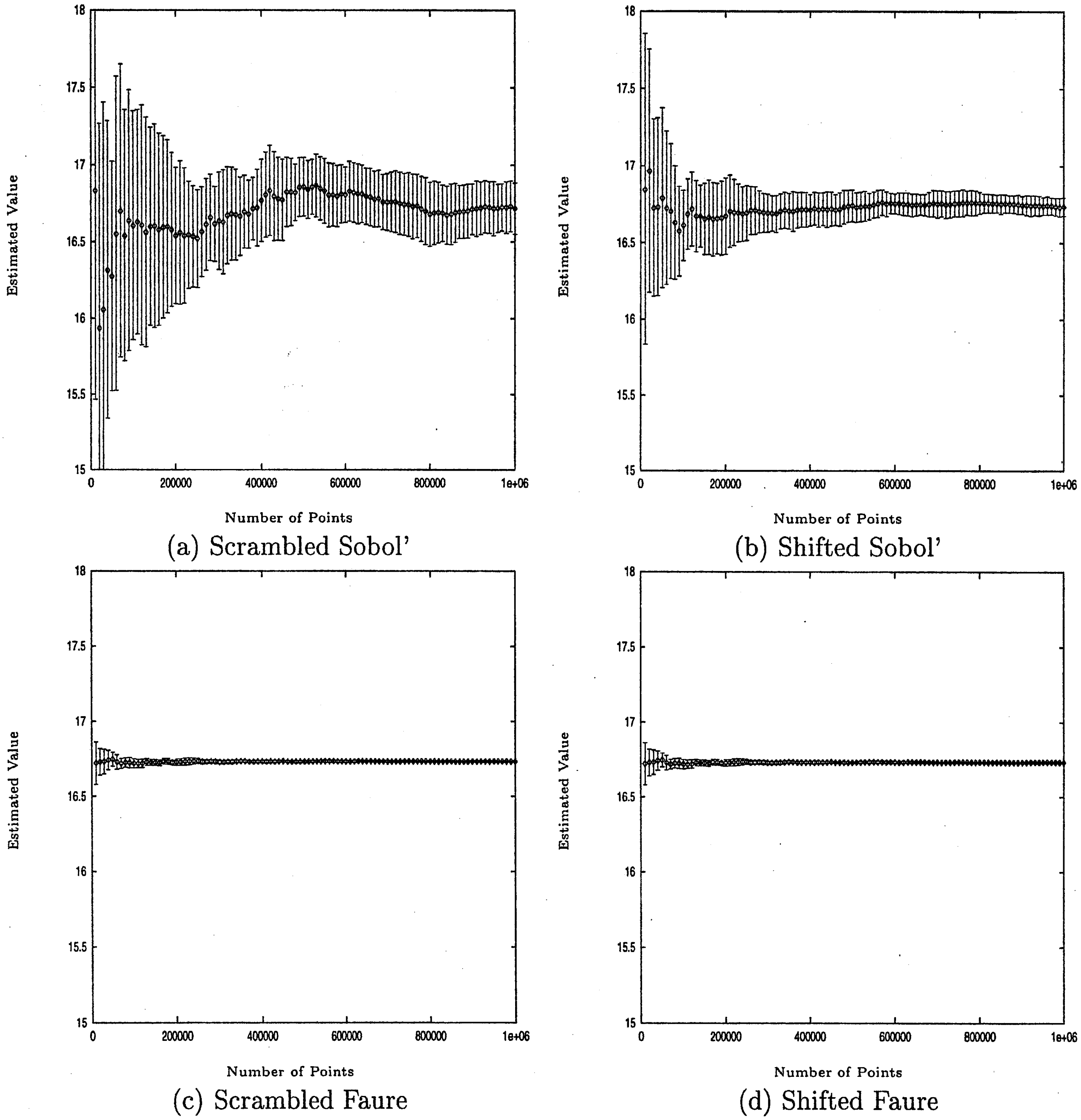


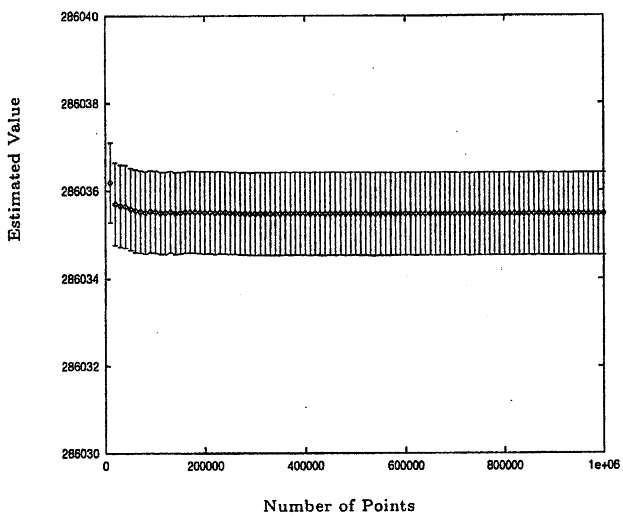
(a) Scrambled Sobol'

(b) Shifted Sobol'

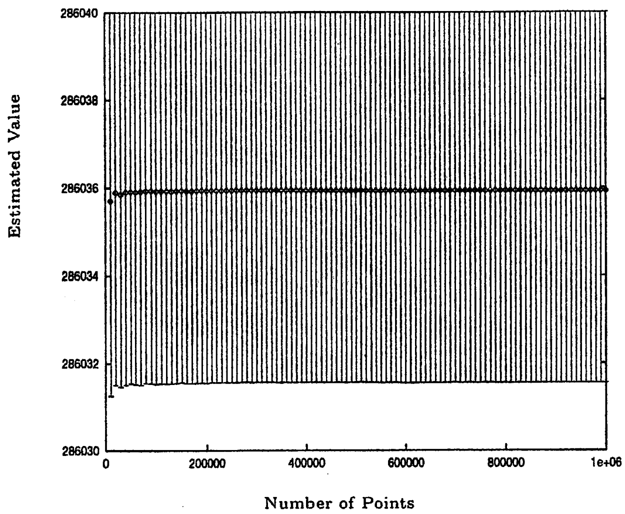
(c) Scrambled Faure

(d) Shifted Faure

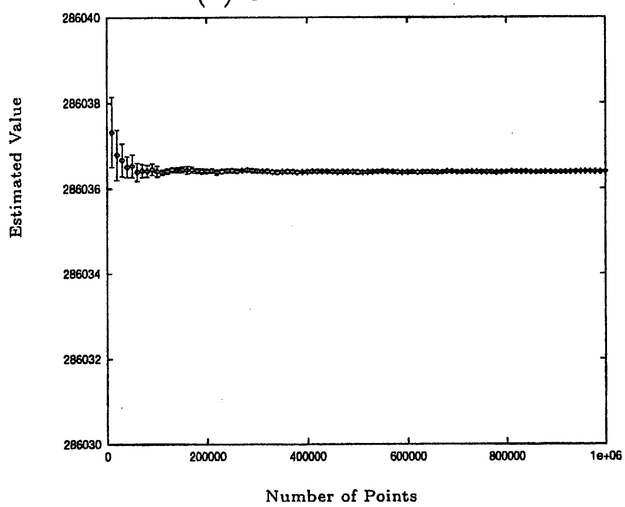
Figure 1: Pricing European call option and its error estimate.



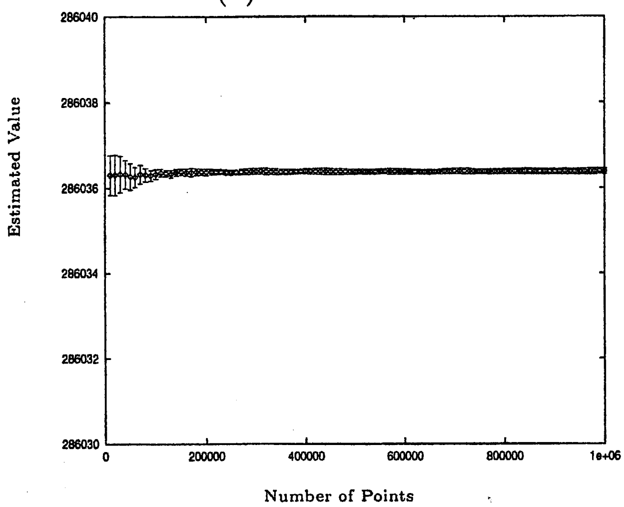
(a) Scrambled Sobol'



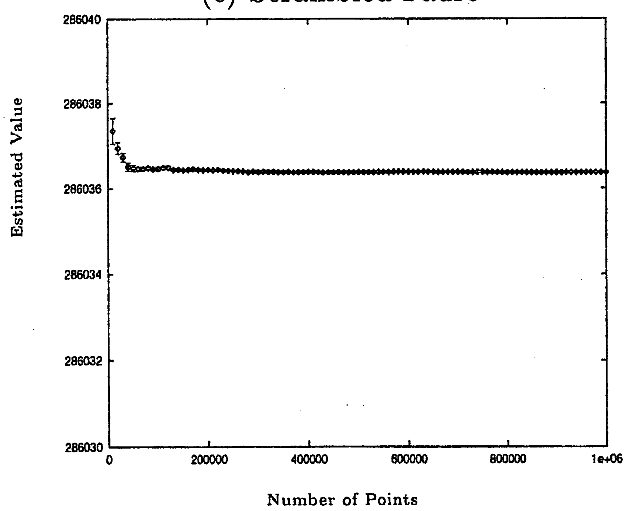
(b) Shifted Sobol'



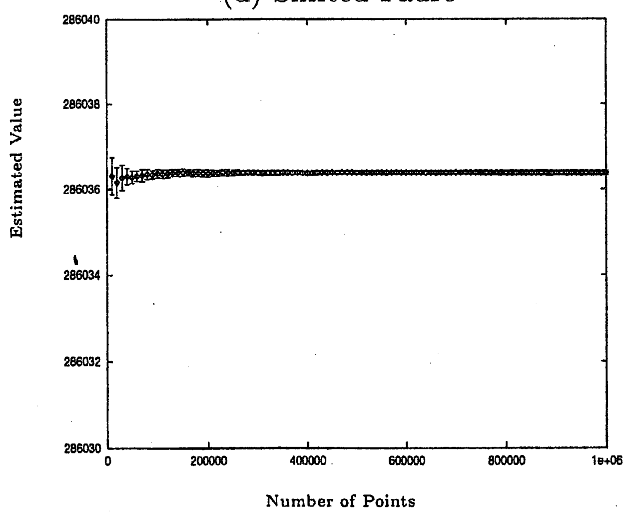
(c) Scrambled Faure



(d) Shifted Faure

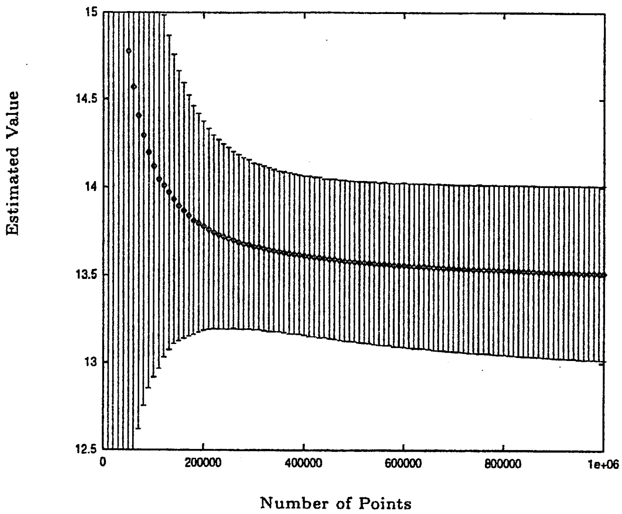


(e) Scrambled Sobol'(with permuting)

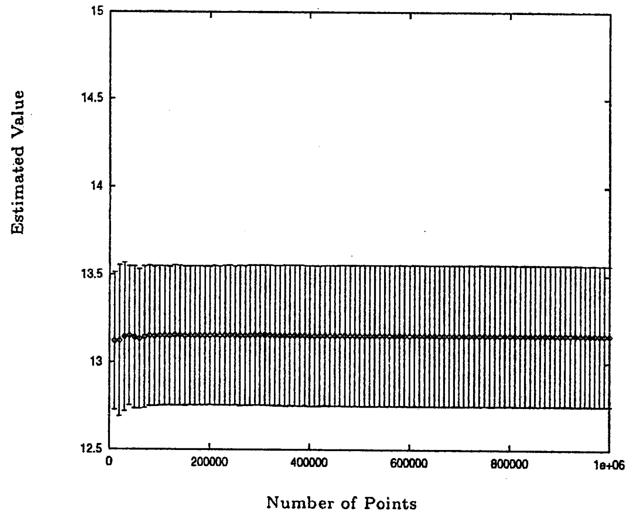


(f) Shifted Sobol'(with permuting)

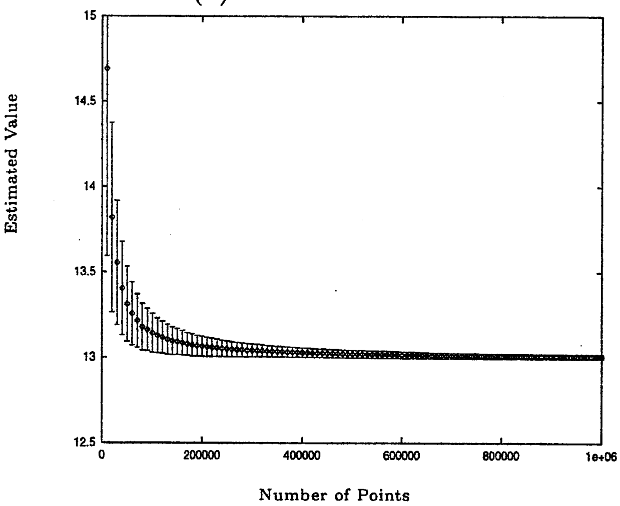
Figure 2: Pricing MBS and its error estimate.



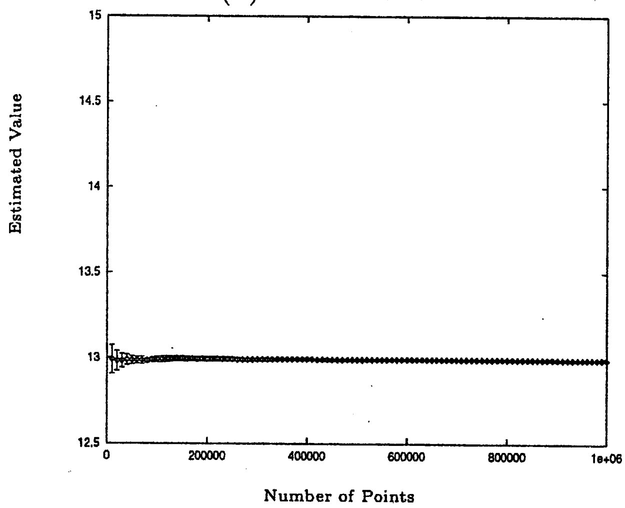
(a) Scrambled Sobol'



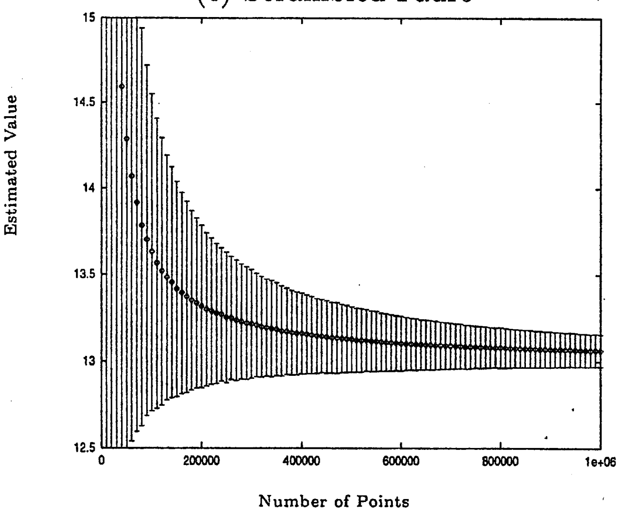
(b) Shifted Sobol'



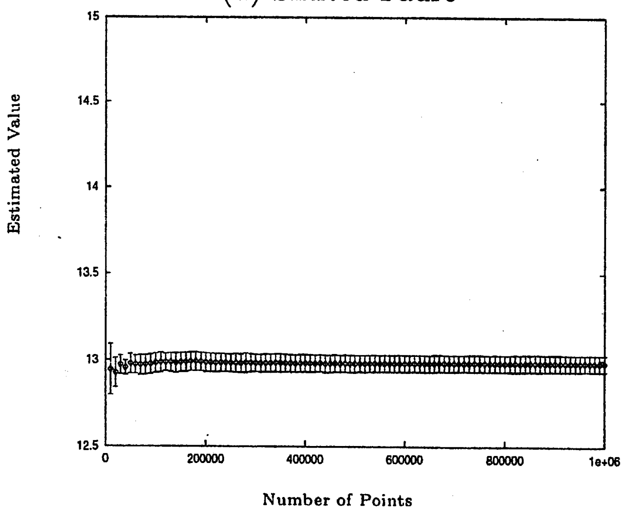
(c) Scrambled Faure



(d) Shifted Faure



(e) Scrambled Sobol'(with permuting)



(f) Shifted Sobol'(with permuting)

Figure 3: Pricing Asian option and its error estimate.